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Generalizations of Precupanu's inequality for orthornormal families of vectors in inner product spaces (**)

1 - Introduction

In 1976, T. Precupanu [**6**] obtained the following result related to the Schwarz inequality in a real inner product space $(H; \langle \cdot, \cdot \rangle)$:

Theorem 1. *For any* $a, b \in H$, $x, y \in H \backslash \{0\}$, *we have the inequality:*

$$\frac{-\|a\|\,\|b\| + \langle a, b\rangle}{2} \leqslant \frac{\langle x, a\rangle \langle x, b\rangle}{\|x\|^2} + \frac{\langle y, a\rangle \langle y, b\rangle}{\|y\|^2} - 2\cdot \frac{\langle x, a\rangle \langle y, b\rangle \langle x, y\rangle}{\|x\|^2\,\|y\|^2} \leqslant \frac{\|a\|\,\|b\| + \langle a, b\rangle}{2}\,. \tag{1.1}$$

In the right-hand side or in the left-hand side of (1.1) *we have equality if and only if there are* $\lambda, \mu \in \mathbb{R}$ *such that*

$$\lambda \frac{\langle x, a\rangle}{\|x\|^2}\cdot x + \mu \frac{\langle y, b\rangle}{\|y\|^2}\cdot y = \frac{1}{2}(\lambda a + \mu b)\,. \tag{1.2}$$

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Note for instance that [**6**], if $y \perp b$, i.e., $\langle y, b\rangle = 0$, then by (1.1) one may deduce:

$$\frac{-\|a\|\,\|b\| + \langle a, b\rangle}{2}\,\|x\|^2 \leqslant \langle x, a\rangle\langle x, b\rangle \leqslant \frac{\|a\|\,\|b\| + \langle a, b\rangle}{2}\,\|x\|^2 \tag{1.3}$$

for any $a, b, x \in H$, an inequality that has been obtained previously by U. Richard [**7**]. The case of equality in the right-hand side or in the left-hand side of (1.3) holds if and only if there are $\lambda, \mu \in \mathbb{R}$ with

$$2\lambda\langle x, a\rangle\, x = (\lambda a + \mu b)\|x\|^2. \tag{1.4}$$

For $a = b$, we may obtain from (1.1) the following inequality [**6**]

$$0 \leqslant \frac{\langle x, a\rangle^2}{\|x\|^2} + \frac{\langle y, a\rangle^2}{\|y\|^2} - 2\cdot\frac{\langle x, a\rangle\langle y, a\rangle\langle x, y\rangle}{\|x\|^2\,\|y\|^2} \leqslant \|a\|^2. \tag{1.5}$$

This inequality implies [**6**]:

$$\frac{\langle x, y\rangle}{\|x\|\,\|y\|} \geqslant \frac{1}{2}\left[\frac{\langle x, a\rangle}{\|x\|\,\|a\|} + \frac{\langle y, a\rangle}{\|y\|\,\|a\|}\right]^2 - \frac{3}{2}\,. \tag{1.6}$$

In [**5**], M.H. Moore pointed out the following reverse of the Schwarz inequality

$$|\langle y, z\rangle| \leqslant \|y\|\,\|z\|\,, \qquad y, z \in H\,, \tag{1.7}$$

where some information about a third vector x is known:

Theorem 2. *Let $(H; \langle\cdot, \cdot\rangle)$ be an inner product space over the real field $\mathbb{R}$ and $x, y, z \in H$ such that:*

$$|\langle x, y\rangle| \geqslant (1-\varepsilon)\|x\|\,\|y\|\,, \qquad |\langle x, z\rangle| \geqslant (1-\varepsilon)\|x\|\,\|z\|\,, \tag{1.8}$$

where ε is a positive real number, reasonably small. Then

$$|\langle y, z\rangle| \geqslant \max\left\{1 - \varepsilon - \sqrt{2\varepsilon},\, 1 - 4\varepsilon,\, 0\right\}\|y\|\,\|z\|\,. \tag{1.9}$$

Utilising Richard's inequality (1.3) written in the following equivalent form:

$$2\cdot\frac{\langle x, a\rangle\langle x, b\rangle}{\|x\|^2} - \|a\|\,\|b\| \leqslant \langle a, b\rangle \leqslant 2\cdot\frac{\langle x, a\rangle\langle x, b\rangle}{\|x\|^2} + \|a\|\,\|b\| \tag{1.10}$$

for any $a, b \in H$ and $x \in H\backslash\{0\}$, Precupanu has obtained the following Moore's type result:

Theorem 3. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real inner product space. If* $a, b, x \in H$ *and* $0 < \varepsilon_1 < \varepsilon_2$ *are such that:*

$$\varepsilon_1 \|x\|\,\|a\| \leq \langle x, a\rangle \leq \varepsilon_2 \|x\|\,\|a\|,$$
$$\varepsilon_1 \|x\|\,\|b\| \leq \langle x, b\rangle \leq \varepsilon_2 \|x\|\,\|b\|, \tag{1.11}$$

then

$$(2\varepsilon_1^2 - 1)\|a\|\,\|b\| \leq \langle a, b\rangle \leq (2\varepsilon_1^2 + 1)\,\|a\|\,\|b\|. \tag{1.12}$$

Remark that the right inequality is always satisfied, since by Schwarz's inequality, we have $\langle a, b\rangle \leq \|a\|\,\|b\|$. The left inequality may be useful when one assumes that $\varepsilon_1 \in (0, 1]$. In that case, from (1.12), we obtain

$$-\|a\|\,\|b\| \leq (2\varepsilon_1^2 - 1)\|a\|\,\|b\| \leq \langle a, b\rangle \tag{1.13}$$

provided $\varepsilon_1 \|x\|\,\|a\| \leq \langle x, a\rangle$ and $\varepsilon_1 \|x\|\,\|b\| \leq \langle x, b\rangle$, which is a refinement of Schwarz's inequality

$$-\|a\|\,\|b\| \leq \langle a, b\rangle.$$

In the complex case, apparently independent of Richard, M.L. Buzano obtained in [**2**] the following inequality

$$|\langle x, a\rangle\langle x, b\rangle| \leq \frac{\|a\|\,\|b\| + |\langle a, b\rangle|}{2}\cdot\|x\|^2, \tag{1.14}$$

provided x, a, b are vectors in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$.

In the same paper [**6**], Precupanu, without mentioning Buzano's name in relation to the inequality (1.14), observed that, on utilising (1.14), one may obtain the following result of Moore type:

Theorem 4. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a (real or) complex inner product space. If* $x, a, b \in H$ *are such that*

$$|\langle x, a\rangle| \geq (1-\varepsilon)\|x\|\,\|a\|, \qquad |\langle x, b\rangle| \geq (1-\varepsilon)\|x\|\,\|b\|, \tag{1.15}$$

then

$$|\langle a, b\rangle| \geq (1 - 4\varepsilon + 2\varepsilon^2)\|a\|\,\|b\|. \tag{1.16}$$

Note that the above theorem is useful when, for $\varepsilon \in (0, 1]$, the quantity $1 - 4\varepsilon + 2\varepsilon^2 > 0$, i.e., $\varepsilon \in \left(0, 1 - \frac{\sqrt{2}}{2}\right]$.

Remark 1. *When the space is real, the inequality* (1.16) *provides a better lower bound for* $|\langle a, b\rangle|$ *than the second bound in Moore's result* (1.9). *However, it is not known if the first bound in* (1.9) *remains valid for the case of complex spaces. From Moore's original proof, apparently, the fact that the space* $(H;\langle\cdot,\cdot\rangle)$ *is real plays an essential role.*

Before we point out some new results for orthonormal families of vectors in real or complex inner product spaces, we state the following result that complements the Moore type results outlined above for real spaces:

Theorem 5. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real inner product space and* $a, b, x, y \in H\backslash\{0\}$.

(i) *If there exist* $\delta_1, \delta_2 \in (0,1]$ *such that*

$$\frac{\langle x, a\rangle}{\|x\|\,\|a\|} \geq \delta_1, \qquad \frac{\langle y, a\rangle}{\|y\|\,\|a\|} \geq \delta_2$$

and $\delta_1 + \delta_2 \geq 1$, *then*

$$\frac{\langle x, y\rangle}{\|x\|\,\|y\|} \geq \frac{1}{2}(\delta_1+\delta_2)^2 - \frac{3}{2} \qquad (\geq -1). \tag{1.17}$$

(ii) *If there exist* $\mu_1(\mu_2) \in \mathbb{R}$ *such that*

$$\mu_1\|a\|\,\|b\| \leq \frac{\langle x, a\rangle\langle x, b\rangle}{\|x\|^2}\,(\leq \mu_2\|a\|\,\|b\|)$$

and $1 \geq \mu_1 \geq 0$ $(-1 \leq \mu_2 \leq 0)$, *then*

$$[-1 \leq]\,2\mu_1 - 1 \leq \frac{\langle a, b\rangle}{\|a\|\,\|b\|}\,(\leq 2\mu_2 + 1[\leq 1]). \tag{1.18}$$

The proof is obvious by the inequalities (1.6) and (1.10). We omit the details.

2 - Inequalities for orthonormal families

We recall that the finite family $\{e_i\}_{i\in I}$ is orthonormal in $(H;\langle\cdot,\cdot\rangle)$, a real or complex inner product space, if

$$\langle e_i, e_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

where $i, j \in I$.

The following result may be stated.

Theorem 6. *Let* $\{e_i\}_{i\in I}$ *and* $\{f_j\}_{j\in J}$ *be two finite families of orthonormal vectors in* $(H;\langle\cdot,\cdot\rangle)$. *For any* $x, y \in H\backslash\{0\}$ *one has the inequality*

$$(2.1)\qquad \Big|\sum_{i\in I}\langle x, e_i\rangle\langle e_i, y\rangle + \sum_{j\in J}\langle x, f_j\rangle\langle f_j, y\rangle - 2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle f_j, y\rangle\langle e_i, f_j\rangle - \frac{1}{2}\langle x, y\rangle\Big| \le \frac{1}{2}\|x\|\,\|y\|\,.$$

The case of equality holds in (2.1) *if and only if there exists a* $\lambda\in\mathbb{K}$ *such that*

$$(2.2)\qquad x-\lambda y = 2\Big(\sum_{i\in I}\langle x, e_i\rangle\, e_i - \lambda\sum_{j\in J}\langle y, f_j\rangle f_j\Big).$$

Proof. We know that, if $u, v\in H$, $v\neq 0$, then

$$(2.3)\qquad \left\|u-\frac{\langle u, v\rangle}{\|v\|^2}\cdot v\right\|^2 = \frac{\|u\|^2\|v\|^2 - |\langle u, v\rangle|^2}{\|v\|^2}$$

showing that, in Schwarz's inequality

$$(2.4)\qquad |\langle u, v\rangle|^2 \le \|u\|^2\|v\|^2,$$

the case of equality, for $v\neq 0$, holds if and only if

$$(2.5)\qquad u=\frac{\langle u, v\rangle}{\|v\|^2}\cdot v\,,$$

i.e. there exists a $\lambda\in\mathbb{K}$ such that $u=\lambda v$.

Now, let $u := 2\sum_{i\in I}\langle x, e_i\rangle\, e_i - x$ and $v := 2\sum_{j\in J}\langle y, f_j\rangle f_j - y$.

Observe that

$$\begin{aligned}\|u\|^2 &= \Big\|2\sum_{i\in I}\langle x, e_i\rangle\, e_i\Big\|^2 - 4\,\mathrm{Re}\Big\langle\sum_{i\in I}\langle x, e_i\rangle\, e_i, x\Big\rangle + \|x\|^2\\ &= 4\sum_{i\in I}|\langle x, e_i\rangle|^2 - 4\sum_{i\in I}|\langle x, e_i\rangle|^2 + \|x\|^2 = \|x\|^2,\end{aligned}$$

and, similarly

$$\|v\|^2=\|y\|^2.$$

Also,

$$\langle u, v\rangle = 4 \sum_{i\in I, j\in J} \langle x, e_i\rangle\langle f_j, y\rangle\langle e_i, f_j\rangle + \langle x, y\rangle - 2\sum_{i\in I}\langle x, e_i\rangle\langle e_i, y\rangle - 2\sum_{j\in J}\langle x, f_j\rangle\langle f_j, y\rangle .$$

Therefore, by Schwarz's inequality (2.4) we deduce the desired inequality (2.1). By (2.5), the case of equality holds in (2.1) if and only if there exists a $\lambda \in \mathbb{K}$ such that

$$2\sum_{i\in I}\langle x, e_i\rangle e_i - x = \lambda\Big(2\sum_{j\in J}\langle y, f_j\rangle f_j - y\Big),$$

which is equivalent to (2.2). ■

Remark 2. *If in* (2.2) *we choose* $x = y$, *then we get the inequality:*

$$\left|\sum_{i\in I}|\langle x, e_i\rangle|^2 + \sum_{j\in J}|\langle x, f_j\rangle|^2 - 2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle f_j, x\rangle\langle e_i, f_j\rangle - \frac{1}{2}\|x\|^2\right| \leqslant \frac{1}{2}\|x\|^2 \tag{2.6}$$

for any $x \in H$.

If in the above theorem we assume that $I = J$ *and* $f_i = e_i$, $i \in I$, *then we get from* (2.1) *the Schwarz inequality* $|\langle x, y\rangle| \leqslant \|x\|\,\|y\|$.

If $I\cap J = \emptyset$, $I \cup J = K$, $g_k = e_k$, $k\in I$, $g_k = f_k$, $k \in J$ *and* $\{g_k\}_{k\in K}$ *is orthonormal, then from* (2.1) *we get:*

$$\left|\sum_{k\in K}\langle x, g_k\rangle\langle g_k, y\rangle - \frac{1}{2}\langle x, y\rangle\right| \leqslant \frac{1}{2}\|x\|\,\|y\|, \qquad x, y \in H \tag{2.7}$$

which has been obtained earlier by the author in [3].

If I and J reduce to one element, namely $e_1 = \frac{e}{\|e\|}$, $f_1 = \frac{f}{\|f\|}$ with $e, f \neq 0$, then from (2.1) we get

$$\left|\frac{\langle x, e\rangle\langle e, y\rangle}{\|e\|^2} + \frac{\langle x, f\rangle\langle f, y\rangle}{\|f\|^2} - 2\cdot\frac{\langle x, e\rangle\langle f, y\rangle\langle e, f\rangle}{\|e\|^2\|f\|^2} - \frac{1}{2}\langle x, y\rangle\right| \leqslant \frac{1}{2}\|x\|\,\|y\|, \quad x, y\in H \tag{2.8}$$

which is the corresponding complex version of Precupanu's inequality (1.1).

If in (2.8) we assume that $x = y$, then we get

$$\left| \frac{|\langle x, e\rangle|^2}{\|e\|^2} + \frac{|\langle x, f\rangle|^2}{\|f\|^2} - 2\cdot \frac{\langle x, e\rangle\langle f, e\rangle\langle e, f\rangle}{\|e\|^2\|f\|^2} - \frac{1}{2}\|x\|^2 \right| \leq \frac{1}{2}\|x\|^2. \tag{2.9}$$

The following corollary may be stated:

Corollary 1. *With the assumptions of Theorem* 6, *we have:*

$$\begin{aligned} &\left| \sum_{i\in I}\langle x, e_i\rangle\langle e_i, y\rangle + \sum_{j\in J}\langle x, f_j\rangle\langle f_j, y\rangle - 2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle f_j, y\rangle\langle e_i, f_j\rangle \right| \\ &\leq \frac{1}{2}|\langle x, y\rangle| + \left| \sum_{i\in I}\langle x, e_i\rangle\langle e_i, y\rangle + \sum_{j\in J}\langle x, f_j\rangle\langle f_j, y\rangle \right. \\ &\qquad \left. -2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle f_j, y\rangle\langle e_i, f_j\rangle - \frac{1}{2}|\langle x, y\rangle| \right| \\ &\leq \frac{1}{2}[|\langle x, y\rangle| + \|x\|\|y\|]. \end{aligned} \tag{2.10}$$

Proof. The first inequality follows by the triangle inequality for the modulus. The second inequality follows by (2.1) on adding the quantity $\frac{1}{2}|\langle x, y\rangle|$ on both sides. ■

Remark 3. (1) *If we choose in* (2.10), $x = y$, *then we get:*

$$\begin{aligned} &\left| \sum_{i\in I}|\langle x, e_i\rangle|^2 + \sum_{j\in J}|\langle x, f_j\rangle|^2 - 2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle f_j, x\rangle\langle e_i, f_j\rangle \right| \\ &\leq \left| \sum_{i\in I}|\langle x, e_i\rangle|^2 + \sum_{j\in J}|\langle x, f_j\rangle|^2 \right. \\ &\qquad \left. -2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle f_j, x\rangle\langle e_i, f_j\rangle - \frac{1}{2}\|x\|^2 \right| + \frac{1}{2}\|x\|^2 \\ &\leq \|x\|^2. \end{aligned} \tag{2.11}$$

We observe that (2.11) *will generate Bessel's inequality if* $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$ *are disjoint parts of a larger orthonormal family.*

(2) *From* (2.8) *one can obtain:*

$$\left| \frac{\langle x, e\rangle\langle e, y\rangle}{\|e\|^2} + \frac{\langle x, f\rangle\langle f, y\rangle}{\|f\|^2} - 2\cdot \frac{\langle x, e\rangle\langle f, y\rangle\langle e, f\rangle}{\|e\|^2\|f\|^2} \right| \leqslant \frac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|] \tag{2.12}$$

and in particular

$$\left| \frac{|\langle x, e\rangle|^2}{\|e\|^2} + \frac{|\langle x, f\rangle|^2}{\|f\|^2} - 2\,\frac{\langle x, e\rangle\langle f, e\rangle\langle e, f\rangle}{\|e\|^2\|f\|^2} \right| \leqslant \|x\|^2, \tag{2.13}$$

for any $x, y \in H$.

The case of real inner products will provide a natural genearlization for Precupanu's inequality (1.1):

Corollary 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real inner product space and* $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$ *two finite families of orthonormal vectors in* $(H;\langle\cdot,\cdot\rangle)$. *For any* $x, y \in H\setminus\{0\}$ *one has the double inequality:*

$$\begin{aligned}\frac{1}{2}[|\langle x, y\rangle| - \|x\|\,\|y\|] &\leqslant \sum_{i\in I}\langle x, e_i\rangle\langle y, e_i\rangle + \sum_{j\in J}\langle x, f_j\rangle\langle y, f_j\rangle - 2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle y, f_j\rangle\langle e_i, f_j\rangle \\ &\leqslant \frac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|].\end{aligned} \tag{2.14}$$

In particular, we have

$$0 \leqslant \sum_{i\in I}\langle x, e_i\rangle^2 + \sum_{j\in J}\langle x, f_j\rangle^2 - 2\sum_{i\in I, j\in J}\langle x, e_i\rangle\langle x, f_j\rangle\langle e_i, f_j\rangle \leqslant \|x\|^2, \tag{2.15}$$

for any $x \in H$.

Remark 4. *Similar particular inequalities to those incorporated in* (2.7)-(2.13) *may be stated, but we omit them.*

3 - Refinements of Kurepa's inequality

Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space generating the norm $\|\cdot\|$. The *complexification* $H_{\mathbb{C}}$ of H is defined as a complex linear space $H\times H$ of all ordered pairs (x, y), $x, y\in H$ endowed with the operations:

$$(x, y)+(x', y') := (x+x', y+y'), \qquad x, x', y, y' \in H;$$

$$(\sigma+i\tau)\cdot(x, y) := (\sigma x-\tau y, \tau x+\sigma y), \qquad x, y\in H \quad \text{and} \quad \sigma, \tau\in\mathbb{R}.$$

On $H_{\mathbb{C}} := H\times H$, endowed with the above operations, one can now canonically define the *scalar product* $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ by:

$$\langle z, z'\rangle_{\mathbb{C}} := \langle x, x'\rangle+\langle y, y'\rangle+i[\langle x', y\rangle-\langle x, y'\rangle] \tag{3.1}$$

where $z=(x, y)$, $z'=(x', y')\in H_{\mathbb{C}}$. Obviously,

$$\|z\|_{\mathbb{C}}^2=\|x\|^2+\|y\|^2, \qquad z=(x, y)\in H_{\mathbb{C}}.$$

One can also define the *conjugate* of a vector $z=(x, y)$ by $\bar{z} := (x, -y)$. It is easy to see that, the elements of $H_{\mathbb{C}}$, under defined operations, behave as formal «complex» combinations $x+iy$ with $x, y\in H$. Because of this, we may write $z=x+iy$ instead of $z=(x, y)$. Thus, $\bar{z}=x-iy$.

Under this setting, S. Kurepa [4] proved the following refinement of Schwarz's inequality:

$$|\langle a, z\rangle_{\mathbb{C}}|^2 \leqslant \frac{1}{2}\|a\|^2\left[\|z\|_{\mathbb{C}}^2+|\langle z, \bar{z}\rangle_{\mathbb{C}}|\right] \leqslant \|a\|^2\|z\|_{\mathbb{C}}^2, \tag{3.2}$$

for any $a\in H$ and $z\in H_{\mathbb{C}}$.

This was motivated by generalising the de Bruijn result for sequences of real and complex numbers obtained in [**1**].

The following result may be stated.

Theorem 7. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real inner product space and* $\{e_i\}_{i\in I}, \{f_j\}_{j\in J}$ *two finite families in* H. *If* $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$ *is the complexification of* $(H;\langle\cdot,\cdot\rangle)$, *then for any* $w\in H_{\mathbb{C}}$, *we have the inequalities*

$$\left|\sum_{i\in I}\langle w, e_i\rangle_{\mathbb{C}}^2 + \sum_{j\in J}\langle w, f_j\rangle_{\mathbb{C}}^2 - 2\sum_{i\in I, j\in J}\langle w, e_i\rangle_{\mathbb{C}}\langle w, f_j\rangle_{\mathbb{C}}\langle e_i, f_j\rangle\right| \tag{3.3}$$

$$\leqslant \frac{1}{2}\left|\langle w, \overline{w}\rangle_{\mathbb{C}}\right| + \left|\sum_{i\in I}\langle w, e_i\rangle_{\mathbb{C}}^2 + \sum_{j\in J}\langle w, f_j\rangle_{\mathbb{C}}^2\right.$$

$$\left. -2\sum_{i\in I, j\in J}\langle w, e_i\rangle_{\mathbb{C}}\langle w, f_j\rangle_{\mathbb{C}}\langle e_i, f_j\rangle - \frac{1}{2}\langle w, \overline{w}\rangle_{\mathbb{C}}\right|$$

$$\leqslant \frac{1}{2}\left[\|w\|_{\mathbb{C}}^2 + \left|\langle w, \overline{w}\rangle_{\mathbb{C}}\right|\right] \leqslant \|w\|_{\mathbb{C}}^2.$$

Proof. Define $g_j\in H_{\mathbb{C}}$, $g_j := (e_j, 0)$, $j\in I$. For any $k, j\in I$ we have

$$\langle g_k, g_j\rangle_{\mathbb{C}} = \langle (e_k, 0), (e_j, 0)\rangle_{\mathbb{C}} = \langle e_k, e_j\rangle = \delta_{kj},$$

therefore $\{g_j\}_{j\in I}$ is an orthonormal family in $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$.

If we apply Corollary 1 for $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$, $x = w$, $y = \overline{w}$, we may write:

$$\left|\sum_{i\in I}\langle w, e_i\rangle_{\mathbb{C}}\langle e_i, \overline{w}\rangle_{\mathbb{C}} + \sum_{j\in J}\langle w, f_j\rangle\langle f_j, \overline{w}\rangle\right. \tag{3.4}$$

$$\left. -2\sum_{i\in I, j\in J}\langle w, e_i\rangle_{\mathbb{C}}\langle f_j, \overline{w}\rangle_{\mathbb{C}}\langle e_i, f_j\rangle\right|$$

$$\leqslant \frac{1}{2}\|w\|_{\mathbb{C}}\|\overline{w}\|_{\mathbb{C}} + \left|\sum_{i\in I}\langle w, e_i\rangle_{\mathbb{C}}\langle e_i, \overline{w}\rangle_{\mathbb{C}} + \sum_{j\in J}\langle w, f_j\rangle\langle f_j, \overline{w}\rangle\right.$$

$$\left. -2\sum_{i\in I, j\in J}\langle w, e_i\rangle_{\mathbb{C}}\langle f_j, \overline{w}\rangle_{\mathbb{C}}\langle e_i, f_j\rangle - \frac{1}{2}\langle w, \overline{w}\rangle_{\mathbb{C}}\right|$$

$$\leqslant \frac{1}{2}\left[\left|\langle w, \overline{w}\rangle_{\mathbb{C}}\right| + \|w\|_{\mathbb{C}}\|\overline{w}\|_{\mathbb{C}}\right].$$

However, for $w := (x, y) \in H_{\mathbb{C}}$, we have $\overline{w} = (x, -y)$ and

$$\langle e_j, \overline{w} \rangle_{\mathbb{C}} = \langle (e_j, 0), (x, -y) \rangle_{\mathbb{C}} = \langle e_j, x \rangle + i \langle e_j, y \rangle$$

and

$$\langle w, e_j \rangle_{\mathbb{C}} = \langle (x, y), (e_j, 0) \rangle_{\mathbb{C}} = \langle x, e_j \rangle + i \langle e_j, y \rangle$$

showing that $\langle e_j, \overline{w} \rangle = \langle w, e_j \rangle$ for any $j \in I$. A similar relation is true for f_j and since

$$\|w\|_{\mathbb{C}} = \|\overline{w}\|_{\mathbb{C}} = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}},$$

hence from (3.4) we deduce the desired inequality (3.3). ■

Remark 5. *It is obvious that, if one family, say* $\{f_j\}_{j \in J}$ *is empty, then, on observing that all sums* $\sum_{j \in J}$ *should be zero, from* (3.3) *one would get* [3]

$$\begin{aligned} &\left| \sum_{i \in I} \langle w, e_i \rangle_{\mathbb{C}}^2 \right| \\ &\leq \frac{1}{2} |\langle w, \overline{w} \rangle_{\mathbb{C}}| + \left| \sum_{i \in I} \langle w, e_i \rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle w, \overline{w} \rangle_{\mathbb{C}} \right| \qquad (3.5) \\ &\leq \frac{1}{2} \left[\|w\|_{\mathbb{C}}^2 + |\langle w, \overline{w} \rangle_{\mathbb{C}}| \right] \leq \|w\|_{\mathbb{C}}^2. \end{aligned}$$

If in (3.5) *one assumes that the family* $\{e_i\}_{i \in I}$ *contains only one element* $e = \frac{a}{\|a\|}$, $a \neq 0$, *then by selecting* $w = z$, *one would deduce*

$$\begin{aligned} |\langle a, z \rangle_{\mathbb{C}}|^2 &\leq \left| \langle a, z \rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle z, \overline{z} \rangle_{\mathbb{C}} \right| + \frac{1}{2} |\langle z, \overline{z} \rangle_{\mathbb{C}}| \\ &\leq \frac{1}{2} \|a\|^2 \left[\|z\|_{\mathbb{C}}^2 + |\langle z, \overline{z} \rangle_{\mathbb{C}}| \right], \end{aligned}$$

which is a refinement for Kurepa's inequality (3.2).

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Abstract

Some generalizations of Precupanu's inequality for orthornormal families of vectors in real or complex inner product spaces and applications related to Buzano's, Richard's and Kurepa's results are given.

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