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About linear combinations of Siegel theta series (**)

1 - Siegel modular forms and theta series

In this Section we briefly recall some well-known notions of the theory of Siegel modular forms. The reader is referred to [**1**] and [8] for the proofs of the statements and for details.

Let us first fix some notation. Let us denote by $\mathbb{H}_n$ the *Siegel upper half-plane of degree n*, that is, the set of all $n \times n$ symmetric complex matrices with positive definite imaginary part, and by $J = J_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ the symplectic involution, where E_n is the $n \times n$ identity matrix. Also, given two suitable matrices A and B, we shall write $A[B]$ for the square matrix ${}^tBAB$. A transitive biholomorphic action of the symplectic group

$$Sp(n, \mathbb{R}) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n, \mathbb{R}) \,|\, J[M] = J \right\} \tag{1}$$

is defined on $\mathbb{H}_n$. To be precise, for any $Z \in \mathbb{H}_n$ and $M \in Sp(n, \mathbb{R})$ the matrix $CZ + D$ is invertible and the action is given by

$$M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}. \tag{2}$$

Let q be a positive integer. The kernel of the mod q reduction homomorphism:

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$Sp(n, \mathbb{Z}) \to Sp(n, \mathbb{Z}/q\mathbb{Z})$ is the *principal congruence subgroup of level q and degree n*, $\Gamma_n[q] \leq Sp(n, \mathbb{Z})$. It turns out that $\Gamma_n[q] \backslash Sp(n, \mathbb{Z}) \cong Sp(n, \mathbb{Z}/q\mathbb{Z})$. A subgroup $\Gamma \leq Sp(n, \mathbb{R})$ is called a *congruence subgroup* if it contains some principal congruence subgroup as a s ubgroup of finite index. Actually, if $n > 1$ every subgroup $\Gamma \leq Sp(n, \mathbb{Z})$ of finite index is a congruence subgroup. The class of Hecke subgroups is of special interest. The *Hecke subgroup* of level q (and degree n) is

$$\Gamma_{n,0}[q] := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}) \,|\, C \equiv 0 \bmod q \right\}. \tag{3}$$

Definition 1.1. Let $\Gamma \leq Sp(n, \mathbb{Z})$ be a congruence subgroup. Consider a character v of Γ with values in a finite subgroup of $\mathbb{C}^\times$, a finite dimensional vector space $\mathcal{Z}$ over $\mathbb{C}$ and a rational representation $\varrho : GL(n, \mathbb{C}) \to GL(\mathcal{Z})$. A holomorphic function f: $\mathbb{H}_n \to \mathcal{Z}$, is called a *Siegel modular form with respect to* Γ, ϱ *and the character* v, if it satisfies the following condition

$$\begin{gathered} (f|_\varrho M)(Z) := \varrho(CZ + D)\, f(M\langle Z \rangle) = v(M)\, f(Z) \\ \forall Z \in \mathbb{H}_n,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \end{gathered} \tag{4}$$

with the extra requirement for $n = 1$ that f is holomorphic at all «cusps». That is, $\forall \alpha \in SL(2, \mathbb{Z})$ the function $f|_\varrho \alpha$ is holomorphic at $i\infty$. In other words, there are no negative powers in the power series expansion at $i\infty$ of $f|_\varrho \alpha$. For $n > 1$, by the *Koecher principle* the regularity at the cusps follows from the functional equation. If r is the largest integer such that $\varrho(G)\det(G)^{-r}$ is a polynomial representation, then $\frac{r}{2}$ is called the *weight* of f. We shall consider mainly integral weights. When r is odd «multiplier systems» play the role of characters. In any case, the Siegel modular forms defined above span finite dimensional vector spaces, which we denote by $[\Gamma, \varrho, v]$.

The existence of (non-constant) modular forms is by no means trivial, yet they arise naturally in many contexts in mathematics. Of the many possible and interesting examples we shall here describe only those that are constructed via theta series.

Main example 1.2. Let $r = 2k$ be a positive even integer. Let us consider the following data:

(1) a reduced representation (i.e. a polynomial representation not vanishing on the «$\det(A)=0$» hyper-surface) $\varrho_0 \colon GL(n, \mathbb{C}) \to GL(\mathcal{Z})$;
(2) a harmonic form with respect to ϱ_0, that is, a polynomial function $P : \mathbb{C}^{(r,n)} \to \mathcal{Z}$ such that:
   (i) $P(G\,{}^tA) = \varrho_0(A)\,P(G)\ \forall A \in GL(n, \mathbb{C})$;
   (ii) P is harmonic (i.e., $\Delta P = 0$, where Δ is the usual Laplacian operator, applied componentwise);
(3) a pair of rational *characteristics* $U, V \in \mathbb{Q}^{(r,n)}$;
(4) a positive definite quadratic form $S = {}^tS \in \mathbb{Q}^{(r,r)}$.

We define the *theta series* as the following functions on $\mathbb{H}_n$:

$$\vartheta_P^{(n)}\begin{bmatrix} U \\ V \end{bmatrix}(S, Z) := \sum_{G \in \mathbb{Z}^{(r,n)}} P(S^{\frac{1}{2}}(G+U)) \exp(\pi i \sigma(S[G+U]Z + 2\,{}^tV(G+U))).$$

Then $\vartheta_P^{(n)}\begin{bmatrix} U \\ V \end{bmatrix}(S, Z)$ are holomorphic functions and, denoting $\varrho_0 \otimes \det^{\frac{r}{2}}$ by ϱ, there exist a congruence subgroup $\Gamma \leq Sp(n, \mathbb{Z})$ and a character $v = v_S$ such that $\vartheta_P^{(n)}\begin{bmatrix} U \\ V \end{bmatrix}(S, Z) \in [\Gamma, \varrho, v_S]$.

Vice versa, under suitable assumptions on S, U, V (and P), it is possible to obtain families of modular forms with respect to the most significant subgroups of the symplectic group. An important parameter is the so-called *level* of the quadratic form. We say that a quadratic form $Q \in \mathbb{Q}^{(r,r)}$ is *even* if it is integral and its entries on the main diagonal are even. The level of an even quadratic form Q, $lev(Q)$, is the smallest positive integer N such that NQ^{-1} is again even. Explicit examples of modular forms may be produced as follows. Let q be a positive integer; given any even quadratic form S such that $lev(S) \mid q$ and $U \in \mathbb{Z}^{(r,n)}$ satisfying $SU \equiv 0 \bmod q$, then

$$\vartheta_P^{(n)}(S, Z) := \vartheta_P^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(S, Z) \in [\Gamma_{n,0}[q], \varrho, v_S],$$

and

$$\vartheta_P^{(n)}(Z, S \mid U) := \vartheta_P^{(n)}\begin{bmatrix} \dfrac{U}{q} \\ 0 \end{bmatrix}(S, Z) \in [\Gamma_n[q], \varrho, 1].$$

It is also useful to take into account the theta series $\vartheta_P^{(n)}\begin{bmatrix} 0 \\ \dfrac{V}{q} \end{bmatrix}\left(\dfrac{S}{q}, Z\right)$ for even quadratic forms S of level dividing q and integral characteristics $V = V^{(r,n)}$ such that $S^{-1}[V]$ is even and $qS^{-1}V$ is integral. Such theta series lie in the space $[\Gamma_n[q], \varrho, v_{S/q}]$, for a suitable character (multiplier) $v_{S/q}$. When q is odd, it is proved in [**6**] that their span coincides with the span of the above theta series $\vartheta_P(Z, Q | U)$ (Q even of level dividing q).

2 - Problems concerning theta series

We have seen that theta series are a powerful tool for constructing modular forms, so the following question is quite natural. Given a congruence subgroup $\Gamma \leqslant Sp(n, \mathbb{Z})$, a rational finite dimensional representation ϱ of $GL(n, \mathbb{C})$ and a multiplier v, and a non-empty set $\mathcal{T}\mathcal{H}$ of theta series contained in $[\Gamma, \varrho, v]$, is it possible to characterize the vector space $\Theta[\Gamma]_{\varrho, v}$ which is spanned by $\mathcal{T}\mathcal{H}$? This is, in rather vague terms, the so-called *basis problem.* The description of all linear relations existing among the elements of $\mathcal{T}\mathcal{H}$ is another important problem. Both issues are in general quite difficult; nevertheless, they have been considered in many variants and there are plenty of results relevant to many particular cases. Even a rough outline of these results would carry us much too far. We shall only emphasize two points which are important for our purposes. First, let us recall some results concerning *singular weights*. Given a modular form $f(Z)$ with Fourier expansion $f(Z) = \sum a(T) \exp(\pi i \sigma(TZ))$, then $f(Z)$ is called singular if

$$a(T) \neq 0 \Rightarrow \det T = 0 .$$

It can be shown that a non-zero modular form in $[\Gamma, \varrho, v]$ (with $\Gamma \leqslant Sp(n, \mathbb{R})$, and ϱ having weight $r/2$) is singular if and only if $n > r$. Moreover, spaces of singular modular forms with respect to the more significant congruence subgroups of $Sp(n, \mathbb{Z})$ have been described (mainly by E. Freitag) in terms of theta series. The general picture is that, for suitable $\Theta[\Gamma]_{\varrho, v}$,

$$n > r \Rightarrow [\Gamma, \varrho, v] = \Theta[\Gamma]_{\varrho, v}. \tag{5}$$

For instance, (under some technical hypotheses on multipliers) this holds for Hecke subgroups. It is believed that the analogous statement should also hold for principal congruence subgroups, but the only published proof actually requires the stronger condition $n > 2r$, see [8].

The second point is that some partial answers to the basis problem may be obtained by means of the adelic theory of automorphic representations. We are not able to give here a complete introduction to such a deep and vast topic. We shall just mention some facts which will serve as a motivation for the main question we are going to discuss below. A much more in-depth presentation of these facts can be found in [**14**]. Further references we could recommend in this respect are [**9**], [**11**] , and [**4**] for the basic notions. A very general *Theta Correspondence* between automorphic forms on adelic orthogonal groups $O(V)(\mathbb{A})$ and adelic symplectic groups $Sp(W)(\mathbb{A})$ may be defined via the so-called *Weil representation* of the group $O(V)(\mathbb{A}) \times Sp(W)(\mathbb{A})$. In particular, this correspondence yields, by an appropriate choice of some parameters, symplectic automorphic forms that, with the usual identification, are linear combinations of Siegel theta series. Roughly speaking, given an irreducible automorphic representation $\mathcal{C}$ on one side of the correspondence, the natural questions one has to deal with in this context are:

(i) whether $\mathcal{C}$ is in the image under the theta correspondence of a representation space of automorphic forms on the other group;

(ii) whether the image of $\mathcal{C}$ under the correspondence is zero or not.

In some cases it is possible to translate results coming from this adelic approach back to classical terms. Such a translation process usually leads to solutions to the classical basis problem in a very vague form: one gets expressions of modular forms as linear combinations of theta series belonging to very large families (relevant to quite arbitrary lattices, possibly having characteristics varying in infinite dimensional spaces, hence being modular forms with respect to «small» congruence subgroups).

The previous discussion suggests that it could be interesting to take into account a refined version of the basis problem. For a pair of congruence subgroups $\Gamma' \supset \Gamma$ and a pair of multiplier systems v' on Γ' and v on Γ such that $v'|_{\Gamma} = v$ we may ask if the following relation holds:

$$\Theta[\Gamma]_{\varrho, v} \cap [\Gamma', \varrho, v'] = \Theta[\Gamma']_{\varrho, v'} . \tag{6}$$

It is possible to deal with such a problem by introducing a suitable symmetrization map which is called the trace operator.

Definition 2.1. With the notation just introduced, the *trace of* $f \in [\Gamma, \varrho, v]$ is defined as:

$$Tr^{\Gamma, v}_{\Gamma', v'} f = \frac{1}{[\Gamma' : \Gamma]} \sum_{g \in \Gamma \backslash \Gamma'} v'(g)^{-1} f|_{\varrho} g . \tag{7}$$

It is obvious that the trace is a surjective linear map from $[\Gamma, \varrho, v]$ onto $[\Gamma', \varrho, v']$. We may also easily illustrate how the trace operator can be useful to understand if a given modular form can be expressed as a linear combination of suitable theta series. Let $f \in \Theta[\Gamma]_{\varrho, v} \cap [\Gamma', \varrho, v']$, and let $\{\vartheta_i\}_{i \in I}$ be a set of series in $\Theta[\Gamma]_{\varrho, v}$ such that f is a linear combination of these ϑ_i's. If, for every ϑ_i, we have that $Tr_{\Gamma', v'}^{\Gamma, v} \vartheta_i$ can be expressed as a linear combination of theta series $\theta_j \in [\Gamma', \varrho, v']$, then, clearly, there will be such an expression for $f = Tr_{\Gamma', v'}^{\Gamma, v} f$, too.

The problem for general pairs of subgroups seems to be rather hard to handle. In fact, we shall address our attention to the cases which are of special interest, namely, the cases when the involved congruence subgroups are either principal congruence subgroups or Hecke subgroups. Then, several different strategies can be used to study the action of the trace operator on theta series. For example, the effect of the trace operator can be computed explictly by using either the p-adic theory of lattices or an adelic method based on the interpretation of the trace operator as an integral operator (convolution). In fact, these two approaches are used for the case of pairs of Hecke subgroups of different level in the first part of [**3**] and in [**10**], respectively. We wish to describe in some detail a different technique, namely the one applied in the second part of [**3**] (see also [**2**]). This technique can be considered as an application of the theory of singular modular forms. Indeed, it is based on the simple remark that, keeping in mind (5), if $n > r$ and $\vartheta \in \Theta[\Gamma]_{\varrho, v}$ then $Tr_{\Gamma', v'}^{\Gamma, v}(\vartheta) \in \Theta[\Gamma']_{\varrho, v'}$. Thus, it would be useful to reduce the problem to the singular case. In order to explain how this can be accomplished, it is necessary to introduce the so-called *Siegel Φ operator*. For a function $f: \mathbb{H}_n \to \mathcal{Z}$, we define

$$\Phi(f): \quad \mathbb{H}_{n-1} \to \mathcal{Z}$$

$$\Phi(f)(Z) := \quad \lim_{t \to \infty} f\begin{pmatrix} it & 0 \\ 0 & Z \end{pmatrix}$$

whenever the limit exists. Siegel operator can be applied termwise on Siegel modular forms and more generally on functions f having a Fourier expansion of the type:

$$f(Z) = \sum_{T \in L^*} a(T) \exp(2\pi i \sigma(TZ)) \qquad (Z \in \mathbb{H}_n)$$

where L is some rational lattice of symmetric $n \times n$ matrices, $L^* := \{T = {}^tT \mid \sigma(TX) \in \mathbb{Z}\, \forall X \in L\}$, and $a(T) \neq 0$ only if T is semi-positive definite. To be precise, we obtain

$$\Phi(f)(Z) = \sum_{\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix} \in L^*} a\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix} \exp(2\pi i \sigma(TZ)). \tag{8}$$

By applying the Siegel operator on spaces of modular forms $[\Gamma, \varrho, v]$, where Γ has degree n, we get maps $\Phi: [\Gamma, \varrho, v] \to [\Gamma|_\Phi, \varrho|_\Phi, v|_\Phi]$. We refer to [8] for the general definitions of the congruence subgroup $\Gamma|_{\Phi} \leqslant Sp(n-1, \mathbb{R})$, the representation $\varrho|_{\Phi}$, and the multiplier $v|_{\Phi}$. Here we would like just to notice that $\Gamma_n[q]|_{\Phi} = \Gamma_{n-1}[q]$, and $\Gamma_{n,0}[q]|_{\Phi} = \Gamma_{n-1,0}[q]$.

Besides, theta series are stable under the action of the Siegel operator. In fact, for $U = (\overbrace{u_1}^{1}\,\overbrace{U_2}^{n-1})$, $V = (\overbrace{v_1}^{1}\,\overbrace{V_2}^{n-1})$, and $P_0(X) := P(0, X)$, $\Phi(\vartheta_P^{(n)}\begin{bmatrix} U \\ V \end{bmatrix}(S)) = \vartheta_{P_0}^{(n-1)}\begin{bmatrix} U_2 \\ V_2 \end{bmatrix}(S)$ if $u_1 = 0$ and it vanishes otherwise.

Next, the key step for the reduction of the problem to the singular case consists in establishing commutation formulas between the trace operator and the Siegel operator. Let $\Gamma_{(n)}$ and $\Gamma'_{(n)}$ form the pair of subgroups we are to consider. Suppose that we could prove formulas like

$$\Phi \circ Tr^{\Gamma_{(n+1)},\, v}_{\Gamma'_{(n+1)},\, v'}(\vartheta^{(n+1)}) = Tr^{\Gamma_{(n)},\, v}_{\Gamma'_{(n)},\, v'} \circ \Phi(\vartheta^{(n+1)}) \tag{9}$$

for suitable $\vartheta^{(n+1)}$ such that $\Phi(\vartheta^{(n+1)}) = \vartheta^{(n)}$. Then, we could repeat the procedure many times and finally reach the singular case, obtaining, for $k \gg 0$,

$$Tr^{\Gamma_{(n)},\, v}_{\Gamma'_{(n)},\, v'}\vartheta^{(n)} = \Phi^k \circ Tr^{\Gamma_{(n+k)},\, v}_{\Gamma'_{(n+k)},\, v'}(\vartheta^{(n+k)}) = \Phi^k\Big(\sum_{i \in I_{n+k}} c_i\,\theta_i^{(n+k)}\Big) = \sum_{i \in I_n} c_i\,\theta_i^{(n)}$$

for «convenient» families of theta series $\theta_i^{(n+k)} \in \Theta[\Gamma'_{(n+k)}]_{\varrho,\, v'}$.

The idea of investigating commutation properties of symmetrizations and Siegel operator is due to Salvati Manni, [**13**], who used this technique for the study of the surjectivity of the Siegel operator on spaces of particular theta series known as thetanullwerte.

Unfortunately, simple commutation formulas like (9) do not hold in general. Anyway, once we have a «good» (i.e. compatible with the Siegel operator) set of representatives for the cosets in $\Gamma\backslash\Gamma'$, we can always begin to evaluate the expression

$$\sum_{g \in \Gamma_{(n+1)}\backslash\Gamma'_{(n+1)}} v'(g)^{-1}\,\Phi(\vartheta^{(n+1)}|_{\varrho}\, g) = \ldots \tag{10}$$

trying to recognize in the r.h.s. above summands like $Tr^{\Gamma_{(n)},\, v}_{\Gamma'_{(n)},\, v'}\vartheta^{(n)}$. Then, evaluating (10) yields an expression of the following kind:

$$\Phi \circ Tr^{\Gamma_{(n+1)},\, v}_{\Gamma'_{(n+1)},\, v'}(\vartheta^{(n+1)}) = C \cdot Tr^{\Gamma_{(n)},\, v}_{\Gamma'_{(n)},\, v'}(\vartheta^{(n)}) + R$$

where C is some complex number depending on the involved parameters (levels, degree, quadratic form, etc.) and R is a remainder term of the type

$R = \sum_j Tr^{\Gamma_{(n)}, v}_{\Gamma'_{(n)}, v'}(\vartheta^{(n)}_j)$. It is usually possible to control these remainder terms and prove that they lie in $\Theta[\Gamma'_{(n)}]_{\varrho, v'}$ by means of some inductive argument on the «rank» of the characteristic of $\vartheta^{(n)}$. The constant C may vanish, giving rise to possible obstructions to our procedure. However, such vanishings of C have quite natural interpretations. A delicate situation, revealed by this kind of obstructions, occurs when we try to compute the trace of a theta series whose associated character does not agree with the fixed character v' on Γ'. This is at least quite meaningful from the point of view of theta-liftings since that machinery usually produces combinations of theta series attached to lattices belonging to a fixed quadratic space, hence having a fixed discriminant and associated character. Another difficulty which is detected by these obstructions appears when one wants to get rid of a prime in the level. In that case the constant has a factor of the type: $1 + s_p(\mathbb{Q}^r)\,\det_p(S)^{-\frac{1}{2}}\,p^{\lambda}$. In the last expression $s_p(\mathbb{Q}^r)$ denotes the Witt invariant, normalized as in [**12**], of the quadratic space $\mathbb{Q}^r_p$ (with the quadratic form induced by the fixed rational quadratic space), $\det_p(S)$ is the p-power occurring in $\det(S)$ and λ is a positive integer. It has been pointed out in [**3**] that, if the ambient quadratic space $\mathbb{Q}^r$ contains even lattices of level q, the condition $s_p(\mathbb{Q}^r) = 1$ is equivalent to the fact that $\mathbb{Q}^r$ carries even lattices of level N. Therefore, keeping in mind the equivalence between lattices and quadratic forms, if $s_p(\mathbb{Q}^r) \neq 1$, the «not-vanishing» condition implies the vanishing of the trace. On the other hand, if even lattices of level N do exist inside the given quadratic space $\mathbb{Q}^r$, then a relation like (6) holds.

We may now state in precise terms the results we have proved in [**5**] and [**6**] by using the strategy sketched above. The reader is referred to [**3**] and [**10**] for the analogous results relevant to pairs of Hecke subgroups. We keep the same notation used throughout the paper, that is, we let $r = 2k$, k a positive integer and ϱ_0 a rational reduced representation of $GL(n, \mathbb{C})$, $\varrho = \varrho_0 \otimes \det^{\frac{r}{2}}$.

Theorem 2.2. *Let* $\Theta[\Gamma_{n,0}[q]]_{\varrho, v_q} \subset [\Gamma_{n,0}[q], \varrho, \chi_q]$ *be the vector space spanned by the theta series of type* $\vartheta^{(n)}_P(Q, Z)$ *and let* $\Theta[\Gamma_n[q]]_{\varrho, \chi_q} \subset \left[\Gamma_n[q], \varrho, v_{\frac{S}{q}}\right]$ *be the vector space spanned by the theta series of the type* $\vartheta^{(n)}_P\begin{bmatrix} 0 \\ \frac{V}{q} \end{bmatrix}\left(\frac{S}{q}, Z\right)$ *(with* χ_q *the Dirichlet character* mod q *that extends* v_S*), then*:

$$Tr^{\Gamma_n[q],\, v_{\frac{S}{q}}}_{\Gamma_{n,0}[q],\, \chi_q}\, \vartheta^{(n)}_P\begin{bmatrix} 0 \\ \frac{V}{q} \end{bmatrix}\left(\frac{S}{q}, Z\right) \in \Theta[\Gamma_{n,0}[q]]_{\varrho, \chi_q}.$$

Therefore:

$$\Theta[\Gamma_n[q]]_{\varrho,\chi_q} \cap [\Gamma_{n,0}[q], \varrho, \chi_q] = \Theta[\Gamma_{n,0}[q]]_{\varrho,\chi_q}.$$

Theorem 2.3. *Let q be an odd integer $q = N \cdot p$, with p prime such that $p \mid N$ and let $\Theta[\Gamma_n[N]]_\varrho \subset [\Gamma_n[N], \varrho, 1]$ be the vector space spanned by the theta series of weight k of type $\vartheta_P^{(n)}\begin{bmatrix} 0 \\ \dfrac{V}{N} \end{bmatrix}\left(\dfrac{S}{N}, Z\right)$. Similarly, let $\vartheta_P^{(n)}\begin{bmatrix} 0 \\ \dfrac{V}{q} \end{bmatrix}\left(\dfrac{S}{q}, Z\right)$ be a theta series in $\Theta[\Gamma_n[q]]_\varrho \subset [\Gamma_n[q], \varrho, 1]$, then*

$$Tr_{\Gamma_n[N],1}^{\Gamma_n[q],1} \vartheta_P^{(n)}\begin{bmatrix} 0 \\ \dfrac{V}{q} \end{bmatrix}\left(\dfrac{S}{q}, Z\right) \in \Theta[\Gamma_n[N]]_\varrho.$$

Therefore:

$$\Theta[\Gamma_n[q]]_\varrho \cap [\Gamma_n[N], \varrho, 1] = \Theta[\Gamma_n[N]]_\varrho.$$

Theorem 2.4. *Let q be an odd integer $q = N \cdot p$ with p prime and $(N, p) = 1$. Let $\Theta[\Gamma_n[N]]_\varrho \subset [\Gamma_n[N], \varrho, 1]$ be the vector space spanned by theta series of type $\vartheta_P^{(n)}(Z, Q \mid T)$. Furthermore, let us consider $\vartheta_P^{(n)}(Z, S \mid T) \in [\Gamma_n[q], \varrho, 1]$ such that $v_S \mid_{\Gamma^0_{n,0}[q] \cap \Gamma_n[N]} = 1$. ($\Gamma^0_{n,0}[q]$ is the subgroup of $\Gamma_{n,0}[q]$ consisting of the matrices with the upper-right part $B \equiv 0 \bmod q$).*

The trace of $\vartheta_P^{(n)}(Z, S \mid T)$ vanishes of if $S[T]$ is not 0 modulo qp. Moreover, if we denote by ι the rank of $T \bmod p$, the following condition

$$1 + s_p(\mathbb{Q}^r) \det_p(S)^{-\frac{1}{2}} p^{\xi+1} \neq 0 \quad \forall n \leqslant \xi \leqslant r + \iota,$$

implies that

$$Tr_{\Gamma_n[N],1}^{\Gamma_n[q],1} \vartheta_P^{(n)}(Z, S \mid T) \in \Theta[\Gamma_n[N]]_\varrho.$$

Remark 2.5. For the sake of simplicity, the half-integral case was not included in the above statements. Basically, in order to extend the previous results to quadratic forms Q in an odd number of variables one has to consider the companion quadratic forms $Q \perp (2)$.

Of course, it would be nice to have an overall result including even levels for the principal congruence subgroups. Some investigations we carried out suggest that the general picture is not different from the odd level one (see [**7**]).

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Abstract

We give an elementary introduction to theta series from the point of view of the theory of Siegel modular forms and we discuss some results concerning the expression of modular forms for congruence subgroups as linear combinations of theta series of the appropriate level.

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