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**On certain positive integer sequences (\*\*)**

**1 - Introduction**

The aim of this paper is to survey some topics in elementary number theory. The contents are based on the talk given in Parma at the occasion of the Second Italian Meeting of Number Theory in november 2003. All these topics are of arithmetical nature and, as is often the case, no special knowledge is required. Still today improvements and new developments are possible, so this makes this paper suitable for students who wish to begin or to continue their studies in elementary number theory.

Each section contains a problem involving a suitable sequence of integers or family of sequences with a short review of known results, open problems, and some new results.

Section 2 is consecrated to practical numbers, i.e., those numbers  $m$  such that the set of all distinct positive divisors sums contains all positive integers not exceeding  $m$ . Section 3 concerns sum-free sequences, i.e., increasing sequences of positive integers such that each term is never a sum of distinct preceding terms. Section 4 presents some results on density of certain sets whose elements are sums of distinct powers of positive integers. We provide a counterexample to a conjecture of Burr, Erdős, Graham and Wen-Ching Li [2]. Section 5 deals with certain families of positive integer sequences whose digital expansion of elements is suitably related to the digital expansion of their powers.

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**2 - Practical numbers**

A positive integer  $m$  is a practical number if every positive integer  $n < m$  is a sum of distinct positive divisors of  $m$ .

Let  $P(x)$  be the counting function of practical numbers and let  $P_2(x)$  be the function that counts practical numbers  $m \leq x$  such that  $m + 2$  is also a practical number. Stewart [18] proved that a positive integer  $m \geq 2$ ,  $m = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_k^{\alpha_k}$ , with primes  $q_1 < q_2 < \dots < q_k$  and integers  $\alpha_i \geq 1$ , is practical if and only if  $q_1 = 2$  and, for  $i = 2, 3, \dots, k$ ,

$$q_i \leq \sigma(q_1^{\alpha_1} q_2^{\alpha_2} \dots q_{i-1}^{\alpha_{i-1}}) + 1,$$

where  $\sigma(n)$  denotes the sum of the positive divisors of  $n$ . A wide survey of results and conjectures on practical numbers is given by Margenstern [9].

Practical numbers appear to be a prime-like sequence. Saias [15], using suitable sieve methods introduced by Tenenbaum provided a good estimate in terms of a Chebishev-type theorem:

*Theorem 1. For suitable constants  $c_1$  and  $c_2$ ,*

$$c_1 \frac{x}{\log x} < P(x) < c_2 \frac{x}{\log x}.$$

The author [10] solved the Golbach problem by proving that every even positive integer is a sum of two practical numbers. The proof used an auxiliary increasing sequence  $m_n$  of practical numbers such that for every  $n$ ,  $m_n + 2$  is also a practical number and  $m_{n+1}/m_n$  bounded by an absolute constant, and a corollary of Stewart's theorem, namely if  $m$  is a practical number and  $n \leq 2m$ , then  $mn$  is a practical number too. Every pair of twin practical numbers yields a suitable interval of even numbers expressible as a sum of two practical numbers and intervals overlap.

A local property which does not appear in primes holds for practical numbers: there exist infinitely many practical numbers  $m$  such that both  $m - 2$  and  $m + 2$  are practical, as one can check by taking  $m = 2 \cdot 3^{3^k \cdot 70}$ , for  $k \in \mathbb{N}$ .

Further, infinitely many 5-tuples of practical numbers of the form  $m - 6, m - 2, m, m + 2, m + 6$  exist under a suitable but still unproved hypothesis [13].

Twenty years ago Margenstern conjectured that for suitable  $\lambda_1$  and  $\lambda_2 > 0$   $P(x) \sim \lambda_1 \frac{x}{\log x}$  and  $P_2(x) \sim \lambda_2 \frac{x}{(\log x)^2}$ . He empirically proposed  $\lambda_1 \approx 1.341$  and  $\lambda_2 \approx 1.436$ . Such conjectures appear far to be proved. However it should be inte-

resting to prove that  $\lim_{x \rightarrow \infty} P(2x)/P(x) = 2$ , a somewhat weaker result conjectured by Erdős [5].

Concerning the counting function of twin practical numbers a recent result of the author [12] is the following

**Theorem 2.** *Let  $k > 2 + \log(3/2)$ . For sufficiently large  $x$ ,*

$$P_2(x) > \frac{x}{\exp \{k(\log x)^{1/2}\}} .$$

In particular this implies that for every  $\alpha < 1$ ,  $P_2(x) \gg x^\alpha$ . The proof uses the fact that if  $m_1$  and  $m_2$  are two practical numbers, with  $0.5 < m_1/m_2 < 2$ , and with  $\gcd\{m_1, m_2\} = 2$ , there exist  $r$  and  $s$ , not exceeding respectively  $2m_1$  and  $2m_2$ , such that  $m_1 r$  and  $m_2 s$  are a pair of twin practical numbers. It is possible to build  $m_1$  and  $m_2$  by picking the primes of their factorization in suitable intervals in order to control their mutual size. One has to count all possible pairs and divide by the maximal number of repetitions.

Many other open problems on practical numbers and related questions have been raised by Erdős in [6].

### 3 - Sum-free sequences

An increasing sequence of positive integers  $\{n_1, n_2, \dots\}$  is called a sum-free sequence if each term is never a sum of distinct smaller terms. This definition is due to Erdős [4] who proved certain related results and raised several problems. In his paper he proved that if  $\{n_k\}$  is a sum-free sequence then it has zero asymptotic density. In other words, for every  $\varepsilon > 0$ , and for sufficiently large  $k$ ,  $n_k > k^{1+\varepsilon}$ . By the same argument he proved that for every  $\beta < (\sqrt{5} + 1)/2$ , for infinitely many  $k$ ,  $n_k > k^\beta$ .

Until 1996, all known sum-free sequences had a gap, namely

$$\limsup_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} > 1 .$$

In [3], Deshouillers, Erdős and the author gave some examples of infinite sum-free sequences with no gap. They also proved that for every positive  $\delta$ , there exists a sum-free sequence  $\{n_k\}$  such that  $n_k \sim k^{3+\delta}$ .

The best extremal result concerning sum-free sequences is due to Łuczak and Schoen [8]. They proved that for every positive  $\delta$ , there exists a sum-free sequence  $\{n_k\}$  such that  $n_k \sim k^{2+\delta}$  and that the exponent 2 is the best possible.

However an interesting problem remains open. Erdős proved that for any sum-free sequence  $\{n_k\}$  one has

$$\sum_{j=1}^{\infty} \frac{1}{n_j} < 103.$$

It is natural to define

$$R = \sup_{\{n_k\} \text{ sum-free}} \left\{ \sum_{j=1}^{\infty} \frac{1}{n_j} \right\}.$$

The best known lower bound is due to Abbott [1] who proved that  $R > 2.064$ . Levine and O'Sullivan [7] proved the actual best known upper bound  $R < 4$ .

#### 4 - Sums of distinct powers

A sequence  $S = \{s_1, s_2, \dots\}$  of positive integers is a *complete* sequence, if

$$\Sigma(S) := \left\{ \sum_{i=1}^{\infty} \varepsilon_i s_i, \text{ for } \varepsilon_i \in \{0, 1\}, \sum_{i=1}^{\infty} \varepsilon_i < \infty \right\}$$

contains all sufficiently large integers. In this section we deal with sequences  $S$  whose elements are powers of positive integers. Let  $s \geq 1$  and  $A$  be a (finite or infinite) set of integers greater than 1. Let  $\text{Pow}(A; s)$  be the nondecreasing sequence of positive integers of the form  $a^k$  with  $a \in A$  and  $k \geq s$ .

Burr, Erdős, Graham and Wen-Ching Li [2] proved several results providing sufficient conditions in order that  $\text{Pow}(A; s)$  is complete. They also conjectured that for any  $s \geq 1$ ,  $\text{Pow}(A; s)$  is complete if and only if

- (i)  $\sum_{a \in A} 1/(a-1) \geq 1$ ,
- (ii)  $\gcd\{a \in A\} = 1$ .

The following proposition disproves the «only if» part of their conjecture. We provide counterexamples using suitable infinite sets  $A$ . However for finite sets the problem is open.

**Proposition 1.** *Let  $\varepsilon > 0$ . There exists a set  $A$  of integers  $\geq 2$  such that:*

- (i)  $\sum_{a \in A} 1/(a-1) < \varepsilon$ ,
- (ii) *for every  $s \geq 1$ ,  $\text{Pow}(A; s)$  is complete.*

**Proof.** Let  $p \geq 3$  be a prime, and let  $R_p := \{n^2 p, n \in \mathbb{N}\}$ . We have that

$\text{Pow}(R_p; s) \supseteq \{n^{2s}p^s, n \in \mathbb{N}\}$ . Since every sufficiently large integer is a sum of distinct  $2s$ -th powers of positive integers [17], there exists  $r_{2s}$  such that

$$\Sigma(\text{Pow}(R_p; s)) \supseteq \{np^s, n > r_{2s}\}.$$

If  $Q_p$  is such that  $Q_p \cap R_p = \emptyset$ , we get  $\Sigma(\text{Pow}(R_p \cup Q_p; s)) = \Sigma(\text{Pow}(R_p; s)) + \Sigma(\text{Pow}(Q_p; s))$ . Since  $\Sigma(\text{Pow}(R_p; s))$  contains all sufficiently large multiples of  $p^s$ , in order that  $\text{Pow}(R_p \cup Q_p; s)$  is complete, it suffices to provide a set  $Q_p$  such that  $\Sigma(\text{Pow}(Q_p; s))$  contains at least one element from each congruence class modulo  $p^s$ . Let  $Q_p = \{p + 1\}$ . It is clear that  $\text{Pow}(Q_p; s)$  contains infinitely many elements  $\equiv 1 \pmod{p^s}$ , so  $\Sigma(\text{Pow}(Q_p; s))$  contains at least one element from each congruence class modulo  $p^s$ .

By the above arguments this implies that for  $A := R_p \cup Q_p$ ,  $\text{Pow}(A; s)$  is complete. Note that the elements of  $A$  do not depend on  $s$ , and that for sufficiently large  $p$ ,

$$\sum_{a \in A} \frac{1}{a-1} = \frac{1}{p} + \sum_{n=1}^{\infty} \frac{1}{n^2p-1} < \varepsilon. \quad \blacksquare$$

A related open question is the following. Consider the sequence  $\{n_k\}$  of positive integers that are a sum of distinct powers of 3 and of 4. Erdős asked for a proof that  $n_k \ll k$ . The best known result is  $n_k \ll k^{1.0353}$  as shown in [11].

More generally we propose the following conjecture.

**Conjecture 1.** *Let  $s \geq 1$  and let  $A$  be a sequence of integers  $\geq 2$ . If for every  $a_1, a_2 \in A$ ,  $\gcd\{a_1, a_2\} = 1$  and  $\sum_{a \in A} \frac{1}{\log a} > \log 2$  then  $\Sigma(\text{Pow}(A; s))$  has a positive lower asymptotic density.*

Note that if we replace the condition «for every  $a_1, a_2 \in A$ ,  $\gcd\{a_1, a_2\} = 1$ » by « $\gcd\{a \in A\} = 1$ » the statement is not true. For the set  $A = \{3, 9, 81, 104\}$ , we have  $\gcd\{a \in A\} = 1$ ;  $\sum_{a \in A} \frac{1}{\log a} > \log 2$ , but  $\Sigma(\text{Pow}(A; s))$  has zero lower asymptotic density [11].

### 5 - Simultaneous binary expansions

Let  $B(n)$  be the sum of digits of the positive integer  $n$  written on base 2. A natural question of some interest is to describe the sequence of positive integers  $n$  such that  $B(n) = B(n^2)$ . Other related questions can be easily raised. For example it can be of interest to study the sequence of the positive integers  $n$  such

that  $2B(n) = B(n^2)$ , i.e., those  $n$  such that  $n$  and  $n^2$  have the same «density» of ones in their binary expansion.

**Definition 1.** *Let  $k \geq 2, l \geq 1, m \geq 2$  be positive integers. We say that a positive integer  $n$  is a  $(k, l, m)$ -number if the sum of digits of  $n^m$  in its expansion in base  $k$  is  $l$  times the sum of the digits of the expansion in base  $k$  of  $n$ .*

The above sequences respectively represent the  $(2, 1, 2)$ -numbers and the  $(2, 2, 2)$ -numbers.

The simplest case is  $(k, l, m) = (2, 1, 2)$ , which corresponds to the positive integers  $n$  for which the numbers of ones in their binary expansion is equal to the number of ones in  $n^2$ .

The list of  $(2, 1, 2)$ -numbers as well as the list of  $(2, 2, 2)$ -numbers shows several interesting facts. The distribution is not regular. A huge amount of questions, most of which of elementary nature, can be raised.

In spite of their elementary definition, as far as we know these sequences do not appear in literature. Several questions, concerning both the structure properties and asymptotic behaviour, can be raised. Is there a necessary and sufficient condition to assure that a number is of type  $(2, 1, 2)$ ? of type  $(2, 2, 2)$ ? What is the asymptotic behaviour of the counting function of  $(2, 1, 2)$ -numbers? of  $(2, 2, 2)$ -numbers?

The irregularity of distribution does not suggest a clear answer to these questions.

Let  $p_{(k, l, m)}(n)$  be the number of  $(k, l, m)$ -numbers which do not exceed  $n$ . By elementary arguments one can prove the following.

**Theorem 3.** *Let  $p_{(2, 1, 2)}(n)$  be the counting function of the  $(2, 1, 2)$ -numbers. We have*

$$p_{(2, 1, 2)}(n) \gg n^{0.025}.$$

The proof uses the fact that for every  $n$  it is possible to construct a set of  $n$  distinct  $(2, 1, 2)$ -numbers not exceeding  $An^{40}$ . To do this, one uses the fact that for every  $n < 2^v$ ,  $B(n(2^v - 1)) = v$ . The construction uses an arbitrary number not exceeding  $n$ : by adding in its binary expansion a suitable finite sequence of zeros and ones, with a special attention for the control of the function  $B$  for simultaneously the new number and its square, one obtains a  $(2, 1, 2)$ -number not exceeding  $An^{40}$ . Further details can be found in [14].

By an analogous procedure it is possible to prove the following theorem.

**Theorem 4.** *Let  $p_{(2,2,2)}(n)$  be the counting function of the  $(2, 2, 2)$ -numbers. We have*

$$p_{(2,2,2)}(n) \gg n^{0.0909}.$$

Concerning upper bounds for counting functions, Sándor [16] recently announced that

$$p_{(2,1,2)}(n) \ll n^{0.9183}.$$

Some conjectures on counting functions can be proposed. Apart from small intervals centered on powers of 2, the binary expansions of  $n$  and  $n^2$  appear as random and uncorrelated sequences of zeros and ones. Using this appearance and considering  $B(n)$  and  $B(n^2)$  as independent random variables, a heuristic approach suggests the following conjectures.

**Conjecture 2.**

$$p_{(2,1,2)}(n) = n^{\alpha + o(1)}$$

where  $\alpha = \log 1.6875 / \log 2 \approx 0.7548875$ .

**Conjecture 3.** *For each  $k$  one has:*

$$p_{(2,k,k)}(n) = \frac{n}{(\log n)^{1/2}} G_k + R(n),$$

where  $G_k = \sqrt{\frac{2 \log 2}{\pi(k^2 + k)}}$  and  $R(n) = o(n/(\log n)^{1/2})$ .

A detailed discussion of the above conjectures can be found in [14]. Computations show that the above conjectures describe quite well the behaviour of counting functions for  $n < 10^8$ .

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#### Abstract

*A survey of recent results in elementary number theory is presented in this paper. Special attention is given to structure and asymptotic properties of certain families of positive integers. In particular, a conjecture on complete sequences of Burr, Erdős, Graham and Wen-Ching Li is disproved.*

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