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Algebraically compactness of Sylow p-groups in abelian group rings of characteristic p (**)

1 - Introduction

The study of the algebraic compactness starts by us in [**1**]. Later on, Mollov and Nachev have argued in [**4**] some extensions to theorems of [**1**]. However, in [**2**], we proved more general results than [**4**]. The purpose of this paper is to finish the investigation of this theme and to formulate the results in a final form.

As usual, suppose RG is the group ring of an abelian group G over a commutative ring R with identity of prime characteristic p. For any arbitrary subgroup H of G, we let $I(RG; H)$ denote the relative augmentation ideal of RG with respect to H, and let $I_p(RG; H)$ designate its nil-radical. For simplicity of the exposition, we assume that $1 + I_p(RG; H)$ denotes the p-group $S(RG; H)$. Evidently, $S(RG) = S(RG; G)$ is the normalized Sylow p-subgroup in RG.

As we have above remarked, our main aim here is to find a criterion for $S(RG; H)$ to be algebraically compact only in terms of R and G.

The definitions of algebraically compact groups, divisible groups and bounded groups are the standard ones and follow essentially [**3**]. The notation and terminology not explicitly defined herein are the same as in [**3**]. For example, G_p is the p-component of G, and G_d and G_r are the maximal divisible subgroup (= the divisible part) and the reduced part of G, respectively. Throughout the paper we assume that $N(R)$ is the Baer's nil-ideal of R.

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2 - Algebraic compact p-components of modular abelian group rings

Following [3], the abelian p-group A is said to be algebraically compact if there exists a non-negative integer n such that $A^{p^n} = A^{p^{n+1}}$, i.e. if A^{p^n} is divisible for some $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An arbitrary abelian group B with such a property that $B^{p^n} = B^{p^{n+1}}$ for any fixed $n \in \mathbb{N}$ is called weakly p-divisible.

Before proving the main attainment, we first need a useful technical lemma.

Lemma 1. *Let G be an abelian group with subgroups C and H, and let R be a commutative unitary ring with prime characteristic p and with subrings P and K which contain the same identity. If*

(a) $N(P) = 0$, *then* $I_p(PG; H) = 0 \Leftrightarrow G_p = 1$ *or* $H = 1$.

(b) $N(P) \neq 0$, *then* $I_p(PG; H) = 0 \Leftrightarrow H = 1$.

(a′) $N(P) = 0$, *then* $S(PG; H) = 1 \Leftrightarrow G_p = 1$ *or* $H = 1$.

(b′) $N(P) \neq 0$, *then* $S(PG; H) = 1 \Leftrightarrow H = 1$.

(c) $G_p \neq 1$, *then* $I_p(PG; H) = I_p(KG; C) \Leftrightarrow P = K$ *and* $H = C$, *when* $H_p \neq G_p$ *and* $C_p \neq G_p$ *or when* $H = C = G$ *or when* $H = H_p$ *and* $C = C_p$ *or when* $N(P) \neq 0$ *and* $N(K) \neq 0$, *but* $I_p(PG; H) = I_p(KG; C) \Leftrightarrow P = K$ *and* $H \subset C$ *when* $H = G_p$, $C \neq G_p$ *and* $N(P) = 0$; *or* $P = K$ *and* $H \neq C$ *when* $G_p = H_p = C_p$ *and* $N(P) = 0$. *In the remaining case when* $G_p = H_p \neq C_p$, *we have* $I_p(PG; H) \neq I_p(KG; C) = 0$.

(d) $G_p = 1$ *and* $N(P) = N(K) = 0$, *then* $I_p(PG; H) = I_p(KG; C)$.

(e) $G_p = 1$, $H \neq 1$, $C \neq 1$ *and* $N(P) \neq 0$, $N(K) \neq 0$, *then* $I_p(PG; H) = I_p(KG; C) \Leftrightarrow N(P) = N(K)$ *and* $H = C$.

Proof. (a) $G_p = 1$. We will prove that $I_p(PG; G) = 0$, therefore $0 = I_p(PG; H) \subseteq I_p(PG; G)$. Indeed, given $x \in I_p(PG; G)$. Thus we write $x = \sum_{g \in \Pi} \sum_{h \in G_p} f_{gh}\, gh$, where $\Pi = \Pi(G/G_p)$ is a complete family of coset representatives of the group G with respect to its subgroup G_p (containing the identity of G), and $\sum_{h \in G_p} f_{gh} = y_g \in N(P) = 0$ for each $g \in \Pi$, where $f_{gh} \in P$. So, $\sum_{g \in \Pi} y_g = 0$ and x can be represented in the form $x = \sum_{g \in \Pi} \sum_{h \in G_p} f_{gh}\, g(h-1)$. Finally, we obviously detect that $x = 0$ since $G_p = 1$, which proves the first half.

Let now $I_p(PG; H) = 0$ and $G_p \neq 1$, $H \neq 1$. Thus, $1 \neq g_p \in G_p$, $1 \neq h \in H$ and $x_{gh} = (1 - g_p)(1 - h) \in I_p(PG; H)$. But $(1 - g_p)(1 - h) = 1 - g_p - h + g_p h$ is a canonical element (because $g_p \notin H$ otherwise; $g_p \in H \cap G_p = H_p \subseteq 1 + I_p(PG; H) = 1$, hence $1 = g_p \in H_p = 1$). It is elementarily to see that $x_{gh} \neq 0$ ($x_{gh} = 0$ only when $g_p = 1$ or $h = 1$), which completes (a).

(b) If $H = 1$, we derive $I_p(PG; H) \subseteq I(PG; H) = 0$. For the reverse, assume now that $I_p(PG; H) = 0$ and $0 \neq \delta \in N(P)$. Consequently $\delta(1 - h) \in I_p(PG; H)$ and $\delta(1 - h) = \delta - \delta h = 0$, i.e. $h = 1$ for every $h \in H$. That is why $H = 1$.

(c) First of all, we presume at this point that $0 \neq \gamma \in P$, $1 \neq h \in H$ and $1 \neq g_p \in G_p \setminus (H \cup C)$ (if $G_p \subseteq H \cup C$, then $G_p = H_p$ or $G_p = C_p$; for example, the reader can see cf. [3], p. 14). Clearly $0 \neq \gamma\ (1-g_p)(1-h) \in I_p(KG;\, C)$. Furthermore, we obtain $\gamma \in K$ and $h \in C$, whence $P \subseteq K$ and $H \subseteq C$. Similarly for $K \subseteq P$ and $C \subseteq H$, i.e. $P = K$ and $H = C$.

Now, suppose $H_p = G_p$, $C_p = G_p$ (analogically for the cases $H_p \neq G_p$, $C_p \neq G_p$ or $H_p = G_p$, $C_p \neq G_p$) as well as suppose $N(P) \neq 0$ and $N(K) \neq 0$. If $0 \neq \gamma \in P$, we deduce that $\gamma(1-g_p) \in I_p(PG;\, H) = I_p(KG;\, C)$, hence $\gamma \in K$ and $P \subseteq K$. Analogous to the preceding situation we yield $K \subseteq P$, i.e. $P = K$. For $1 \neq h \in H$, we conclude $r(1-h) \in I_p(KG;\, C)$, where $r \in N(P)$. Thus $h \in C$, i.e. $H \subseteq C$. Similarly for $C \subseteq H$, i.e. $H = C$.

Let now $H = G_p$, $C \neq G_p$ (respective $G_p = C_p$ or $G_p \neq C_p$) and $N(P) = 0$. Certainly, $I(PG;\, G_p) = I_p(PG;\, G)$ since $S(PG) = 1 + I(PG;\, G_p) = 1 + I_p(PG;\, G)$, whence $I(PG;\, G_p) = I_p(KG;\, C)$ only when $P = K$ and $G_p \subset C$ (notice that $1 - g_p \in I_p(PG;\, C)$). Besides, we note that $P = K$, $I_p(PG;\, H) \subseteq I_p(PG;\, G) = I(PG;\, G_p)$ and $I_p(KG;\, C) \subseteq I_p(KG;\, G) = I(KG;\, G_p)$, since $N(P) = N(K) = 0$, that verifies (c) in all generality.

(d) Referring to (a), it is trivial.

(e) Let $0 \neq \beta \in N(P)$ and $h \in H$. Therefore, $\beta(1-h) \in I_p(PG;\, H) = I_p(KG;\, C)$ and therefore $\beta \in K$, $h \in C$. Thus $\beta \in N(P) \cap K = N(P) \cap N(K)$, i.e. $\beta \in N(K)$ and $N(P) \subseteq N(K)$. Moreover $H \subseteq C$. Similarly $C \subseteq H$ and $N(K) \subseteq N(P)$.

Suppose now $C = H$ and $N(P) = N(K)$. We will check that $I_p(PG;\, H) = I_p(KG;\, H)$ provided $G_p = 1$. Indeed, for $x \in I_p(PG;\, H)$, we extract that $x = \sum_{g \in \Pi} \sum_{h \in G_p} \varphi_{gh}\, gh$, where $\Pi = \Pi(G/G_p)$ is a full system of coset representatives of the group G with respect to its maximal p-subgroup of torsion G_p, and $\sum_{h \in G_p} \varphi_{gh} = \psi_g \in N(P)$ for every $g \in \Pi$, where $\varphi_{gh} \in P$. So, $\sum_{g \in \Pi} \psi_g = 0$ and $x = \sum_{g \in \Pi} \sum_{h \in G_p} \varphi_{gh}\, g(h-1) + \sum_{g \in \Pi \setminus \{1\}} \psi_g (g-1) = \sum_{g \in \Pi \setminus \{1\}} \psi_g (g-1)$, where $\psi_g \in N(P) = N(K)$. Hence clearly $x \in I_p(KG;\, H)$, because $\psi_g = \varphi_{g1} \in N(K)$ and $x \in I_p(PG;\, H) \cap I_p(KG;\, G) = I_p((P \cap K)\, G;\, H) \subseteq I_p(KG;\, H)$. We continue by analogy also for the other relation. The proof of the lemma is finished.

We are now ready to formulate the following assertion.

Theorem 2. *Suppose G is an abelian group with a non-identity subgroup H and R is a commutative ring with unity of prime characteristic p. Then $S(RG;\, H)$ is divisible if and only if at least one from the following conditions of the table (*) is fulfilled:*

(1) $G_p = 1,\ N(R) = 0$;

(2) $G_p = 1,\ N(R) = N(R^p) \neq 0,\ G = G^p,\ H = H^p$;

(3) $G_p \neq 1,\ N(R) \neq 0,\ R = R^p,\ G = G^p,\ H = H^p$;
(*) (4) $G_p \neq 1,\ N(R) = 0,\ R = R^p,\ G = G^p,\ H \neq H^p,\ G_p = H_p = (H_p)^p$;
(5) $G_p \neq 1,\ N(R) = 0,\ R = R^p,\ G = G^p,\ H = H^p,\ G_p \neq H_p$;
(6) $G_p \neq 1,\ N(R) = 0,\ R = R^p,\ G = G^p,\ H = H^p = H_p$;
(7) $G_p \neq 1,\ N(R) = 0,\ R = R^p,\ G = G^p,\ H = H^p = G$.

Proof. Since $S^p(RG; H) = S(R^p G^p; H^p) = S(RG; H)$, we can employ the previous technical affirmation to complete the proof.

Substituting G by G^{p^n}, H by H^{p^n} and R by R^{p^n} we may obtain a valuable criterion for algebraic compactness, namely:

Central Theorem 3. *Suppose G is an abelian group with a non-trivial subgroup H and suppose R is a commutative ring with identity of prime characteristic p. Then $S(RG; H)$ is algebraically compact if and only if for some $n \in N$ at least one from the following conditions of a table (**) holds valid:*
(1) $G_p^{p^n} = 1,\ N(R^{p^n}) = 0$;
(2) $G_p^{p^n} = 1,\ N(R^{p^n}) = N(R^{p^{n+1}}) \neq 0,\ G^{p^n} = G^{p^{n+1}},\ H^{p^n} = H^{p^{n+1}}$;
(**) (3) $G_p^{p^n} \neq 1,\ N(R^{p^n}) \neq 0,\ R^{p^n} = R^{p^{n+1}},\ G^{p^n} = G^{p^{n+1}},\ H^{p^n} = H^{p^{n+1}}$;
(4) $G_p^{p^n} \neq 1,\ N(R^{p^n}) = 0,\ R^{p^n} = R^{p^{n+1}},\ G^{p^n} = G^{p^{n+1}},\ H^{p^n} \neq H^{p^{n+1}},\ G_p^{p^n} = H_p^{p^n} = H_p^{p^{n+1}}$;
(5) $G_p^{p^n} \neq 1,\ N(R^{p^n}) = 0,\ R^{p^n} = R^{p^{n+1}},\ G^{p^n} = G^{p^{n+1}},\ H^{p^n} = H^{p^{n+1}},\ G_p^{p^n} \neq H_p^{p^n}$;
(6) $G_p^{p^n} \neq 1,\ N(R^{p^n}) = 0,\ R^{p^n} = R^{p^{n+1}},\ G^{p^n} = G^{p^{n+1}},\ H^{p^n} = H^{p^{n+1}} = H_p^{p^n}$;
(7) $G_p^{p^n} \neq 1,\ N(R^{p^n}) = 0,\ R^{p^n} = R^{p^{n+1}},\ G^{p^n} = G^{p^{n+1}},\ H^{p^n} = H^{p^{n+1}} = G^{p^n}$.

Proof. It is straightforward utilizing the foregoing theorem.

The next consequence appeared in [4] and [2] too.

Corollary 4. *Under the above circumstances, $S(RG)$ is algebraically compact if and only if for some $n \in \mathbb{N}$ it is fulfilled:*
(1) $N^{p^n}(R) = 0,\ G_p^{p^n} = 1$;
(2) $G_p^{p^n} \neq 1,\ G^{p^n} = G^{p^{n+1}},\ R^{p^n} = R^{p^{n+1}}$;
(3) $N^{p^n}(R) \neq 0,\ G_p^{p^n} = 1,\ G^{p^n} \neq 1,\ G^{p^n} = G^{p^{n+1}},\ N^{p^n}(R) = N^{p^{n+1}}(R)$;
(4) $N^{p^n}(R) \neq 0,\ G^{p^n} = 1$.

Proof. Putting $H = G$ and taking into account the already mentioned fact that $S(RG; G) = S(RG)$, the major theorem is applicable to complete the proof.

In some particular cases, we are in a position to find certain more explicit criteria (see cf. [1], [2] as well). Foremost, we state one more definition, above listed too.

Definition. An abelian group G is termed weakly p-divisible if $G^{p^i} = G^{p^{i+1}}$ for an arbitrary fixed $i \in \mathbb{N}$, i.e. if G^{p^i} is p-divisible for this $i \in \mathbb{N}$. Evidently, in such a case, for the maximal p-divisible subgroup G^* of G, we have $G^* = G^{p^i}$ for that $i \in \mathbb{N}$.

Proposition 5. *Suppose R is without nilpotents.*

(i) If *G_p is not reduced, then $S(RG)$ is algebraically compact if and only if G is weakly p-divisible and R is perfect.*

(ii) *If G_p is reduced, then $S(RG)$ is algebraically compact if and only if G_p is bounded.*

Proof. Follows immediately from Corollary 4. The proof is over.

The following is a direct consequence of the foregoing one, but for the sake of completeness we include a new independent proof.

Corollary 6. *Let G be torsion and R be without nilpotents.*

(j) *If G_p is reduced, then $S(RG)$ is algebraically compact if and only if G_p is algebraically compact.*

(jj) *If G_p is not reduced, then $S(RG)$ is algebraically compact if and only if G_p is algebraically compact and R is perfect.*

Proof. (j) We know that, applying [2], the subgroup G_p being reduced implies that $S(RG)$ is reduced. And so, $S(RG)$ is reduced algebraically compact, i.e. $S(RG)$ is bounded via [3]. This is equivalent to G_p is bounded, i.e. G_p is algebraically compact again by [3].

(jj) Suppose $G = G_p \times M$ where $M = \coprod_{q \neq p} G_q$. Thus, (see [1]), $S(RG) \cong S((RM)\, G_p)$, where RM is an abelian ring with 1 and prime characteristic p without nilpotent elements. In fact, to verify the latter, choose $w = \sum_i v_i f_i \in RM (v_i \in R,\ f_i \in M)$ and $w^p = 0$, i.e. $\sum_i v_i^p f_i^p = 0$. But $f_{j-1}^p \neq f_j^p$ $(j = 2, \dots, n+1,\ f_{n+1} = f_n)$ since otherwise if we assume that $f_{j-1}^p = f_j^p$, then $(f_{j-1} f_j^{-1})^p = 1$, i.e. $f_{j-1} . f_j^{-1} \in M_p = 1$ or equivalently $f_{j-1} = f_j$ — a contradiction. Hence $v_i^p = 0$, i.e. $v_i = 0$. Finally $w = 0$. Thus $S(RG)$ is algebraically compact if and only if $S((RM)\, G_p)$ is algebraically compact, that is, from ([1], Theorems 16 or 17), if and only if RM is perfect and G_p is algebraically compact. The last is equivalent to R is perfect, since M is p-divisible (cf. [1]), and G_p is algebraically compact. So, the proof is completed.

Corollary 7. *Let G be torsion and R be perfect. Then $S(RG)$ is algebraically compact if and only if G_p is algebraically compact.*

Proof. Since, invoking to [1], $S(RG) \cong S(RM) \times S((RM)\, G_p)$ where RM is perfect and since $S(RM)$ is divisible whence algebraically compact, $S(RG)$ is algebraically compact if and only if the same is $S((RM)\, G_p)$, i.e., if and only if G_p is algebraically compact (cf. [1], Theorem 19). So, the proof is finished.

Corollary 8. *Let G be an abelian group whose reduced part is torsion and let R be perfect. Then $S(RG)$ is algebraically compact if and only if G_p is algebraically compact.*

Proof. Because $G = G_d \times G_r$, we derive that $S(RG) \cong S(RG_d) \times S((RG_d)\, G_r)$ by complying with [1]. But $S(RG_d)$ is divisible and hence it is algebraically compact. Besides RG_d is a perfect ring. Consequently, employing Corollary 7, $S((RG_d)\, G_r)$ is algebraically compact if and only if so is $(G_r)_p$. Finally, $S(RG)$ is algebraically compact (e.g. [3]) if and only if so does G_p since $G_p = (G_d)_p \times (G_r)_p$ and $(G_d)_p$ is divisible. So, the proof is over.

Lemma 9. $(G_p)_d = (G_d)_p$.

Proof. It is easy to verify that $(G_p)_d \subseteq G_d$ and $(G_p)_d \subseteq G_p$. Thus $(G_p)_d \subseteq (G_d)_p$. Certainly $(G_p)_d = (G^*)_p$, where as above G^* is the maximal p-divisible subgroup of G. But $G_d \subseteq G^*$ since $G_d = G_d^{p^\tau} \subseteq G^{p^\tau} = G^*$ for some ordinal τ (the first ordinal with $G^{p^\tau} = G^{p^{\tau+1}}$). Hence, $(G_d)_p \subseteq (G^*)_p = (G_p)_d$. Finally, $(G_p)_d = (G_d)_p$. So, the lemma is true.

Let now we write $G = G_d \times G_r$ and $G_p = (G_p)_d \times (G_p)_r$. Therefore, the first equality assures that $G_p = (G_d)_p \times (G_r)_p$. We see that Lemma 9 implies $(G_p)_r \cong (G_r)_p$.

Proposition 10. *Let R be perfect. Then if*

(k) *G_p is not reduced, or*

(kk) *G_p is reduced and R has nilpotent elements,*

$S(RG)$ is algebraically compact if and only if G is weakly p-divisible.

(kkk) *G_p is reduced and R has no nilpotent elements, $S(RG)$ is algebraically compact if and only if G_p is bounded.*

Proof. In virtue of Lemma 9, $(G_d)_p = (G_p)_d$.

(k), (kk) If $(G_d)_p \neq 1$, or $(G_d)_p = 1$ and R is a ring with nilpotent elements, then the group ring RG_d possesses nilpotent elements — for instance these are the elements $0 \neq \alpha = 1 - x_{dp}$, where $x_{dp} \in (G_d)_p$ or $0 \neq \beta \in R$ with $\beta^p = 0$, respectively. Moreover, $S(RG_d)$ is divisible and hence it is algebraically compact. Furthermore,

$S(RG)$ is algebraically compact $\Leftrightarrow$ $S((RG_d)\,G_r)$ is algebraically compact, i.e. $\Leftrightarrow$ $S((RG_d)\,G_r)_r$ is bounded where $S((RG_d)\,G_r) = S((RG_d)D) \times S((RG_d)\,G_r)_r$ and D is the maximal p-divisible subgroup of the group G_r. Consequently, $S((RG_d)\,G_r)_r$ is bounded if and only if $S((RG_d)\,G_r)/S((RG_d)\,D)$ is bounded, that is $S((RG_d)\,G_r^{p^i}) = S((RG_d)\,D)$ for some $i \in \mathbb{N}$, which is equivalent to $G_r^{p^i} = D$ for this $i \in \mathbb{N}$. Thereby G_r is weakly p-divisible, i.e. G is a weakly p-divisible group. This holds because $G^{p^i} = G_d \times G_r^{p^i} = G_d \times G_r^{p^{i+1}} = G^{p^{i+1}}$ for some $i \in \mathbb{N}$ iff $G_r^{p^i} = G_r^{p^{i+1}}$ since $G_r^{p^{i+1}} \subseteq G_r^{p^i}$.

(kkk) Apparently $S(RG_d) = 1$ and RG_d is a ring with no nilpotent elements. Hence $S(RG) \cong S((RG_d)\,G_r)$ is reduced since $G_p = (G_p)_r = (G_r)_p$ is reduced. Therefore $S(RG)$ is algebraically compact only if it is bounded by using [**3**], i.e. only if G_p is bounded.

Henceforth, the proposition is proved.

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Abstract

Let G be an abelian group, let H be a non-identity subgroup of G and let R be a commutative ring with 1 of prime characteristic p. Necessary and sufficient conditions are established for the p-group S(RG; H) in the group algebra RG to be algebraically compact. These claims supersede a statement due to Mollov-Nachev (*Compt. Rend. Acad. Bulg. Sci.,* 1994) *and completely exhaust the problem (see also the author's paper in Compt. Rend. Acad. Bulg. Sci.*, 1995) *as well.*

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