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Discrete time uniformly recurrent Markov additive processes: two simplified models and large deviations (**)

1 - Introduction

The literature on Markov additive processes is extensive and a source of references can be found in [1] (chapter 2, section 5, page 47).

The aim of this paper is to present a discrete time version of some results presented for the continuous time case in [7], even if in this paper the environment state space could be not finite.

In [7] (subsection 4.2) the inequalities between rate functions are an immediate consequence of a result with a cumbersome proof (Theorem 3.1 in [7]) while in this paper we use a different and simpler procedure. In order to explain the procedure used here, let us point out that the rate functions in this paper are the Legendre transforms κ^*, κ_F^* and κ_A^* of suitable functions κ, κ_F and κ_A respectively (see (5), (9) and (12)). Then the proofs of the inequalities between rate functions in this paper are an immediate consequence of some other inequalities which are easy to check, namely the opposite inequalities between the functions κ, κ_F and κ_A.

In [7] the author defined the fluid model and the averaged parameters model as two simplified Markov additive processes derived in a suitable way from a ge-

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neral continuous time Markov additive process with a finite environment state space and a real valued additive part (see [**7**], subsection 4.1). In particular, when the environment is an irreducible Markov chain, the author proved some inequalities between rate functions having a common unique zero; the interest of such inequalities consists to say that, in some sense, a convergence is not faster than another one (see [**7**], subsection 4.2). Furthermore the inequalities between rate functions can also allow to compare some decay rate for level crossing probabilities (see [**7**], section 5).

The definition and some results in [**7**] concerning averaged parameters model are inspired by the content of [**2**]. Anyway we point out that, when we deal with a discrete Markov additive process, the additive part is a generalization of two kinds of sequences of random variables: a sequence which evolves as a deterministic function of the environment; a random walk independent of the environment. Then, for a given discrete time Markov additive process, it is natural to study fluid model and averaged parameters model as sequences of the first kind and the second kind respectively. Indeed, under suitable hypotheses, the additive part of the Markov additive process (with suitable normalization) converges; moreover the analogous limits for the simplified models exist and are equal.

The paper is organized as follows. Section 2 is devoted to recall some preliminaries. In particular we present the uniformly recurrence hypothesis in [**5**] (section 3) which plays a crucial role in large deviations results. Furthermore in section 2 we also recall a known characterization presented in [**4**]; such a characterization (see Proposition 2.2 in this paper) provides a concrete presentation of discrete time Markov additive processes which can be seen as a natural extension of the description in [**1**] (page 40) for the discrete time case with a finite environment state space.

Section 3 in this paper is devoted to define fluid model and averaged parameters model in discrete time case together with the proof of the same inequalities proved for the continuous time case in [**7**]. This will be done even if the environment state space is not finite and the additive part is $\mathbb{R}^d$-valued (for some $d \geqslant 1$). Thus we have the same consequences in terms of comparison between convergences of Markov additive processes and, when $d = 1$, between decay rates of the corresponding level crossing probabilities.

At the end of this section we point out what follows. All the Markov processes in this paper are homogeneous; we denote the scalar product in some $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$ and the environment state space by E (as we shall see in the next Definition 2.1), so that it will be useful to consider the notation

$$E^h = \underbrace{E \times \cdots \times E}_{h \text{ times}}.$$

2 - Preliminaries

Let us recall the definition of Markov additive process and let us consider the discrete time case.

Definition 2.1. *Let* $(E, \mathcal{E})$ *be a measurable space and let* $(\mathcal{F}_n)$ *be a filtration. Then a sequence of* $E \times \mathbb{R}^d$*-valued random variables* $((J_n, S_n))$ *adapted to* $(\mathcal{F}_n)$ *is a Markov additive process if*

$$(1) \qquad P(J_n \in B, S_n - S_{n-1} \in \Gamma | \mathcal{F}_{n-1}) = P(J_n \in B, S_n - S_{n-1} \in \Gamma | J_{n-1}) =: P(J_{n-1}, B \times \Gamma)$$

for each $B \in \mathcal{E}$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ *and* $n \geq 1$. *In such a case* (J_n) *is called environment,* (S_n) *is called additive part and the kernel* $(P(y, \cdot \times \cdot) : y \in E)$ *is called MA kernel.*

In order to have a simpler presentation, the additive parts of all Markov additive processes in this paper start at the origin $0 \in \mathbb{R}^d$. Moreover we refer to the following characterization which can be extended to Markov additive processes with additive part taking values in a Hilbert space (see e.g. [4], section 2, Corollary 1).

Proposition 2.2. *Let* $(E, \mathcal{E})$ *be a measurable space and let* $((J_n, S_n))$ *be a sequence of* $E \times \mathbb{R}^d$*-valued random variables adapted to a filtration* $(\mathcal{F}_n)$. *Moreover let* $\mathcal{F}^J_\infty = \sigma(J_n : n \geq 0)$ *be the σ-field generated by* $J = (J_n)$. *Then the two following statements are equivalent:*

(i) $((J_n, S_n))$ *is a Markov additive process with MA kernel* $(P(y, \cdot \times \cdot) : y \in E)$;

(ii) there exist kernels $(H(y, y', \cdot) : y, y' \in E)$ *and* $(Q(y, \cdot) : y \in E)$ *such that*

$$P(S_n - S_{n-1} \in \Gamma | \mathcal{F}_{n-1} \vee \mathcal{F}^J_\infty) \equiv H(J_{n-1}, J_n, \Gamma);$$

$$P(J_n \in B | \mathcal{F}_{n-1}) \equiv Q(J_{n-1}, B);$$

$$P(y, B \times \Gamma) \equiv \int_B H(y, y', \Gamma)\, Q(y, dy').$$

Thus, roughly speaking, we can say what follows:
$J = (J_n)$ is a E-valued Markov process with kernel $(Q(y, \cdot) : y \in E)$;
for each $n \geq 1$ the conditional distribution of $S_n - S_{n-1}$ given J depends on (J_{n-1}, J_n) only and it is $H(J_{n-1}, J_n, \cdot)$.

Now let us introduce the k-th power of the MA kernel $(P(y, \cdot \times \cdot) : y \in E)$ defined by

$$P^{k}(y, B \times \Gamma) \equiv \int_{E^{k-1} \times B} H(y, y_1, \cdot)^{*} \cdots ^{*} H(y_{k-1}, y_k, \cdot)](\Gamma)\, Q(y, dy_1) \cdots Q(y_{k-1}, dy_k)$$

(where $*$ denotes convolution); in other words we are considering a generalization of (1):

$$P(J_n \in B, S_n - S_{n-k} \in \Gamma \,|\, \mathcal{F}_{n-k}) = P(J_n \in B, S_n - S_{n-k} \in \Gamma \,|\, J_{n-k}) =: P^{k}(J_{n-k}, B \times \Gamma)$$

for each $B \in \mathcal{E}$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ and $n \geq k \geq 1$.

In view of the large deviations results presented below, throughout this paper we always deal with Markov additive processes satisfying the following hypotheses presented in [**5**] (section 3).

Hypothesis (INN). *Let (J_n) be an irreducible and aperiodic Markov process with respect to some maximal irreducibility measure. Assume that there exist a probability measure ν on $E \times \mathbb{R}^d$, an integer $m_0 \geq 1$ and some numbers $0 < a < b < \infty$ such that*

$$a\nu(B \times \Gamma) \leq P^{m_0}(y, B \times \Gamma) \leq b\nu(B \times \Gamma)$$

for each $y \in E$, $B \in \mathcal{E}$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. Moreover let

$$\mathcal{D} = \left\{ \theta \in \mathbb{R}^d : \int_{E \times \mathbb{R}^d} e^{\langle \theta, s \rangle} \nu(dy, ds) < \infty \right\}$$

and let $\mathcal{S}$ be the convex hull of the support of $\nu(E \times \cdot)$; then we assume that $\mathcal{D}$ is open and $\mathcal{S}^{\circ} \neq \emptyset$.

Furthermore it is useful to consider the moment generating functions $(\widehat{H}(y, y', \cdot) : y, y' \in E)$ defined as follows (for $\theta \in \mathbb{R}^d$)

$$\widehat{H}(y, y', \theta) \equiv \int_{\mathbb{R}^d} e^{\langle \theta, s \rangle} H(y, y', ds)$$

and the function $\widehat{P}$ defined by

$$\widehat{P}(y, B, \theta) \equiv \int_B \widehat{H}(y, y', \theta)\, Q(y, dy').$$

It is also useful to consider the k-th power of $\widehat{P}$ defined by

$$\widehat{P}^k(y, B, \theta) \equiv \int_{E^{k-1}\times B} \widehat{H}(y, y_1, \theta)\cdots\widehat{H}(y_{k-1}, y_k, \theta)\, Q(y, dy_1)\cdots Q(y_{k-1}, dy_k) \tag{2}$$

and, in other words, we are considering the following equalities:

$$\mathbb{E}[e^{\langle\theta, S_n - S_{n-k}\rangle}1_B(J_n)\,|\,\mathcal{F}_{n-k}] \equiv \mathbb{E}[e^{\langle\theta, S_n - S_{n-k}\rangle}1_B(J_n)\,|\,J_{n-k}] \equiv \widehat{P}^k(J_{n-k}, B, \theta) \tag{3}$$

for each $B \in \mathcal{E}$, $\theta \in \mathbb{R}^d$ and $n \geq k \geq 1$.

The next Proposition 2.5 provides the LDP for $\left(\frac{S_n}{n}\right)$ with a rate function which does not depend on the initial distribution π of J (namely the distribution of J_0). Thus, in view of what follows, we use the notation $\mathbb{E}^{(\pi)}[\cdot]$ when the initial distribution of J is π. Before presenting the LDP, the next Lemma is needed (see [**5**]: section 3 for (i) and (ii); section 4 for (iii)).

Lemma 2.3. *Assume* (INN) *holds. Then:*

(i) *for each fixed $\theta \in \mathcal{D}$, $\widehat{P}(\theta)$ has a simple, maximal and positive eigenvalue $e^{\kappa(\theta)}$ with associated right eigenfunction $r(\cdot, \theta)$ which is uniformly positive and bounded;*

(ii) *the function κ is analytic, strictly convex and essentially smooth on $\mathcal{D}$;*

(iii) *there exists a stationary distribution $\pi^{(J)}$ for J.*

Remark 2.4. *We could have $\mathcal{D} \neq \mathbb{R}^d$ and, in such a case, we think to extend the definition of κ on all $\mathbb{R}^d$ by setting $\kappa(\theta) = \infty$ for $\theta \notin \mathcal{D}$.*

We remark that, by (i) in Lemma 2.3, for all $n \geq 1$ we have

$$\int_{E^n} \widehat{H}(y, y_1, \theta)\cdots\widehat{H}(y_{n-1}, y_n, \theta)\, r(y_n, \theta)\, Q(y, dy_1)\cdots Q(y_{n-1}, dy_n) \equiv e^{n\kappa(\theta)} r(y, \theta)\,. \tag{4}$$

Proposition 2.5. *Assume* (INN) *holds. Then, whatever is the initial distribution π for J, $\left(\frac{S_n}{n}\right)$ satisfies the LDP with rate function κ^* defined by*

$$\kappa^*(s) \equiv \sup_{\theta\in\mathbb{R}^d}[\langle\theta, s\rangle - \kappa(\theta)]\,. \tag{5}$$

Proof. Let $n \geq 1$, $y \in E$ and $\theta \in \mathcal{D}$ be arbitrarily fixed; indeed we can neglect the case $\theta \notin \mathcal{D}$ by Remark 2.4 and (5). Moreover let us consider (2) with

$k = n$ and $B = E$:

$$\widehat{P}^n(y, E, \theta) \equiv \int_{E^n} \widehat{H}(y, y_1, \theta) \cdots \widehat{H}(y_{n-1}, y_n, \theta)\, Q(y, dy_1) \cdots Q(y_{n-1}, dy_n).$$

Then, by (i) in Lemma 2.3, we have

$$\int_{E^n} \widehat{H}(y, y_1, \theta) \cdots \widehat{H}(y_{n-1}, y_n, \theta) \frac{r(y_n, \theta)}{\sup_{z \in E} r(z, \theta)} Q(y, dy_1) \cdots Q(y_{n-1}, dy_n) \leqslant \widehat{P}^n(y, E, \theta)$$

$$\leqslant \int_{E^n} \widehat{H}(y, y_1, \theta) \cdots \widehat{H}(y_{n-1}, y_n, \theta) \frac{r(y_n, \theta)}{\inf_{z \in E} r(z, \theta)} Q(y, dy_1) \cdots Q(y_{n-1}, dy_n);$$

thus, by (4), we obtain

$$e^{n\kappa(\theta)} \frac{r(y, \theta)}{\sup_{z \in E} r(z, \theta)} \leqslant \widehat{P}^n(y, E, \theta) \leqslant e^{n\kappa(\theta)} \frac{r(y, \theta)}{\inf_{z \in E} r(z, \theta)}. \tag{6}$$

Now let π be any initial distribution for J and, by (3) with $k = n$ and $B = E$, we have

$$\mathbb{E}^{(\pi)}[e^{\langle \theta, S_n \rangle}] \equiv \int_E \widehat{P}^n(y, E, \theta)\, \pi(dy). \tag{7}$$

In conclusion

$$e^{n\kappa(\theta)} \frac{\int_E r(y, \theta)\, \pi(dy)}{\sup_{z \in E} r(z, \theta)} \leqslant \mathbb{E}^{(\pi)}[e^{\langle \theta, S_n \rangle}] \leqslant e^{n\kappa(\theta)} \frac{\int_E r(y, \theta)\, \pi(dy)}{\inf_{z \in E} r(z, \theta)},$$

follows from (6) and (7), so that we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}^{(\pi)}[e^{\langle \theta, S_n \rangle}] \equiv \kappa(\theta). \tag{8}$$

Then the proof is complete as a consequence of Gärtner Ellis Theorem (see e.g. [3], chapter 2, section 3) and by (ii) in Lemma 2.3. ■

Remark 2.6. *One can check that* $\frac{S_n}{n}$ *converges to the unique zero* ℓ *of* κ^*, *where*

$$\ell = \int_E \left[\int_E \nabla \widehat{H}(y, y', \theta) \,|_{\theta = 0} Q(y, dy') \right] \pi^{(J)}(dy)$$

and

$$\nabla \widehat{H}(y, y', \theta) \,|_{\theta = 0} \equiv \int_{\mathbb{R}^d} s H(y, y', ds) .$$

Remark 2.7. *Let us consider the case* $d = 1$ *and assume the following condition holds:*

$(\mathbf{L})$: there exists $w > 0$ such that $\kappa(w) = 0$ and $\kappa'(w) > 0$.

Then we can derive the limits below as in [**6**] (*Theorem* 3.1 *and* (3.4) *in Theorem* 3.2) *which are proved in* [**6**] *when* E *is finite and the initial distribution of* (J_n) *is deterministic. Indeed, when* (INN) *holds, we have a conjugate family of kernels with the same structure as in* [**6**] (*see* [**5**], *section* 4); *then the proofs can be easily adapted to our case since we have*

$$\sup_{y' \in E} \frac{\int_E r(y, w)\, \pi(dy)}{r(y', w)} < \infty .$$

The limits concern the level crossing probabilities $\psi(b)$ *and the second moment of importance sampling estimator* $\eta(b)$ (*here* $b > 0$ *is the level*) *and we have:*

$$\lim_{b \to \infty} \frac{1}{b} \log \psi(b) = -w \quad \text{and} \quad \lim_{b \to \infty} \log \eta(b) = -2w .$$

Finally we point out the equality $w = \inf_{s > 0} \frac{\kappa^*(s)}{s}$.

3 - Two simplified models derived from $((J_n, S_n))$

In this section we present two Markov additive processes derived in a suitable way from $((J_n, S_n))$: the fluid model $((J_n^{(F)}, S_n^{(F)}))$ and the averaged parameters model $((J_n^{(A)}, S_n^{(A)}))$. In order to have a simpler presentation we assume that

$(J_n^{(F)}) = (J_n^{(A)}) = (J_n)$; more precisely we could say that $(J_n^{(F)})$ and $(J_n^{(A)})$ have the same distribution of (J_n).

3.1 - *Fluid model*

In order to define fluid model we introduce the kernel $(H_F(y, y', \cdot): y, y' \in E)$ which plays the role of $(H(y, y', \cdot): y, y' \in E)$ when we have fluid model in place of $((J_n, S_n))$. In general $H_F(y, y', \cdot)$ is the distribution of the constant random variable equal to the mean value $\int_{\mathbb{R}^d} sH(y, y', ds)$. Thus the moment generating functions $(\widehat{H}_F(y, y', \cdot): y, y' \in E)$ are defined by

$$\widehat{H}_F(y, y', \theta) \equiv \exp\left(\left\langle \theta, \int_{\mathbb{R}^d} sH(y, y', ds)\right\rangle\right).$$

The large deviations results presented for $((J_n, S_n))$ can be adapted to fluid model. Thus let κ_F be the function which plays the role of κ when we have the fluid model in place of $((J_n, S_n))$; then $\left(\frac{S_n^{(F)}}{n}\right)$ satisfies the LDP with rate function κ_F^* defined by

$$\kappa_F^*(s) \equiv \sup_{\theta \in \mathbb{R}^d}[\langle \theta, s\rangle - \kappa_F(\theta)]. \tag{9}$$

It is easy to check that ℓ is the unique zero of κ_F^*; indeed we have

$$\nabla \widehat{H}_F(y, y', \theta)\,|_{\theta = 0} \equiv \int_{\mathbb{R}^d} sH(y, y', ds) \equiv \nabla \widehat{H}(y, y', \theta)\,|_{\theta = 0}.$$

Thus $\frac{S_n^{(F)}}{n}$ converges to ℓ which is the same limit of $\frac{S_n}{n}$ (see Remark 2.6).

Proposition 3.1. *Assume* (INN) *holds. Then we have* $\kappa_F^*(s) \geq \kappa^*(s)$ *for all* $s \in \mathbb{R}^d$.

Proof. In general we have

$$\widehat{H}(y, y', \theta) \geq \exp\left(\int_{\mathbb{R}^d} \langle \theta, s\rangle\, H(y, y', ds)\right)$$

$$= \exp\left(\left\langle \theta, \int_{\mathbb{R}^d} sH(y, y', ds)\right\rangle\right) = \widehat{H}_F(y, y', \theta)$$

by Jensen inequality. Then, given any initial distribution π for J, for all $\theta \in \mathbb{R}^d$ we have

$$\mathbb{E}^{(\pi)}[e^{\langle \theta, S_n \rangle}] = \int_E \left[\int_{E^n} \widehat{H}(y, y_1, \theta) \cdots \widehat{H}(y_{n-1}, y_n, \theta)\, Q(y, dy_1) \cdots Q(y_{n-1}, dy_n) \right] \pi(dy)$$

$$\geq \int_E \left[\int_{E^n} \widehat{H}_F(y, y_1, \theta) \cdots \widehat{H}_F(y_{n-1}, y_n, \theta)\, Q(y, dy_1) \cdots Q(y_{n-1}, dy_n) \right] \pi(dy)$$

$$= \mathbb{E}^{(\pi)}[e^{\langle \theta, S_n^{(F)} \rangle}].$$

Thus, by (8) in Proposition 2.5 (for $((J_n, S_n))$ and for fluid model), we obtain

$$\text{(10)} \quad \kappa(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}^{(\pi)}[e^{\langle \theta, S_n \rangle}] \geq \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}^{(\pi)}[e^{\theta, S_n^{(F)} \rangle}] = \kappa_F(\theta) \quad (\forall \theta \in \mathbb{R}^d).$$

In conclusion the proof is complete by (5), (9) and (10). ■

Remark 3.2. *Let us consider the content of Remark* 2.7 (*thus in particular* $d = 1$). *Moreover assume that the analogous of* (**L**) *for fluid model also holds, namely:*

$$\text{there exists } w_F > 0 \text{ such that } \kappa_F(w_F) = 0 \text{ and } \kappa'_F(w_F) > 0 .$$

Then, by taking into account the latter statement in Remark 2.7, *we have*

$$w_F = \inf_{s > 0} \frac{\kappa_F^*(s)}{s} \geq \inf_{s > 0} \frac{\kappa^*(s)}{s} = w$$

by Proposition 3.1; *from a different point of view* $w_F \geq w$ *also follows from* (10). *If we adapt final part of section* 5 *in* [7] *to our situation, we have the same consequences of* $w_F \geq w$ *concerning level crossing probabilities and second moment of importance sampling estimator.*

3.2 - *Averaged parameters model*

In order to define averaged parameters model we introduce the kernel $(H_A(y, y', \cdot) : y, y' \in E)$ which plays the role of $(H(y, y', \cdot) : y, y' \in E)$ when we have averaged parameters model in place of $((J_n, S_n))$. In general $H_A(y, y', \cdot)$ is a suitable distribution which does not depend on (y, y'); more precisely the moment generating functions $(\widehat{H}_A(y, y', \cdot) : y, y' \in E)$ are all equal to

$$\widehat{H}_A(\theta) \equiv \exp\left(\int_E \left(\int_E \log \widehat{H}(y, y', \theta)\, Q(y, dy')\right) \pi^{(J)}(dy)\right). \tag{11}$$

Remark 3.3. *We are considering an implicit assumption, namely the function* $\widehat{H}_A$ *defined in* (11) *is a moment generating function. This happens when all the distributions* $(H(y, y', \cdot) : y, y' \in E)$ *are infinitely divisible. This remark has some analogy with Remark* 4.1 *in* [7].

We point out that, since the distributions $(H_A(y, y', \cdot) : y, y' \in E)$ do not depend on (y, y'), (J_n) and $(S_n^{(A)})$ are independent.

The large deviations results presented for $((J_n, S_n))$ can be adapted to averaged parameters model. Thus let κ_A be the function which plays the role of κ when we have the averaged parameters model in place of $((J_n, S_n))$; then $\left(\frac{S_n^{(A)}}{n}\right)$ satisfies the LDP with rate function κ_A^* defined by

$$\kappa_A^*(s) \equiv \sup_{\theta \in \mathbb{R}^d} [\langle \theta, s \rangle - \kappa_A(\theta)]. \tag{12}$$

It is easy to check that κ_A coincides with $\log \widehat{H}_A$ and the unique zero of κ_A^* is $\nabla \kappa_A(\theta)|_{\theta=0}$ which coincides with ℓ. Thus $\frac{S_n^{(A)}}{n}$ converges to ℓ which is the same limit of $\frac{S_n}{n}$ (see Remark 2.6).

Proposition 3.4. *Assume* (INN) *holds. Then we have* $\kappa_A^*(s) \geq \kappa^*(s)$ *for all* $s \in \mathbb{R}^d$.

Proof. In general we have

$$\mathbb{E}^{(\pi^{(J)})}[e^{\langle \theta, S_n \rangle}]$$

$$= \int_E \left[\int_{E^n} \widehat{H}(y, y_1, \theta) \cdots \widehat{H}(y_{n-1}, y_n, \theta)\, Q(y, dy_1) \cdots Q(y_{n-1}, dy_n)\right] \pi^{(J)}(dy)$$

$$\geq \exp\left(\int_E \left[\int_{E^n} \{\log \widehat{H}(y, y_1, \theta) + \cdots +\right.\right.$$

$$\left.\left.\log \widehat{H}(y_{n-1}, y_n, \theta)\}\, Q(y, dy_1) \cdots Q(y_{n-1}, dy_n)\right] \pi^{(J)}(dy)\right)$$

by Jensen inequality. Then, since $\pi^{(J)}$ is stationary for J, by (11) we have

$$\mathbb{E}^{(\pi^{(J)})}[e^{\langle \theta, S_n \rangle}] \geqslant \exp\left(n \int_E \left(\int_E \log \widehat{H}(y, y', \theta)\, Q(y, dy')\right) \pi^{(J)}(dy)\right) = (\widehat{H}_A(\theta))^n .$$

Thus, since κ_A coincides with $\log \widehat{H}_A$, by (8) in Proposition 2.5 with $\pi = \pi^{(J)}$ and by the latter inequality we have

$$\kappa(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}^{(\pi^{(J)})}[e^{\langle \theta, S_n \rangle}] \geqslant \log \widehat{H}_A(\theta) = \kappa_A(\theta) \quad (\forall \theta \in \mathbb{R}^d). \tag{13}$$

In conclusion the proof is complete by (5), (12) and (13). ■

Remark 3.5. *Let us consider the content of Remark* 2.7 (*thus in particular* $d = 1$). *Moreover assume that the analogous of* (**L**) *for averaged parameters model also holds, namely:*

$$\text{there exists } w_A > 0 \text{ such that } \kappa_A(w_A) = 0 \text{ and } \kappa'_A(w_A) > 0 .$$

Then, by taking into account the latter statement in Remark 2.7, *we have*

$$w_A = \inf_{s > 0} \frac{\kappa_A^*(s)}{s} \geqslant \inf_{s > 0} \frac{\kappa^*(s)}{s} = w$$

by Proposition 3.4; *from a different point of view* $w_A \geqslant w$ *also follows from* (13). *If we adapt final part of section* 5 *in* [7] *to our situation, we have the same consequences of* $w_A \geqslant w$ *concerning level crossing probabilities and second moment of importance sampling estimator.*

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Summary

In this paper we consider a discrete time uniformly recurrent Markov additive process $((J_n, S_n))$ *according to the presentation in* [**5**]. *As in* [**7**] (*which deals with continuous time Markov additive processes with finite state space environment*) *we define the fluid model and the averaged parameters model as two simplified Markov additive processes derived from* $((J_n, S_n))$ *in a suitable way. In this paper we prove some inequalities between rate functions which coincide with the analogous inequalities in* [**7**].

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