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Honest subgroups in commutative group rings (**)

Introduction

Throughout the rest of this research paper RG will denote the group ring of an abelian group G over a commutative ring R with identity of prime characteristic p, $V(RG)$ the group of all normalized units (i.e. units which coefficient sum is equal to 1), and $S(RG)$ its Sylow p-subgroup. For such a group G, tG is the torsion component of G and G_p is its Sylow p-subgroup. Our notations and terminology are standard and follow essentially [K] where all unexplained concepts and unreferenced facts from the abelian group theory can be found. Nevertheless, for the sake of completeness and a convenience of the readers, we include the definition of so-called *honest* subgroups ([AR] and [K], p. 578, too), namely the subgroup A of a multiplicative abelian group G is said to be honest if for every cyclic subgroup $<g>$ of G it holds $\langle g\rangle \subseteq A$ or $\langle g\rangle \cap A = 1$.

In [AR], Abian and Rinehart (1963) have established a characterization of so-defined honest subgroups of abelian groups. They have proved in [AR] the following necessary and sufficient condition.

Criterion (Abian-Rinehart, 1963). *Let G be an abelian group and let A be its nontrivial subgroup. Then A is honest in G if and only if precisely one from the following conditions is fulfilled*:

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(°) *$A < tG$, the torsion part tG is p-primary, $A^p = 1$ and A is a direct factor of G*

or

(°°) *$tG \leqslant A$ and G/A is torsion-free.*

We continue with the statement of the central assertions.

Main results

Before stating and proving the major attainments, we need some preliminaries.

Following [Da], we let (RG, RA) to denote the pair of R-group algebras whenever A is a subgroup of G, and $I(RG; A)$ the relative augmentation ideal of G with respect to the subgroup A of G. A problem of some importance in the commutative group algebras theory is of whether the properties of A in G are preserved by (RG, RA). In a more exclusive form it asserts thus:

Problem. *Assume that $A \leqslant G$ has the property $\mathcal{P}$ in G. Does it follow that the same property $\mathcal{P}$ can be inherited for $B \leqslant H$ in H, provided (RG, RA) and (RH, RB) are R-isomorphic?*

The main goal that motivates this article is to extract that the property of A being honest in G is an invariant property for the pair of commutative group R-algebras (RG, RA). Before doing this, we need a brief discussion and a review of our recent results, appeared in ([D]-[Dan]), as the first two ones are only for a complete information:

(1) In [Da] we have proved that $RG = RH$ implies $I(RG; D) = I(RH; D')$ assuming G is p-mixed, R is a field and D and D' are the maximal divisible subgroups of G and H respectively.

(2) In [D] we have argued that $RG = RH$ does not yield $I(RG; B^{(p)}) = I(RH; B'^{(p)})$ when $B^{(p)}$ and $B'^{(p)}$ are the p-basic subgroups of G and H respectively, even if $G = H$ are p-primary groups.

(3) In [M] (see [Dan] as well) was obtained that $FG = FH$ assures $I(FG; tG) = I(FH; tH)$, provided F is a field such that the algebra FG is semisimple, that is, $char(F) = 0$, or $char(F) = p$ but $G_p = 1$.

In the case of fields of characteristic $p > 0$, Ullery ([U], Lemma 2.2) precised this dependence by showing that $RG = RH$ ensures $I(RG; \coprod_{q \neq p} G_q) = I(RH; \coprod_{q \neq p} H_q)$. Of course, $tG = \coprod_{\forall q\text{-primes}} G_q$, hence $I(RG; tG) = I(RG; G_p) + I(RG; \coprod_{q \neq p} G_q)$. Utilizing [Da], Lemma 3, RG recaptures $I(RG; G_p)$ whence by the Ullery's result RG determines $I(RG; tG)$.

Since $I(RG;\coprod_{q\neq p} G_q) = \sum_{q\neq p} I(RG;G_q)$, it is interesting to ask whether RG recovered $I(RG;G_q)$.

(4) Lemma ([D], Lemma 2). *Assume* $A \leqslant B \leqslant G$. *Then* $I(RG;A) = I(RG;B) \Leftrightarrow A = B$.

And so, we can attack the first central affirmation.

Theorem (Invariances). *Suppose* $(RG, RA) \cong (RH, RB)$ *as* R*-algebras. If* A *is a honest subgroup of* G, *then* B *is a honest subgroup of* H.

Proof. Since by definition the isomorphism between RG and RH implies a restricted isomorphism between RA and RB, we may harmlessly assume that $RG = RH$ and $RA = RB$, as well as that R is a field (see, for instance, cf. [Da]).

Foremost, suppose (°) holds valid. Hence $A < tG$, tG is a p-group, $A^p = 1$ and A is pure in G. The implication $RG = RH \Rightarrow tH$ is a p-group is a well-known fact (see, for example, [Da]) it follows directly from the quoted below proposition as well. Since $RG = RH$ and $RA = RB$, we extract $V(RG) = V(RH)$ and $V(RA) = V(RB)$. Therefore, $1 = V(R^pA^p) = V^p(RA) = V^p(RB) \supseteq B^p$. Thus $B^p = 1$. Moreover, because $A \cap G^{p^n} = A^{p^n}$ for each natural number n, we detect that $V(RA) \cap V^{p^n}(RG) = V(RA) \cap V(R^{p^n}G^{p^n}) = V(R^{p^n}(A \cap G^{p^n})) = V(R^{p^n}A^{p^n}) = V^{p^n}(RA)$. Consequently, $V(RB) \cap V^{p^n}(RH) = V^{p^n}(RB)$, hence $B \cap H^{p^n} \subseteq V^{p^n}(RB) \cap H = V(R^{p^n}B^{p^n}) \cap H = B^{p^n}$. That is why B is a pure subgroup of H i.e., invoking [K], B is a direct factor of H.

What is enough to prove in conclusion is that $B < tH$. In fact, by what we have just shown in (3), $I(RH;B) = RH.I(RB;B) = RG.I(RA;A) = I(RG;A) \subset I(RG;tG) = I(RH;tH)$, whence $I(RH;B) \subset I(RH;tH)$. Furthermore $B \subseteq 1 + I(RH;B) \subset 1 + I(RH;tH)$, i.e. $B \subset (1 + I(RH;tH)) \cap H = tH$. If we presume that $B = tH$, then we derive $I(RG;A) = I(RG;tG)$ and so the above listed lemma is applicable to obtain that $A = tG$, which contradicts with the text of (°). Finally, point (°) may be employed in a reverse order to infer that B is honest in H.

Let now (°°) be true. As we have above observed, $R(G/A) \cong RG/I(RG;A) = RH/I(RH;B) \cong R(H/B)$ and thus bearing in mind a classical result from [M], we find that $H/B \cong G/A$ must be torsion-free.

Since $tG \leqslant A$, by what we have seen above in [3], $I(RH;tH) = I(RG;tG) \subseteq I(RG;A) = I(RG;B)$. Thus, we deduce $tH \subseteq 1 + I(RH;tH) \subseteq 1 + I(RG;B)$, i.e. $tH \subseteq [1 + I(RG;B)] \cap H = B$. So, taking into account in a converse order point (°°), we conclude that B is a honest subgroup of H. This completes the proof.

Before finding criteria for certain subgroups in $V(RG)$ to be honest, we shall generalize some well-known particular variants of the structure of normed torsion units, i.e. of the torsion elements in $V(RG)$.

Proposition. *Let R be a field of characteristic p. Then*

$$tV(RG) = S(RG)\, tV(R(tG)).$$

Proof. For any $x \in tV(RG)$ we write $x = r_1 g_1 + \ldots + r_t g_t$ with $r_1 + \ldots + r_t = 1$; $t \in \mathbf{N}$. First of all, suppose that the group $\langle g_1, \ldots, g_t\rangle$ contains no element of order p, hence the group algebra $R\langle g_1, \ldots, g_t\rangle$ is semisimple. So, consulting with [Dan], we yield $r_1 g_1 + \ldots + r_t g_t \in tV(R\langle g_1, \ldots, g_t\rangle) = V(R(t\langle g_1, \ldots, g_t\rangle)) \subseteq V(R(tG))$.

If now $\langle g_1, \ldots, g_t\rangle$ possesses p-elements, one can conclude that there is a positive integer k such that $\langle g_1, \ldots, g_t\rangle^{p^k}$ does not have an element of order p. In that situation,

$$x^{p^k} = r_1^{p^k} g_1^{p^k} + \ldots + r_t^{p^k} g_t^{p^k} \in tV(R^{p^k} G^{p^k}) \cap R^{p^k}\langle g_1, \ldots, g_t\rangle^{p^k} = tV(R^{p^k}\langle g_1, \ldots, g_t\rangle^{p^k})$$
$$= V(R^{p^k} t(\langle g_1, \ldots, g_t\rangle^{p^k})) = V(R^{p^k}(t\langle g_1, \ldots, g_t\rangle)^{p^k}) = V^{p^k}(R(t\langle g_1, \ldots, g_t\rangle)),$$

where the second equality holds referring again to [Dan]. Thereby,

$$x \in S(RG)\, V(R(t\langle g_1, \ldots, g_t\rangle)) \subseteq S(RG)\, V(R(tG)),$$

which substantiates our claim. The proof is finished.

Now, we are ready to proceed by proving the second major statement.

Theorem (Structure). *Suppose A is a p-group and $A \leqslant G$. Then $1 + I(RG; A)$ is a honest subgroup of $V(RG)$ if and only if A is a honest subgroup of G, provided R is without zero divisors.*

Proof. Given that $1 + I(RG; A)$ is honest in $V(RG)$. In virtue of the Criterion, we differ two cases:

(a) $1 + I(RG; A) \subset S(RG)$, $1 + I(RG; A) \neq S(RG)$ and $[1 + I(RG; A)] \cap V^{p^n}(RG) = 1$ for all natural numbers n;

(b) $S(RG) \subseteq 1 + I(RG; A)$ and $V(RG)/[1 + I(RG; A)] \cong V(R(G/A))$ is torsion-free.

First, we examine (a). Thus $A \subset S(RG) = 1 + I(RG; G_p)$ whence $A \subset [1 + I(RG; G_p)] \cap G = G_p$, i.e. $A \subset G_p$ and $A \neq G_p$. Moreover, $A \cap G^{p^n} \subseteq [1 + I(RG; A)] \cap V^{p^n}(RG) = 1$. Finally, according to (°), A is honest in G. After this, we investigate (b). Foremost, we observe that $G_p \subseteq [1 + I(RG; A)] \cap G = A$ and G/A is torsion-free. Henceforth, owing to (°°), A is a honest subgroup of G.

Oppositely, assume that A is honest in G, i.e. (°) and (°°) are satisfied. In the first case, $A < tG = G_p$ implies $1 + I(RG; A) < 1 + I(RG; G_p) = S(RG)$, and the

Proposition gives $tV(RG) = S(RG)$. Besides, $[1 + I(RG; A)]^p = 1 + I^p(RG; A) = 1 + I(R^p G^p; A^p) = 1$, and $A \cap G^{p^n} = A^{p^n}$ $\forall n \in \boldsymbol{N}$ together with an intersection claim documented in [D] ensure that

$$[1 + I(RG; A)] \cap V^{p^n}(RG) = [1 + I(RG; A)] \cap V(R^{p^n} G^{p^n})$$

$$= 1 + I(R^{p^n} G^{p^n}; G^{p^n} \cap A) = 1 + I(R^{p^n} G^{p^n}; A^{p^n}) = [1 + I(RG; A)]^{p^n}.$$

Next, $tG \leqslant A = A_p$ and the Proposition guarantee that $tV(RG) = S(RG) = 1 + I(RG; G_p) \subseteq 1 + I(RG; A)$. Moreover [M] along with the isomorphism $V(RG)/[1 + I(RG; A)] \cong V(R(G/A))$ insure the torsion-free property.

So, in both situations, $1 + I(RG; A)$ is honest in $V(RG)$.

The proof is completed.

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Summary

We use a result for honest abelian subgroups due to Abian-Rinehart (1963), *published in the Rendiconti Palermo, to prove an invariance theorem via pairs of commutative group algebras of such abelian groups. We also establish an explicit formula for the normed torsion elements in modular group rings with prime characteristic.*

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