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Fixed point theorems for Lipschitz type maps (**)

1 - Introduction

Since its advent the notion of compatible maps due to Jungck [**10**], [**11**] proved useful and has emerged as an area of intense research activity in fixed point considerations. Pant [**21**] also underlined the usefulness of compatible maps in the study of common fixed points of contractive type maps and observed that the compatibility of maps often requires the assumption of their continuity and the completeness of the space on a metric space. Indeed, compatible maps are necessarily continuous at their common fixed points. This is also true for maps involving ϕ-conditions studied, among others, by Berinde [**2**], [**3**], Chang [**5**], Danes [**6**], Djoudi [**7**], Jachymski [**9**], Kang et al. [**13**], Kasahara [**14**], [**15**], Khan and Imdad [**16**], Kim et al. [**17**], Matkowski [**18**], Park [**26**], Park and Rhoades [**27**], Popa [**29**] and Singh and Meade [**34**]. For an excellent discussion on continuity of maps on their fixed points, one may refer to Rhoades [**30**] (see also Hicks and Rhoades [**8**]). However, there is a good scope for the study of common fixed points of non-compatible maps, and recently Pant [**21**], [**22**], [**23**], [**24**] (see also [**25**]), Aamri and Moutawakil [**1**] and others have initiated work on these lines. It is interesting to note that non-compatible maps need not be continuous at their common fixed points (see [**20**], [**22**], [**23**] and Examples 2-4 below).

The following fixed point theorem for noncompatible type maps is due to Pant [**21**].

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Theorem P. *Let A and S be non-compatible pointwise R-weakly commuting self-maps of the metric space (X, d) satisfying*

(i) $\overline{AX} \subset SX$,

(ii) $d(Ax, Ay) \leq kd(Sx, Sy)$, $k \geq 0$, *and*

(iii) $d(Ax, A^2x) \neq \max\{d(Ax, SAx), d(A^2x, SAx)\}$

whenever the right-hand side is nonzero. Then A and S have a common fixed point.

Pant [**21**] calls Lipschitz type maps for the pair A, S satisfying the condition (ii). Notice that (ii) is satisfied for all $x, y \in X$.

The maps $A, S : X \to X$ satisfying the condition (1.2) (cf. Theorem 1 below) or (1.2a) may be called generalized Lipschitz pair of maps on a metric space. We also consider generalized Lipschitz type quadruplet of maps on an arbitrary set with values in a metric space and study the conditions under which they (cf. (2.2) below) have coincidences and fixed points.

In the present paper, first we obtain fixed point theorems for a generalized Lipschitz pair of maps replacing the R-weak commutativity by the commutativity of the maps just at a coincidence point. Towards the end of the paper we demonstrate, by means of examples, that the conditions (1.2), (1.2a) and (2.2) (cf. Theorems 1, 1 Bis and 2 respectively) apply to a wider class of maps.

2 - Fixed point theorems

Throughout this paper, let Y be an arbitrary nonempty set, (X, d) a metric space and $C(A, S) = \{u : Au = Su\}$, the collection of coincidence points of A and S.

Definition 1 [**10**]. Self-maps A and S of a metric space (X, d) are compatible if $\lim_{n \to \infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t$ for some $t \in X$. They are compatible maps of type (B) [**28**] if $\lim_{n \to \infty} d(SAx_n, AAx_n) \leq [\lim_{n \to \infty} d(SAx_n, St) + \lim_{n \to \infty} d(St, SSx_n)]/2$ and $\lim_{n \to \infty} d(ASx_n, SSx_n) \leq [\lim_{n \to \infty} d(ASx_n, At) + \lim_{n \to \infty} d(At, AAx_n)]/2$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t$ for some $t \in X$.

Definition 2 [**19**]. Self-maps A and S of a metric space (X, d) are R-weakly commuting at a point $x \in X$ if $d(ASx, SAx) \leq Rd(Ax, Sx)$ for some $R > 0$. They are pointwise R-weakly commuting on X if given $x \in X$ there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$.

Notice that if $R = 1$, we get the definition of weakly commuting maps due to Sessa [**32**]. We remark that compatible maps, R-weakly commuting maps and pointwise R-weakly commuting maps are necessarily weakly compatible [**12**], i.e., commuting at their coincidences. However, the reverse implication is not true (see, for instance, [**33**], p. 488).

Definition 3. Let A and S be maps on Y with values in X. Then A and S will be called to satisfy the (EA) property if there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

If we take $Y = X$ then we get the definition of (EA) property for two self-maps of X studied by Aamri and Moutawakil [**1**]. In such a situation, t is called a tangent point by Sastry and Murthy [**31**].

Example 1. Let $X = [1/4, 32]$ be endowed with the usual metric and $Y = [1/5, 7/4]$. Define $Ax = 8x^2 + 1/2$ and $Sx = 10x^2$ for $x \in Y$. We consider the sequence $\{x_n = 1/2 + 1/n : n \geqslant 1\}$ to see that the maps A and S satisfy the (EA) property.

The following is our main result for noncompatible maps.

Theorem 1. *Let A and S be noncompatible self-maps of a metric space (X, d) such that*

(1.1) $\overline{AX} \subset SX$,

(1.2) $$d(Ax, Ay) \leqslant kd(Sx, Sy) + \max\{\alpha d(Ax, Sx) + d(Ay, Sy), \alpha d(Ax, Sy) + d(Ay, Sx)\},$$

for all $x, y \in X$, where $k \geqslant 0$, $0 \leqslant \alpha < 1$.

Then $C(A, S)$ is nonempty. Further, A and S have a common fixed point provided that A and S commute at (some) $u \in C(A, S)$ and one of the following holds:

(1.3) $$d(Ax, A^2x) \neq \max\{d(Sx, SAx), d(Ax, Sx), d(A^2x, SAx), d(Ax, SAx), d(Sx, A^2x)\}$$

whenever the right-hand side is nonzero for $x \in C(A, S)$;

(1.4) $$d(Sx, S^2x) \neq \max\{d(Ax, ASx), d(Sx, Ax), d(S^2x, ASx), d(Sx, ASx), d(Ax, S^2x)\}$$

whenever the right-hand side is nonzero for $x \in C(A, S)$.

Proof. Since A and S are noncompatible, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent. Since $t \in \overline{AX}$ and $\overline{AX} \subset SX$, there exists a point $u \in X$ such that $t = Su$. Suppose $Au \neq Su$, then by (1.2), $d(Au, Ax_n) \leqslant kd(Su, Sx_n)$

$+ \max\{\alpha d(Au, Su) + d(Ax_n, Sx_n),\ \alpha d(Au, Sx_n) + d(Ax_n, Su)\}$. Making $n \to \infty$ yields $d(Au, Su) \leqslant \alpha d(Au, Su) < d(Au, Su)$, and $Au = Su$. Consequently $C(A, S)$ is nonempty. Further, the commutativity of A and S at u implies $AAu = ASu = SAu = SSu$. So using (1.3) or (1.4) for $x = u$, we immediately see that $Au = Su$ is a common fixed point of A and S.

Under the conditions of Theorem P, (iii) is required to be considered for all $x \in X$. Since only coincidence points of A and S are good candidates to become common fixed points, the condition (iii) need be considered only for $u \in C(A, S)$. Further, (iii) for $x \in C(A, S)$ is included in (1.3).

In view of the above proof, we have another version of Theorem 1.

Theorem 1 Bis. *Theorem* 1 *with* (1.2) *replaced by the following*:

$$d(Ax, Ay) \leqslant k \max\{d(Sx, Sy),\ \alpha[d(Ax, Sx) + d(Ay, Sy)],\ \alpha[d(Ax, Sy) + d(Ay, Sx)]\}, \tag{1.2a}$$

for all $x, y \in X$, *where* $k \geqslant 0$, *and* $\alpha \geqslant 0$ *is chosen such that* $k\alpha < 1$.

The following example shows that the generalized Lipschitz pair of maps are indeed more general than Lipschitz type pair of maps.

Example 2. Let $X = [2, 20]$ be endowed with the usual metric and

$$Ax = x \text{ if } 2 \leqslant x \leqslant 3,\ Ax = 5 \text{ if } 3 < x \leqslant 4,\ Ax = 6 \text{ if } x > 4,$$

and

$$Sx = x \text{ if } 2 \leqslant x \leqslant 3,\ Sx = 16 \text{ if } 3 < x \leqslant 7,\ Sx = (x+1)/2 \text{ if } x > 7.$$

We consider the sequence $\{x_n = 11 + 1/n : n \geqslant 1\}$ to see that the maps A and S are non-compatible. Also, $AX = [2, 3] \cup \{5, 6\}$, $SX = [2, 3] \cup (4, 21/2] \cup \{16\}$, and $\overline{AX} \subset SX$. Notice that A and S satisfy the condition (ii) (cf. Theorem P) for all x, y in X and $k > 4$ except for $x \in (3, 4]$, $y \in (4, 7]$, since in this situation $d(Ax, Ay) = 1 > 0 = kd(Sx, Sy)$. On the other hand, for $x \in (3, 4]$, $y \in (4, 7]$ and $\alpha \in [1/21, 1/4)$, $\alpha[d(Ax, Sx) + d(Ay, Sy)] = 21\alpha \geqslant d(Ax, Ay)$. So A and S satisfy (1.2a) with $k > 4$ and $k\alpha < 1$, and it is easily seen that Theorem 1 Bis is applicable. It is easily verified that Theorem 1 is also applicable with $k > 4$ and $0 \leqslant \alpha < 1$. It is also clear that maps A and S are not pointwise R-weakly commuting at $x = 11$.

We remark that the generalized Lipschitz maps may have discontinuity at their common fixed points. In the above example, A and S have several common fixed points and there is a discontinuity at their common fixed point $x = 3$. This observation is significant since it is well known that fixed or common fixed points of various non-continuous contractions and contraction type maps are points of continuity (see [**30**]). Further, conditions (iii), (1.3) and (1.4) in the above theorems are required just to ensure that a coincidence point u of maps A and S becomes a common fixed point of A and S provided that $ASu = SAu$. Indeed, the commutativity requirement in Theorems 1 and 1 Bis, viz., $ASu = SAu$ is minimal and cannot be relaxed. For example, for $X = [0, \infty)$, $Ax = x^3 + 2/9$ and $Sx = 3x^3$, conditions (1.1), (1.2), (1.2*a*) and (1.3) are satisfied. Notice that for $u = 9^{-1/3}$, $Au = Su = 1/3$, $ASu \neq SAu$, and the coincidence point u is not a common fixed point of A and S.

Now we present coincidence theorems for maps on an arbitrary (nonempty) set Y with values in a metric space.

Theorem 2. *Let* (X, d) *be a metric space and* $A, B, S, T\colon Y \to X$ *such that*

(2.1) $\overline{AY} \subset TY$ and $\overline{BY} \subset SY$;

(2.2) $d(Ax, By)$

$$\leqslant k \max\{d(Sx, Ty), \alpha[d(Ax, Sx) + d(By, Ty)], \alpha[d(By, Sx) + d(Ax, Ty)]\}$$

for all $x, y \in Y$, *where* $k \geqslant 0$, *and* $\alpha \geqslant 0$ *is chosen such that* $k\alpha < 1$;

(2.3) *one of the pairs* (A, S) *or* (B, T) *satisfies the* (*EA*) *property.*

Then $C(A, S)$ *and* $C(B, T)$ *are nonempty.*

Further, if $Y = X$, *then*

(I) *A and S have a common fixed point provided that A and S commute at (some)* $u \in C(A, S)$ *and one of* (1.3) *or* (1.4) *holds*;

(II) *B and T have a common fixed point provided that B and T commute at (some)* $w \in C(B, T)$ *and one of the following holds*:

(2.4) $d(Bx, B^2x)$

$$\neq \max\{d(Tx, TBx), d(Bx, Tx), d(B^2x, TBx), d(Bx, TBx), d(Tx, B^2x)\}$$

whenever the right-hand side is nonzero for $x \in C(B, T)$;

(2.5) $d(Tx, T^2x)$

$$\neq \max\{d(Bx, BTx), d(Tx, Bx), d(T^2x, BTx), d(Tx, BTx), d(Bx, T^2x)\}$$

whenever the right-hand side is nonzero for $x \in C(B, T)$;

(III) *A, B, S and T have a common fixed point provided* (I) *and* (II) *are true.*

Proof. If the pair (B, T) satisfies the (EA) property, then there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Since $\overline{BY} \subset SY$, for each x_n, there exists y_n in Y such that $Bx_n = Sy_n$, and $Sy_n \to t$ as well. We show that $Ay_n \to t$. If not, there exist a subsequence $\{Ay_m\}$ of

$\{Ay_n\}$, a positive integer M, and a real number $r > 0$ such that for some positive integer $m \geqslant M$, we have

$$d(Ay_m, t) \geqslant r,\ d(Ay_m, Bx_m) \geqslant r,$$

and by (2.2),

$$\begin{aligned} d(Ay_m, Bx_m) \leqslant k \max\{d(Sy_m, Tx_m),\ &\alpha[d(Ay_m, Sy_m) + d(Bx_m, Tx_m)], \\ &\alpha[d(Bx_m, Sy_m) + d(Ay_m, Tx_m)]\} \\ = k\alpha d(Ay_m, Bx_m) < d(Ay_m, Bx_m), \end{aligned}$$

a contradiction, and $Ay_n \to t$.

Since $t \in \overline{BY}$ and $\overline{BY} \subset SY$, there exists an element $u \in Y$ such that $t = Su$. To show that $Au = Su$, we suppose otherwise and use the condition (2.2) to get

$$\begin{aligned} d(Au, Bx_n) \leqslant k \max\{d(Su, Tx_n),\ &\alpha[d(Au, Su) + d(Bx_n, Tx_n)], \\ &\alpha[d(Bx_n, Su) + d(Au, Tx_n)]\}. \end{aligned}$$

Making $n \to \infty$,

$$d(Au, Su) \leqslant k\alpha d(Au, Su), \quad \text{yielding } Au = Su.$$

This proves that $C(A, S)$ is nonempty. Since $\overline{AY} \subset TY$, there exists an element $w \in Y$ such that $Au = Tw$. If $Tw \neq Bw$, then by (2.2),

$$\begin{aligned} d(Au, Bw) \leqslant k \max\{d(Su, Tw),\ &\alpha[d(Au, Su) + d(Bw, Tw)], \\ &\alpha[d(Bw, Su) + d(Au, Tw)]\} = k\alpha d(Au, Bw). \end{aligned}$$

Consequently $Tw = Au = Bw$, and $C(B, T)$ is nonempty.

Now let $Y = X$. The commutativity of A and S at u implies $AAu = ASu = SAu = SSu$. So using (1.3) or (1.4) for $x = u$, we immediately see that Au is a common fixed point of A and S. This proves (I). Similar argument shows that Bw is a common fixed point of B and T, proving (II). Now (III) is immediate.

In case $S = T$ in Theorem 2, we obtain a slightly improved version which we state below.

Theorem 3. *Let* (X, d) *be a metric space and* $A, B, S\colon Y \to X$ *such that* (2.2) *with* $S = T$, *and*

(3.1) $\overline{AY} \cup \overline{BY} \subset SY$;

(3.2) *one of the pairs* (A, S) *or* (B, S) *satisfies the* (EA) *property.*

Then:

(I) *maps* A, B *and* S *have a coincidence point* u (*say*);

(II) *maps* A, B *and* S *have a common fixed point* $z(=Au=Bu=Su)$ *provided that* S *commutes with each of* A *and* B *at* u *and one of* (1.3), (1.4) *or the following holds*:

(3.3) $d(Bx, B^2x)$

$$\neq \max\{d(Sx, SBx), d(Bx, Sx), d(B^2x, SBx), d(Bx, SBx), d(Sx, B^2x)\}$$

whenever the right-hand side is nonzero for $x \in C(B, S)$;

(3.4) $d(Sx, S^2x)$

$$\neq \max\{d(Bx, BSx), d(Sx, Bx), d(S^2x, BSx), d(Sx, BSx), d(Bx, S^2x)\}$$

whenever the right-hand side is nonzero for $x \in C(B, S)$.

Finally, we consider generalized functional conditions and show that Theorems 1, 1 Bis and 2 obtained under tight minimal conditions apply to a class of maps wider than the results involving general nondecreasing functions of five variables. To be specific, let (X, d) be a metric space A, B, S, T: $X \to X$ and $\phi: R^+ \to R^+$ be a nondecreasing function. Following Boyd and Wong [**4**], several fixed point theorems involving the function ϕ have been studied, among others, by Berinde [**2**], [**3**], Chang [**5**], Danes [**6**], Djoudi [**7**], Jachymski [**9**], Kang et al. [**13**], Kasahara [**14**], [**15**], Khan and Imdad [**16**], Kim et al. [**17**], Matkowski [**18**], Park [**26**], Park and Rhoades [**27**], Popa [**29**] and Singh and Meade [**34**].

Here we cite the following result of Djoudi [**7**] wherein $\varphi: (R^+)^5 \to R^+$ is an upper semi-continuous in each coordinate variable and nondecreasing function satisfying the condition:

$$\phi(t) = \max\{\varphi(0, t, 0, 0, t), \varphi(t, 0, 0, t, t), \varphi(t, t, t, 2t, 0), \varphi(0, 0, t, t, 0)\} < t,$$

for any $t > 0$.

Theorem D. *Let* A, B, S *and* T *be self-maps of a complete metric space* (X, d) *satisfying*

(D.1) $AX \subset TX$ *and* $BX \subset SX$;

(D.2) $d(Ax, By) \leq \varphi\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\}$ *for all* $x, y \in X$.

Suppose that one of A, B, S *or* T *is continuous and the pairs* (A, S) *and* (B, T) *are compatible of type* (*B*). *Then* A, B, S *and* T *have a unique common fixed point.*

The following example shows that maps A, B, S and T satisfy all the conditions of Theorem 2 but the condition (D.2) (cf. Theorem D) is not satisfied.

Example 3. Let $X = [2, 20]$ be endowed with usual metric and

$$\begin{array}{llll} Ax = x & \text{if } 2 \leqslant x \leqslant 3, & Ax = 3 & \text{if } x > 3, \\ Bx = x & \text{if } 2 \leqslant x \leqslant 3, & Bx = 6 & \text{if } 3 < x \leqslant 5 \text{ or } x > 15, \\ Bx = 15 & \text{if } 5 < x \leqslant 15, & & \\ Sx = x & \text{if } 2 \leqslant x \leqslant 3, & Sx = 6 & \text{if } 3 < x \leqslant 5 \text{ or } x > 15, \\ Sx = 15 & \text{if } 5 < x \leqslant 15, & \text{and} & \\ Tx = x & \text{if } 2 \leqslant x \leqslant 3, & Tx = 12 & \text{if } 3 < x \leqslant 15, \\ Tx = x - 8 & \text{if } x > 15. & & \end{array}$$

Then maps A, B, S and T have infinitely many common fixed points and all the conditions of Theorem 2 are satisfied. Notice that the condition (D.2) (cf. Theorem D) is not satisfied for $x \in (3, 5]$, $y \in (5, 15]$.

The following example shows that maps A and S satisfy all the conditions of Theorems 1 and 1Bis. However, condition (D.2), with $A = B$ and $S = T$, is not satisfied.

Example 4. Let $X = [2, 20]$ be endowed with the usual metric and

$$Ax = x \text{ if } 2 \leqslant x \leqslant 3, \ Ax = 7 \text{ if } 3 < x \leqslant 4, \ Ax = 8 \text{ if } x > 4,$$

and

$$Sx = x \text{ if } 2 \leqslant x \leqslant 3, \ Sx = 4 \text{ if } 3 < x \leqslant 9, \ Sx = (x+1)/2 \text{ if } x > 9.$$

We consider the sequence $\{x_n = 15 + 1/n : n \geqslant 1\}$ to see that the maps A and S are noncompatible. Also, $\overline{AX} \subset SX$. The condition (1.2) (cf. Theorem 1) is satisfied with $k > 6$ and $k\alpha < 1$. It is easy to see that Theorem 1 Bis is also applicable with $k > 6$ and $0 \leqslant \alpha < 1$. However, the condition (D.2) is not satisfied for $x, y \in [2, 3]$.

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Abstract

The aim of this paper is to obtain common fixed point theorems for generalized Lipschitz pair of non-compatible maps without using the continuity of the maps involved and completeness of the metric space. Coincidence theorems for generalized Lipschitz type quadruplet of maps on an arbitrary set with values in a metric space are also obtained. The remarkable aspect of such maps is that they need not be continuous at their common fixed points.

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