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Bezier variant of a new Durrmeyer type operators (**)

1 - Introduction

Durrmeyer [3] introduced the integral modification of Bernstein polynomials to approximate Lebesgue integrable functions on the interval $[0, 1]$. The operators introduced by Durrmeyer are defined by

$$(1)\qquad B_n(f, x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\, f(t)\, dt, \qquad x \in [0, 1]$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Recently Gupta [6] introduced a slight modification of the operators (1) and studied the rate of convergence for functions of bounded variation. We now define a new type of Durrmeyer operators to approximate Lebesgue integrable functions on the interval $[0, 1]$ as

$$(2)\quad P_n(f, x) = n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)\, f(t)\, dt + (1-x)^n f(0), \qquad x \in [0, 1]$$

where $p_{n,k}$ is as defined in (1).

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Chang [2] studied some approximation properties of Bernstein Bezier polynomials. Zeng and Piriou [7] and Zeng and Chen [8] estimated the rate of convergence for the Bezier variants of Bernstein polynomials and their Kantorovitch and Durrmeyer modifications. In computer aided design Bezier basis functions play an important role. This along with the recent work on Bernstein Bezier type operators, motivated us to study further on some different operators. We observe that the approximation properties of the operators (2) are entirely different from the usual Durrmeyer operators. For a function f defined on $[0,1]$ and $\alpha \geqslant 1$, we introduce the Bezier variant of the operators (2) as

$$P_{n,\alpha}(f,x) = n\sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x)\int_0^1 p_{n-1,k-1}(t)\, f(t)\, dt + Q_{n,0}^{(\alpha)}(x)\, f(0), \tag{3}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$ and $J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x)$.

Some important properties of $J_{n,k}(x)$ are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x),\ k = 0, 1, 2, 3, \dots;$

(ii) $J'_{n,k}(x) = np_{n-1,k-1}(x),\ k = 1, 2, 3, \dots;$

(iii) $J_{n,k}(x) = n\int_0^x p_{n-1,k-1}(u)\, du,\ k = 1, 2, 3, \dots;$

(iv) $J_{n,0}(x) > J_{n,1}(x) > J_{n,2}(x) > \dots > J_{n,n}(x)$.

For every natural number k, $J_{n,k}(x)$ increases strictly from 0 to 1 on $[0,1]$.

Alternatively we may rewrite the operators (3) as

$$P_{n,\alpha}(f,x) = \int_0^1 K_{n,\alpha}(x,t)\, f(t)\, dt, \qquad 0 \leqslant x \leqslant 1 \tag{4}$$

where $K_{n,\alpha}(x,t) = n\sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x)\, p_{n-1,k-1}(t) + Q_{n,0}^{(\alpha)}(x)\, \delta(t)$, $\delta(t)$ being the Dirac delta function.

It is easily verified that $P_{n,\alpha}(f,x)$ are linear positive operators, $P_{n,\alpha}(1,x) = 1$ and for $\alpha = 1$, the operators $P_{n,\alpha}(f,x)$ reduce to the operators (2). For further properties of $Q_{n,k}^{(\alpha)}(x)$, we refer to the readers [7].

In the present paper, we study the rate of point wise convergence of the operators $P_{n,\alpha}(f,x)$ at those points $x \in (0,1)$ at which one sided limits $f(x-)$ and $f(x+)$ exist.

2 - Auxiliary results

In this section we give certain results, which are necessary to prove the main result.

Lemma 1. *For $m \in N^0$ (the set of non-negative integers), if we define*

$$P_n((t-x)^m, x) \equiv A_{n,m}(x) = n \sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)(t-x)^m \, dt + (-x)^m (1-x)^n .$$

Then

$$A_{n,0}(x) = 1, \; A_{n,1}(x) = \frac{-x}{(n+1)}, \; A_{n,2}(x) = \frac{x(1-x)(2n+1) - (1-3x)\,x}{(n+1)(n+2)}$$

and for $m \geq 1$ there holds the recurrence relation:

$$[n+m+1] A_{n,m+1}(x) = x(1-x)[A^{(1)}_{n,m}(x) + 2mA_{n,m-1}(x)] + [m(1-2x) - x] A_{n,m}(x) .$$

Proof. The values of $A_{n,0}(x)$, $A_{n,1}(x)$ easily follows from the definition. We prove the recurrence relation as follows:

$$x(1-x)\, A^{(1)}_{n,m}(x) = n \sum_{k=1}^{n} x(1-x)\, p^{(1)}_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)(t-x)^m \, dt$$

$$- mn \sum_{k=1}^{n} x(1-x)\, p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)(t-x)^{m-1} \, dt$$

$$- \{ n(-x)^m (1-x)^{n-1} + m(-x)^{m-1}(1-x)^n \}\, x(1-x) .$$

Now using the identity $x(1-x)\, p^{(1)}_{n,k}(x) = (k-nx)\, p_{n,k}(x)$, we obtain

$$x(1-x)[A^{(1)}_{n,m}(x) + mA_{n,m-1}(x)]$$

$$= n \sum_{k=1}^{n} (k-nx)\, p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)(t-x)^m \, dt + n(-x)^{m+1}(1-x)^n$$

$$= n \sum_{k=1}^{n} p_{n,k}(x) \int_0^1 [k-1-(n-1)\,t + (n-1)(t-x) + (1-x)]\, p_{n-1,k-1}(t)(t-x)^m \, dt$$

$$+ n(-x)^{m+1}(1-x)^n$$

$$= n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 t(1-t)\, p^{(1)}_{n-1,k-1}(t)(t-x)^m\, dt + (n-1)\, A_{n,m+1}(x) + (1-x)\, A_{n,m}(x) - (-x)^m(1-x)^n$$

$$= n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 [(1-2x)(t-x) + (t-x)^2 + x(1-x)]\, p^{(1)}_{n-1,k-1}(t)(t-x)^m\, dt + (n-1)\, A_{n,m+1}(x) + (1-x)\, A_{n,m}(x) - (-x)^m(1-x)^n$$

$$= -(m+1)(1-2x)[A_{n,m}(x) - (-x)^m(1-x)^n] + (m+2)[A_{n,m+1}(x) - (-x)^{m+1}(1-x)^n]$$

$$-x(1-x)\, m[A_{n,m-1}(x) - (-x)^{m-1}(1-x)^n] + (n-1)\, A_{n,m+1}(x) + (1-x)\, A_{n,m}(x) - (-x)^m(1-x)^n$$

$$= [(1-x) - (m+1)(1-2x)]\, A_{n,m}(x) + (n+m+1)\, A_{n,m+1}(x) - mx(1-x)\, A_{n,m-1}(x)\,.$$

This completes the proof of recurrence relation.

The value of $A_{n,2}(x)$ easily follows from the above recurrence relation.

Remark 1. For n is sufficiently large and $x \in (0, 1)$, it is observed that

$$\frac{x(1-x)}{n} \leqslant A_{n,2}(x) \leqslant \frac{2x(1-x)}{n}\,.$$

Lemma 2. *For every* $0 \leqslant k \leqslant n$, $x \in (0, 1)$ *and for all* $n \in N$, *we have*

$$p_{n,k}(x) \leqslant \frac{1}{\sqrt{2enx(1-x)}}\,.$$

Proof. In order to prove the lemma we have to maximize $\binom{n}{k} x^{k+\frac{1}{2}} \cdot (1-x)^{n-k+\frac{1}{2}}$, for $0 \leqslant k \leqslant n$ and $x \in (0, 1)$.

First $\binom{n}{k} x^{k+\frac{1}{2}}(1-x)^{n-k+\frac{1}{2}}$ is maximum for $x = \dfrac{k+1/2}{(n+1)}$.

Let

$$\nu(n, k) = \frac{\binom{n}{k}(k+1/2)^{k+1/2}(n-k+1/2)^{n-k+1/2}}{(n+1)^{n+1}} = \frac{\Gamma(n+1)}{(n+1)^{n+1}} f_n(k),$$

where

$$f_n(k) = \frac{(k+1/2)^{k+1/2}(n-k+1/2)^{n-k+1/2}}{\Gamma(k+1)\,\Gamma(n-k+1)}.$$

Thus with $x \in [0, n]$, it is easily remarked that $f_n(x) = f_n(n-x)$. We take the logarithm: $\eta_n(x) = \ln f_n(x)$ and then differentiate twice

$$\eta'_n(x) = \ln(x+1/2) - \ln(n-x+1/2) - (\ln\Gamma(x+1))' - (\ln\Gamma(n-x+1))'$$

$$\eta''_n(x) = \left\{\frac{1}{x+1/2} - (\ln\Gamma(x+1))''\right\} + \left\{\frac{1}{n-x+1/2} - (\ln\Gamma(n-x+1))''\right\}$$

$$= \psi(x) + \psi(n-x), \text{ say}.$$

Next $\ln f_n(x)$ is convex on $[0, n]$, that is $\psi(x) + \psi(n-x) \geqslant 0$. To complete the assertion it is sufficient to show that $\psi(x) \geqslant 0$. We have

$$(\ln\Gamma(x+1))'' \sum_{q\geqslant 0} \frac{1}{(x+1+q)^2}.$$

The strike is: $\displaystyle \frac{1}{\mathrm{x}+1/2} = \sum_{p\geqslant 1}\left(\frac{1}{x+p-1/2} - \frac{1}{x+p+1/2}\right)$

$$= \sum_{p\geqslant 1}\frac{1}{(x+p)^2-1/4} = \sum_{q\geqslant 0}\frac{1}{(x+1+q)^2-1/4} > \sum_{q\geqslant 0}\frac{1}{(x+1+q)^2}.$$

Thus with the above it is obvious that f_n attains its maximum at $x=0$ (and $x=n$). So maximum is

$$f_n(0) = \frac{1}{\sqrt{2}}\frac{(n+1/2)^{n+1/2}}{\Gamma(n+1)} \text{ and } \nu(n, k) \text{ is maximum at } k=0.$$

This maximum is

$$\frac{\Gamma(n+1)}{(n+1)^{n+1}} f_n(0) = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2(n+1)}\right)^{n+1}\frac{1}{\sqrt{n+1/2}}$$

$$= \frac{1}{\sqrt{2n}}\left(1 - \frac{1}{2(n+1)}\right)^{n+1}\frac{\sqrt{n}}{\sqrt{n+1/2}}.$$

Here $n+1 \geqslant 1/2$, then $\displaystyle \left(1 - \frac{1}{2(n+1)}\right)^{n+1} \leqslant \frac{1}{\sqrt{e}}$ and $\displaystyle \sqrt{\frac{n}{n+1/2}} \leqslant 1.$

Thus $\varphi(n, k) \leqslant \nu(n, 0) \leqslant \dfrac{1}{\sqrt{2en}}$

This completes the proof of Lemma 2.

Lemma 3. *For all* $x \in (0, 1)$, *there hold*

$$Q_{n,k}^{(\alpha)}(x) \leqslant \alpha \,.\, p_{n,k}(x) \leqslant \frac{\alpha}{\sqrt{2enx(1-x)}} \,.$$

Proof. Using the well known inequality $|a^{\alpha} - b^{\alpha}| \leqslant \alpha |a - b|$, $(0 \leqslant a, b \leqslant 1, \alpha \geqslant 1)$ and by Lemma 2, we obtain

$$Q_{n,k}^{(\alpha)}(x) \leqslant \alpha p_{n,k}(x) \leqslant \frac{\alpha}{\sqrt{2enx(1-x)}} \,.$$

Lemma 4. *Let* $x \in (0, 1)$ *and* $K_{n,\alpha}(x, t)$ *be the kernel defined by* (4). *Then for n sufficiently large, we have*

$$\lambda_{n,\alpha}(x, y) = \int_0^y K_{n,\alpha}(x, t)\, dt \leqslant \frac{2\alpha \,.\, x(1-x)}{n(x-y)^2}\,, \quad 0 \leqslant y < x\,, \tag{5}$$

$$1 - \lambda_{n,\alpha}(x, z) = \int_z^1 K_{n,\alpha}(x, t)\, dt \leqslant \frac{2\alpha \,.\, x(1-x)}{n(z-x)^2}\,, \quad x < z < 1\,. \tag{6}$$

Proof. We first prove (5), as follows

$$\int_0^y K_{n,\alpha}(x, t)\, dt \leqslant \int_0^y K_{n,\alpha}(x, t)\, \frac{(x-t)^2}{(x-y)^2}\, dt$$

$$\leqslant \frac{1}{(x-y)^2} P_{n,\alpha}((t-x)^2, x) \leqslant \frac{\alpha \,.\, P_{n,1}((t-x)^2, x)}{(x-y)^2} \leqslant \frac{2\alpha \,.\, x(1-x)}{n(x-y)^2}\,,$$

by Lemma 1. The proof of (6) is similar.

3 - Main result

In this section we prove the following main theorem

Theorem. *Let f be a function of bounded variation on* $[0, 1]$, $\alpha \geq 1$. *Then for every* $x \in (0, 1)$ *and n sufficiently large, we have*

$$\left| P_{n,\alpha}(f, x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \leq \frac{\alpha}{\sqrt{2enx(1-x)}} |f(x+) - f(x-)|$$

$$+ \frac{5\alpha}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x \\ 0, & t = x \\ f(t) - f(x+), & x < t \leq 1 \end{cases}$$

and $v_a^b(g_x)$ *is the total variation of* g_x *on* [a, b].

Proof. Clearly Following [8], we have

$$\left| P_{n,\alpha}(f, x) - \left\{ \frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right\} \right|$$

(7)

$$\leq |P_{n,\alpha}(g_x, x)| + \frac{1}{2} \left| P_{n,\alpha}(\text{sign}\,(t-x), x) + \frac{\alpha-1}{\alpha+1} \right| |f(x+) - f(x-)| .$$

First, we have

$$P_{n,\alpha}(\text{sign}\,(t-x), x) = \int_x^1 K_{n,\alpha}(x, t)\, dt - \int_0^x K_{n,\alpha}(x, t)\, dt$$

$$= \int_0^1 K_{n,\alpha}(x, t)\, dt - 2 \int_0^x K_{n,\alpha}(x, t)\, dt$$

$$= 1 - 2 \int_0^x K_{n,\alpha}(x, t)\, dt = -1 + 2 \int_x^1 K_{n,\alpha}(x, t)\, dt .$$

Using Lemma 2, Lemma 3 and the fact that $\sum_{j=0}^{k-1} p_{n,j}(x) = n\int_x^1 p_{n-1,k-1}(t)\,dt$, we have

$$P_{n,\alpha}(\text{sign}\,(t-x),\,x) = -1 + 2n\sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x)\int_x^1 p_{n-1,k-1}(t)\,dt + 2\int_x^1 Q_{n,0}^{(\alpha)}(x)\,\delta(t)\,dt\,.$$

Since $\int_x^1 Q_{n,0}^{(\alpha)}(x)\,\delta(t)\,dt = 0$ as $x > 0$, thus

$$\begin{aligned} P_{n,\alpha}(\text{sign}\,(t-x),\,x) &= -1 + 2\sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x)\sum_{j=0}^{k-1} p_{n,j}(x) \\ &= -1 + 2\sum_{j=0}^{n} p_{n,j}(x)\sum_{k=j}^{n} Q_{n,k}^{(\alpha)}(x) = -1 + 2\sum_{j=0}^{n} p_{n,j}(x)\,J_{n,j}^{\alpha}(x)\,. \end{aligned}$$

Therefore we have

$$P_{n,\alpha}(\text{sign}\,(t-x),\,x) + \frac{\alpha-1}{\alpha+1} = 2\sum_{j=0}^{n} p_{n,j}(x)\,J_{n,j}^{\alpha}(x) - \frac{2}{\alpha+1}\sum_{j=0}^{n} Q_{n,j}^{(\alpha+1)}(x)\,.$$

Since $\sum_{j=0}^{n} Q_{n,j}^{(\alpha+1)}(x) = 1$, by mean value theorem, it follows

$$Q_{n,j}^{(\alpha+1)}(x) = J_{n,j}^{\alpha+1}(x) - J_{n,j+1}^{\alpha+1}(x) = (\alpha+1)\,p_{n,j}(x)\,\gamma_{n,j}^{\alpha}(x)\,,$$

where $J_{n,j+1}^{\alpha}(x) < \gamma_{n,j}^{\alpha}(x) < J_{n,j}^{\alpha}(x)$. Hence

$$\begin{aligned} \left| P_{n,\alpha}(\text{sign}\,(t-x),\,x) + \frac{\alpha-1}{\alpha+1} \right| &\leqslant 2\sum_{j=0}^{n} p_{n,j}(x)(J_{n,j}^{\alpha}(x) - \gamma_{n,j}^{\alpha}(x)) \\ &\leqslant 2\sum_{j=0}^{n} p_{n,j}(x)(J_{n,j}^{\alpha}(x) - J_{n,j+1}^{\alpha}(x)) \leqslant 2\alpha\sum_{j=0}^{n} p_{n,j}^{2}(x)\,, \end{aligned}$$

where we have used the inequality $b^{\alpha} - a^{\alpha} < \alpha(b-a)$, $0 \leqslant a, b \leqslant 1$ and $\alpha \geqslant 1$. Applying Lemma 2, we get

$$\left| P_{n,\alpha}(\text{sign}\,(t-x),\,x) + \frac{\alpha-1}{\alpha+1} \right| = \frac{2\alpha}{\sqrt{2enx(1-x)}}\,,\quad x \in (0,\,1)\,. \tag{8}$$

Next we estimate $P_{n,\alpha}(g_x, x)$. By Lebesgue-Stieltjes integral representation, we have

$$P_{n,\alpha}(g_x, x) = \int_0^1 K_{n,\alpha}(x, t)\, g_x(t)\, dt = \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) K_{n,\alpha}(x, t)\, g_x(t)\, dt \tag{9}$$

$$= E_1 + E_2 + E_3, \quad \text{say},$$

where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + (1-x)/\sqrt{n}]$ and $I_3 = [x + (1 - x)/\sqrt{n}, 1]$. We first estimate E_1. Writing $y = x - x/\sqrt{n}$ and using Lebesgue-Stieltjes integration by parts, we have

$$E_1 = \int_0^y g_x(t)\, d_t(\lambda_{n,\alpha}(x, t)) = g_x(y+)\, \lambda_{n,\alpha}(x, y) - \int_0^y \lambda_{n,\alpha}(x, t)\, d_t(g_x(t)).$$

Since $|g_x(y+)| \leqslant v_{y+}^{x}(g_x)$, it follows that

$$|E_1| \leqslant V_{y+}^{x}(g_x)\, \lambda_{n,\alpha}(x, y) + \int_0^y \lambda_{n,\alpha}(x, t)\, d_t(-V_t^x(g_x)).$$

By using (5) of Lemma 4, we get

$$|E_1| \leqslant V_{y+}^{x}(g_x)\, \frac{2\alpha . x(1-x)}{n(x-y)^2} + \frac{2\alpha . x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)).$$

Integrating by parts the last term we have after simple computation

$$|E_1| \leqslant \frac{2\alpha . x(1-x)}{n} \left[\frac{V_0^x(g_x)}{x^2} + 2 \int_0^y \frac{V_t^x(g_x)}{(x-t)^3}\, dt \right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{t}$, we obtain

$$|E_1| \leqslant \frac{2\alpha(1-x)}{nx} \left[V_0^x(g_x) + \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x) \right] \leqslant \frac{4\alpha}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x). \tag{10}$$

Using the similar method and (6) of Lemma 4, we get

$$|E_3| \leqslant \frac{4\alpha}{nx(1-x)} \sum_{k=1}^{n} V_x^{x+(1-x)/\sqrt{k}}(g_x). \tag{11}$$

Finally we estimate E_2. For $t \in [x - x/\sqrt{n}, x + (1 - x/\sqrt{n}]$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x)$$

and therefore

$$|E_2| \leqslant V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}}(g_x) \int_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}} d_t(\lambda_{n,\alpha}(x, t)).$$

Since $\int_a^b d_t \lambda_{n,\alpha}(x, t) \leqslant 1$, for all $(a, b) \subseteq [0, 1]$, therefore

(12) $$|E_2| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x).$$

Collecting the estimates (9)-(12), we have

(13) $$|P_{n,\alpha}(g_x, x)| \leqslant \frac{5\alpha}{nx(1-x)} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}}(g_x).$$

Combining the estimates of (7), (8) and (13), our theorem follows.

Remark 2. By taking the Durrmeyer-Bezier type operators of the form (3) some approximation formulae become simpler (i.e. we do not require the results of the type Lemma 3 and Lemma 4 of [8]), in our case.

Remark 3. Using our Lemma 2, we can improve the main results of [**1**], [**4**] and [**6**] for Bernstein Durrmeyer type operators.

4 - Our estimate is asymptotically optimal

We shall show that our estimate is essentially the best possible for continuity points of bounded variation functions f. If x is continuity point of f, then the conclusion of our Theorem becomes

(14) $$|P_{n,\alpha}(f, x) - f(x)| \leqslant \frac{5\alpha}{nx(1-x)} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}}(g_x), \qquad \alpha \geqslant 1.$$

First we prove the result for $\alpha = 1$. Consider the function $f(t) = |t - x|$ on the closed interval $[0, 1]$, where $0 < x < 1$. By Lemma 1, for any small $\delta > 0$ and n sufficiently large, we have

$$P_n(|t-x|, x) = \left(\int\limits_{|t-x| \leqslant \delta} + \int\limits_{|t-x| > \delta} \right) K_{n,\alpha}(x, t)\, |t-x|\, dt \tag{15}$$
$$\leqslant \delta + \frac{1}{\delta} A_{n,2}(x) \leqslant \delta + \frac{2x(1-x)}{n\delta}$$

and

$$P_n(|t-x|, x) \geqslant \int\limits_{x-\delta}^{x+\delta} K_{n,\alpha}(x, t)\, |t-x|\, dt \geqslant \frac{1}{\delta} \int\limits_{x-\delta}^{x+\delta} K_{n,\alpha}(t)(t-x)^2\, dt \tag{16}$$
$$\geqslant \frac{x(1-x)}{n\delta} - \frac{1}{\delta^3} A_{n,4}(x) = \frac{x(1-x)}{n\delta} - \frac{C_1}{\delta^3 n^2},$$

where C_1 is a constant. Choosing $\delta = 2\sqrt{C_1/nx(1-x)}$, we obtain from (15) and (16) that

$$\frac{3\{x(1-x)\}^{3/2}}{8\sqrt{nC_1}} \leqslant P_n(|t-x|, x) \leqslant \frac{2C_1\{x(1-x)\}^{-1/2} + \{x(1-x)\}^{3/2}}{\sqrt{Cn}}. \tag{17}$$

Also, $V_{x-\beta}^{x+\alpha}(f) = \alpha + \beta$, we get

$$|P_n(f, x) - f(x)| \leqslant \frac{5}{nx(1-x)} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}}(f) \tag{18}$$
$$\leqslant \frac{5}{nx(1-x)} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant \frac{10\{x(1-x)\}^{-1}}{\sqrt{n}}.$$

From (17) and (18) we conclude that the conclusion of our main theorem can not be asymptotically improved for bounded variation functions.

Next when $\alpha \neq 1$, consider the function $f(t) = t$. By (14), we have

$$|P_{n,\alpha}(f, x) - f(x)| \leqslant \frac{5\alpha}{nx(1-x)} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant \frac{10\alpha\{x(1-x)\}^{-1}}{\sqrt{n}}. \tag{19}$$

On the other hand, using the following inequality (see [7], (38)), for n sufficiently large

$$\left| \sum_{k=1}^{n} \frac{k}{n} Q_{n,k}^{(\alpha)}(x) - x \right| \geqslant C_2 \frac{\sqrt{x(1-x)}}{\sqrt{n}}, \tag{20}$$

where C_2 is a positive constant.

Hence for n sufficiently large

$$|P_{n,\alpha}(t, x) - x| = \left| \int_0^1 K_{n,\alpha}(x, t)\, t dt - x \right|$$

$$= \left| \sum_{k=1}^{n} \frac{k}{n+1} Q_{n,k}^{(\alpha)}(x) - x \right| = \left| \frac{n}{n+1} \left[\sum_{k=1}^{n} \frac{k}{n} Q_{n,k}^{(\alpha)}(x) - x \right] + \frac{-x}{n+1} \right| \tag{21}$$

$$\geqslant \left| \frac{n}{n+1} \left[\sum_{k=1}^{n} \frac{k}{n} Q_{n,k}^{(\alpha)}(x) - x \right] \right| - \frac{1}{n+1} \geqslant C_2 \frac{\sqrt{x(1-x)}}{2\sqrt{n}},$$

by (20) and $\dfrac{1}{C_2\sqrt{nx(1-x)}} \leqslant \left(\dfrac{1}{2} - \dfrac{1}{n} \right)$.

Hence by (19) and (21), we conclude that (14) can not be asymptotically improved when $n \to \infty$. Hence for $\alpha \geqslant 1$, our estimate in the main theorem is asymptotically optimal.

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Summary

In this paper, we introduce a Bezier variant of a new type of Durrmeyer operators and estimate the rate of convergence of functions of bounded variation. Our result improves and extends the results of Guo (*J. Approx Theory* **51** (1987), 183-192) *and Zeng and Chen* (*J. Approx. Theory* **102** (2000), 1-12). *In the end we show that our estimate is asymptotically optimal.*

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