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Prime numbers between squares (**)

1 - Introduction

A well known conjecture about the distribution of primes asserts that

Conjecture 1. For every integer n, the interval $[n^2, (n+1)^2]$ contains a prime.

The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis (RH). On the other side it is not difficult to prove unconditionally that Conjecture 1 holds for almost all integers.

More precisely we can prove that, for every $[n^2, (n+1)^2] \subset [1, N]$, we have the expected number of primes with at most $O(N^{1/4+\varepsilon})$ exceptions, and $O((\ln N)^{2+\varepsilon})$ exceptions under the assumption of RH, see Bazzanella [**1**]. To obtain some results in the direction of Conjecture 1 we need to assume hypotheses stronger than RH. Define

$$J(N, h) = \int_1^N (\vartheta(x+h) - \vartheta(x) - h)^2 \, dx ,$$

with $\vartheta(x) = \sum_{p \leq x} \log p$ and p a prime number, and consider the following strong form of Montgomery's pair correlation conjecture.

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Conjecture 2. Let γ be Euler's constant. For any $\varepsilon > 0$ we have

$$J(N, h) = hN \log(N/h) - (\gamma + \log 2\pi)\, hN + o(hN) + O(N),$$

uniformly for $1 \leq h \leq N^{1-\varepsilon}$.

Goldston [**5**] deduced the validity of Conjecture 1 assuming Conjecture 2. The basic idea of this paper is to connect the distribution of primes in intervals of the type $[n^2, (n+1)^2]$ to the exceptional set for the asymptotic formula of the distribution of primes in short intervals, and using the properties of this set, see Bazzanella [**2**] and Bazzanella and Perelli [**3**], to obtain a new conditional proof of Conjecture 1.

With this in mind we state the following conjecture.

Conjecture 3.

$$J(N+Y, h) - J(N, h) = o(hN),$$

uniformly for $1 \leq Y \leq N^{1/2}$ and $N^{1/2} \ll h \ll N^{1/2}$.

Assuming Conjecture 3 we can state our main theorem.

Theorem. *Assume Conjecture* 3. *The intervals of type* $[n^2, (n+1)^2]$ *contain the expected number of primes for* $n \to \infty$.

We note that although Conjecture 3 is weaker than Conjecture 2, our Theorem is stronger than the result of Goldston [**5**], which asserts only the existence of a prime in intervals of type $[n^2, (n+1)^2]$.

2 - Basic lemma

The proof of the Theorem is based on a result about the structure of the exceptional set for the asymptotic formula

$$\psi(x + h(x)) - \psi(x) \sim h(x) \quad \text{as} \quad x \to \infty. \tag{1}$$

Let X be a large positive number, $\delta > 0$, $h(x)$ an increasing function such that $x^{\varepsilon} \leq h(x) \leq x$ for some $\varepsilon > 0$,

$$\Delta(x, h) = \psi(x + h(x)) - \psi(x) - h(x),$$

and

$$E_{\delta}(X, h) = \{X \leq x \leq 2X : |\Delta(x, h)| \geq \delta h(x)\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, h) = \emptyset$ for $X \geqslant X_0(\delta)$. Hence for small $\delta > 0$ and X tending to ∞ the set $E_\delta(X, h)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X, h) \subset E_{\delta'}(X, h) \quad \text{if} \quad 0 < \delta' < \delta .$$

The lemma about the structure of the exceptional set is the following.

Lemma. *Let* $0 < \theta < 1$, $h(x)$ *increasing fuunction such that* $h(x) \asymp x^\theta$, X *be sufficiently large depending on the function* $h(x)$ *and* $0 < \delta' < \delta$ *with* $\delta - \delta' \geqslant \exp(-\sqrt{\log X})$. *If* $x_0 \in E_\delta(X, h)$ *then* $E_{\delta'}(X, h)$ *contains the interval* $[x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, *where* $c = (\delta - \delta')\theta/5$. *In particular, if* $E_\delta(X, h) \neq \emptyset$ *then*

$$|E_{\delta'}(X, h)| \gg_\theta (\delta - \delta')\, h(X).$$

Proof. We will always assume that x and X are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence. From the Brun-Titchmarsh theorem, see Montgomery-Vaughan [7], we have that

$$\psi(x+y) - \psi(x) \leqslant \frac{21}{10}\, y \frac{\log x}{\log y} \tag{2}$$

for $10 \leqslant y \leqslant x$. From (2) we easily see that

$$\psi(x+y) - \psi(x) \leqslant \frac{9}{4\alpha}\, cY \tag{3}$$

for $X \leqslant x \leqslant 3X$ and $0 \leqslant y \leqslant cY$, where $0 < \alpha < 1$, $X^{\alpha - \varepsilon} \leqslant Y \leqslant X$ and

$$\frac{\alpha}{5} \exp(-\sqrt{\log X}) \leqslant c \leqslant 1 .$$

Let $h(x) \asymp x^\theta$, $x_0 \in E_\delta(X, h)$ and $x \in [x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, where c satisfies the above restrictions. We have

$$|\Delta(x, h)| = |\Delta(x_0, h) + \Delta(x, h) - \Delta(x_0, h)|$$

$$\geqslant |\Delta(x_0, h)| - |\psi(x + h(x)) - \psi(x_0 + h(x_0))| - |\psi(x) - \psi(x_0)| - |h(x) - h(x_0)| .$$

Hence from (3) with $\alpha = \theta$ we obtain

$$|\Delta(x, h)| \geq \delta h(x) - \frac{9}{2\theta} ch(X) + O(X^{2\theta - 1 + \varepsilon}) \geq \delta h(x) - \frac{5}{\theta} ch(X) \geq \delta' h(x),$$

by choosing $c = (\delta - \delta')\,\theta/5$, since $h(x)$ is increasing. Hence $x \in E_{\delta'}(X, h)$ and the Lemma follows.

3 - Proof of the theorem

The prime number theorem implies that

$$\psi(x + h(x)) - \psi(x) \sim h(x) \qquad \text{as} \qquad x \to \infty,$$

for $h(x)$ sufficiently large with respect to x. Hence the expected number of primes in intervals of type $[n^2, (n+1)^2]$ is $(n+1)^2 - n^2 \sim 2n$ and then the Theorem asserts that

$$\psi((n+1)^2) - \psi(n^2) \sim 2n \qquad \text{as} \qquad n \to \infty. \tag{4}$$

In order to prove the Theorem we assume that (4) does not hold. Then there exist $\delta_0 > 0$ and a sequence $x_j \to \infty$ with $|\Delta(x_j, h)| \geq \delta_0 h(x_j)$ and

$$h(x) = 2\sqrt{x} + 1. \tag{5}$$

For x_j sufficiently large, choose $\delta' = \delta_0/2$ in the Lemma. Hence

$$|\Delta(x, h)| \geq \frac{\delta_0}{2} h(x) \geq \frac{\delta_0}{2} \sqrt{x_j} \qquad \text{for} \qquad x_j \leq x \leq x_j + \frac{\delta_0}{20}\sqrt{x_j}.$$

From our assumption it follows that

$$x_j^{3/2} \ll \int_{x_j}^{x_j + Y} |\Delta(x, h)|^2 \, dx, \tag{6}$$

where

$$Y = \frac{\delta_0}{20}\sqrt{x_j}. \tag{7}$$

From (5) we see that

$$h(x) = h(x_j) + O(1) \quad \text{uniformly for } x_j \leq x \leq x_j + Y,$$

and therefore

$$\int_{x_j}^{x_j+Y} |\Delta(x, h)|^2\,dx = \int_{x_j}^{x_j+Y} |\psi(x+h(x_j)) - \psi(x) - h(x_j)|^2\,dx + O(x_j^{1/2})\,. \tag{8}$$

Recalling the definitions of the functions $\psi(x)$ and $\vartheta(x)$ we find that

$$\int_{x_j}^{x_j+Y} |\psi(x+h(x_j)) - \psi(x) - h(x_j)|^2\,dx \tag{9}$$
$$= \int_{x_j}^{x_j+Y} |\vartheta(x+h(x_j)) - \vartheta(x) - h(x_j)|^2\,dx + O(x_j^{1/2}\log^2 x_j)\,.$$

From (6), (8) and (9) we can conclude that

$$x_j^{3/2} \ll \int_{x_j}^{x_j+Y} |\vartheta(x+h(x_j)) - \vartheta(x) - h(x_j)|^2\,dx = J(x_j+Y,\, h(x_j)) - J(x_j,\, h(x_j))\,.$$

Assuming Conjecture 3 we get a contradiction, and then the Theorem follows.

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Abstract

A well known conjecture about the distribution of primes asserts that between two consecutive squares there is always at least one prime number. The proof of this conjecture is out of reach at present, even under the assumption of the Riemann Hypothesis. The aim of this paper is to provide a conditional proof of the conjecture assuming a hypothesis about the behavior of Selberg's integral in short intervals.

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