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Two trigonometric matrices (**)

The present work is essentially a shortened version of a wider paper [**9**]; we refer to that paper for the proofs not included here.

1 - Notations

The symbol $\delta_{something}$ assumes the value 1 when *something* holds, 0 otherwise. For $m, n, s \in \mathbb{N}$, (m, n) is the greatest common divisor of m, n; $m|n$ means that m divides n; $m \equiv n(s)$ and $m = n \bmod (s)$ mean that $s|(m-n)$; n is said *square-free* when $p^2 \nmid n$ for every prime p; $\left(\frac{m}{n}\right)$ is the Jacobi symbol, i.e., the completely multiplicative extension of the quadratic character for odd m, n; $\psi := \mu * \phi$ is the Dirichlet convolution of the Möbius and the Euler functions, i.e., $\psi(n) = \sum_{d|n} \mu(n/d)\phi(d)$. At last, we denote by $\operatorname{Im} x$ the imaginary part of x.

2 - Introduction and motivations

Schur introduced the matrix

$$\Phi := \left[\exp\left(\frac{2\pi imn}{s}\right)\right]_{0 \leq m,\, n < s}$$

where s is a positive integer. Since $\Phi^4 = s^2 \mathbb{I}$, the eigenvalues are the numbers $i^\nu \sqrt{s}$ for $0 \leq \nu \leq 3$. Schur has determined the multiplicity of every eigenvalue and

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used such result to evaluate the Gaussian sum $\sum_{n=1}^{s} e^{2\pi i n^2/s}$ which is the trace of Φ (see [**12**]). Such matrix represents also the discrete Fourier transform on s points, therefore it is extensively studied in Approximation Theory and Numerical Analysis. In particular, there exists a broad literature about its eigenvalues and eigenvectors (see [**2**], [**4**], [**5**], [**8**], [**10**]). In this paper we study the four matrices

$$M_{r,s} := \sqrt{2}\left[\sin\left(\frac{rmn\pi}{s}\right)\right]_{0<m,\,n<s}, \qquad M_{2r,s} := \left[\sin\left(\frac{2rmn\pi}{s}\right)\right]_{0<m,\,n<s},$$

$$M'_{r,s} := \sqrt{2}\left[\sin\left(\frac{rmn\pi}{s}\right)\right]_{\substack{0<m,\,n<s\\(mn,\,s)=1}}, \qquad M'_{2r,s} := \left[\sin\left(\frac{2rmn\pi}{s}\right)\right]_{\substack{0<m,\,n<s\\(mn,\,s)=1}},$$

where r, s are odd integers with $(r, s) = 1$ and $s > 1$. In particular we found their eigenvalues, multiplicity included. In this way also their characteristic polynomials are determined since these matrices are symmetric. The normalizating factor $\sqrt{2}$ appearing in the definition of $M_{r,s}$ and $M'_{r,s}$ is introduced by convenience. Moreover, we note that $M_{2(r+s),s} = M_{2r,s}$ and $M'_{2(r+s),s} = M'_{2r,s}$, hence the fact that r is odd is not a true restriction but only a convenient assumption simplifying the proof of the results.

Matrices $M_{\cdot,s}$ and $M'_{\cdot,s}$ are evidently related to Φ, but only $M_{r,s}$, satisfying the identity $M^2_{r,s} = s\mathbb{I}$, has a behavior like that one of Φ. The following examples show that the structure of the characteristic polynomial of the other matrices is strongly influenced by the arithmetical properties of the parameters r and s: in all cases the eigenvalues are 0 and $\pm\sqrt{d}$ where d is a divisor of s but for non-square-free s not every divisor appears and the rule selecting the eigenvalues and their multiplicity is not evident.

$$\begin{aligned}
&\det(x\mathbb{I} - M_{2,7}) = x^3(x^2-7)(x-\sqrt{7})\\
&\det(x\mathbb{I} - M_{6,7}) = x^3(x^2-7)(x+\sqrt{7})\\
&\det(x\mathbb{I} - M_{2,15}) = x^7(x^2-15)^3(x-\sqrt{15})\\
&\det(x\mathbb{I} - M_{14,15}) = x^7(x^2-15)^3(x+\sqrt{15})\\
&\det(x\mathbb{I} - M'_{2,7}) = x^3(x^2-7)(x-\sqrt{7})\\
&\det(x\mathbb{I} - M'_{6,7}) = x^3(x^2-7)(x+\sqrt{7})\\
&\det(x\mathbb{I} - M'_{2,15}) = x^4(x-\sqrt{3})(x^2-5)(x-\sqrt{15})\\
&\det(x\mathbb{I} - M'_{14,15}) = x^4(x-\sqrt{3})(x^2-5)(x+\sqrt{15}),
\end{aligned}$$

s	$\det(x\mathbb{I} - M'_{1,s})$
$3\cdot 5$	$(x^2-3)(x^2-5)^2(x^2-3\cdot 5)$
$5\cdot 11$	$(x^2-5)^2(x^2-11)^5(x^2-5\cdot 11)^{13}$
5^2	$x^4(x^2-5^2)^8$
5^3	$x^{20}(x^2-5^3)^{40}$
7^2	$x^6(x^2-7^2)^{18}$
7^3	$x^{42}(x^2-7^3)^{126}$
11^2	$x^{20}(x^2-11^2)^{50}$
11^3	$x^{110}(x^2-11^3)^{550}$
$3^2\cdot 5$	$x^8(x^2-3^2)^2(x^2-3^2\cdot 5)^6$
$3^3\cdot 5$	$x^{24}(x^2-3^3)^6(x^2-3^3\cdot 5)^{18}$
$3\cdot 5^2$	$x^8(x^2-5^2)^8(x^2-3\cdot 5^2)^8$
$3^2\cdot 5^2$	$x^{56}(x^2-3^2\cdot 5^2)^{32}$
$3^3\cdot 5^2$	$x^{168}(x^2-3^3\cdot 5^2)^{96}$
$3^2\cdot 7$	$x^{12}(x^2-3^2)^2(x^2-3^2\cdot 7)^{10}$
$3^3\cdot 7$	$x^{36}(x^2-3^3)^6(x^2-3^3\cdot 7)^{30}$
$3\cdot 7^2$	$x^{12}(x^2-7^2)^{18}(x^2-3\cdot 7^2)^{18}$
$3^2\cdot 7^2$	$x^{108}(x^2-3^2\cdot 7^2)^{72}$
$3^2\cdot 5\cdot 7$	$x^{48}(x^2-3^2)^2(x^2-3^2\cdot 5)^6(x^2-3^2\cdot 7)^{10}(x^2-3^2\cdot 5\cdot 7)^{30}$.

In Section 3 we will provide some non-trivial preparatory results belonging to the Number Theory, in Section 4 we will prove Theorem 1 giving the characteristic polynomials of the matrices $M_{\cdot,s}$, in Section 5 we will prove Theorems 2-3 giving the characteristic polynomials of the matrices $M'_{\cdot,s}$. Some interesting corollaries are there proved, too.

At last, a word about the origin of our interest for these matrices. Let m, s be integers, $s>1$, s odd and $0<m<s$. For every $a\in\mathbb{N}$ let

$$H^a_{m,s} := \sum_{n=1}^{\infty} \frac{r_{m,s}(n)}{n^{2a+1}}, \quad \text{where} \quad r_{m,s}(n) := \begin{cases} 1 & \text{if } n = m \bmod (2s) \\ -1 & \text{if } n = -m \bmod (2s) \\ 0 & \text{otherwise.} \end{cases}$$

(The series converges conditionally also when $a=0$.) We are looking for a formula giving the value of $H^a_{m,s}$. Let

$$F_a(x) := \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\pi)^{2a+1}} \sin(n\pi x),$$

uniformly convergent on every compact subset of $(-1, 1)$, for every $a\in\mathbb{N}$. A comparison with the known Fourier expansion of the Bernoulli polynomials $B_k(x)$ (see [**13**], Ch. 1.0) shows that for $x \in (-1, 1)$

$$F_a(x) = \frac{(-4)^a}{(2a+1)!} B_{2a+1}\left(\frac{1-x}{2}\right). \tag{1}$$

Since the Bernoulli polynomials can be easily recovered by the identity

$$\sum_{k=0}^{\infty} \frac{B_k(x)}{k!} y^k = \frac{ye^{xy}}{e^y - 1},$$

the values of F_a can be easily calculated. The relevance of F_a in this context comes from the fact that from the definition of $H^a_{m,s}$ we have

$$\sum_{m=1}^{s-1} (-1)^m H^a_{m,s} \sin\left(\frac{mn\pi}{s}\right) = \pi^{2a+1} F_a\left(\frac{n}{s}\right) \qquad \forall n \in \mathbb{Z}, \tag{2}$$

so that by taking $0 < n < s$, we recover a set of $s-1$ linear equations for the $s-1$ numbers $H^a_{m,s}$, with $0 < m < s$.

A second identity can be deduced noting that $H^a_{dm,ds} = d^{-2a-1} H^a_{m,s}$ for every integer d, so that from (2) we have

$$\sum_{d|s} \sum_{\substack{m=1 \\ (m,s/d)=1}}^{s/d} \frac{(-1)^m}{d^{2a+1}} H^a_{m,\frac{s}{d}} \sin\left(\frac{mn\pi}{s/d}\right) = \pi^{2a+1} F_a\left(\frac{n}{s}\right) \qquad \forall n \in \mathbb{Z},$$

that by the Möbius inversion formula (see [**13**], Ch. I.2, Th. 8) gives

$$\sum_{\substack{m=1 \\ (m,s)=1}}^{s} (-1)^m H^a_{m,s} \sin\left(\frac{mn\pi}{s}\right) = \pi^{2a+1} \sum_{d|s} \mu\left(\frac{s}{d}\right)\left(\frac{d}{s}\right)^{2a+1} F_a\left(\frac{n}{d}\right) \quad \forall n \in \mathbb{Z}. \tag{3}$$

Considering this identity for $0 < n < s$, n coprime with s, we get a set of $\phi(s)$ linear equations where only the $\phi(s)$ numbers $H^a_{m,s}$ with $(m, s) = 1$ appear.

At last, we can generalize the previous equations by substituting n by rn in (2) and (3), where r is a fixed integer coprime with s and n runs in $0 < n < s$ (n coprime with s for (3).)

Identities (2) or (3) allow us to recover $H^a_{m,s}$ as linear combination of values of F_a but only if the matrices $M_{r,s}$ and $M'_{r,s}$, respectively, are invertible. For computational purposes we are also interested to find an efficient algorithm for the inverse matrix so that not only the invertibility of those matrices but also the structure of their characteristic polynomials has to be studied.

Actually, using the identity $M_{1,s}^2 = s\mathbb{I}$, from (2) we get the formula we are looking for:

$$(4) \qquad H_{m,s}^a = (-1)^m \frac{2\pi^{2a+1}}{s} \sum_{n=1}^{s-1} F_a\left(\frac{n}{s}\right) \sin\left(\frac{mn\pi}{s}\right) \qquad \text{for} \qquad 0 < m < s\,.$$

Now that the constants $H_{m,s}^a$ have been calculated, we can use them to provide a new proof of the known formula for the values of the Dirichlet L-functions (for the definition see [**3**]). In fact, let χ be a Dirichlet *odd* character modulo $2s$ and let $L(\cdot, \chi)$ be the corresponding Dirichlet L-function, then

$$(5) \qquad L(2a+1, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{2a+1}} = \sum_{m=1}^{s} \chi(m)\, H_{m,s}^a,$$

so that substituting (4) in (5) we get (note that $\chi(m) = 0$ if m is even)

$$L(2a+1, \chi) = -\frac{2\pi^{2a+1}}{s} \sum_{n=1}^{s-1} F_a\left(\frac{n}{s}\right) \sum_{m=1}^{s} \chi(m) \sin\left(\frac{mn\pi}{s}\right).$$

Let χ^* be the character mod s inducing χ and suppose χ^* to be primitive, then a long and a slightly tricky computation proves the identity

$$2i\overline{\chi}^*(2) \sum_{m=1}^{s} \chi(m) \sin\left(\frac{mn\pi}{s}\right) = (-1)^n \overline{\chi}^*(n)\, \tau(\chi^*)\,,$$

where $\tau(\chi^*)$ is the Gaussian sum, so that from the previous formula we deduce

$$L(2a+1, \chi) = \chi^*(2) \frac{\pi^{2a+1} i\tau(\chi^*)}{s} \sum_{n=1}^{s-1} (-1)^n \overline{\chi}^*(n) F_a\left(\frac{n}{s}\right).$$

Substituting (1) in this equation we obtain a formula giving $L(2a+1, \chi)$ in terms of the generalized Bernoulli numbers. Such formula is not new (see for example Theorem 4.2 of [**14**]), but we think that our non-standard deduction is of some interest.

We conclude this section noting that by the orthogonality of the Dirichlet characters modulo $2s$ we can represent $H_{m,s}^a$ as a finite sum of the values of Dirichlet L-functions, i.e.,

$$H_{m,s}^a = \frac{2}{\phi(s)} \sum_{\substack{\chi \bmod 2s \\ \chi \text{ odd}}} \overline{\chi}(m)\, L(2a+1, \chi)\,,$$

therefore to determinate $H^a_{m,s}$ and to determinate the values of $L(\cdot, \chi)$ at odd integers are equivalent problems.

3 - Tools from Number Theory

Proposition 1. *Let $s > 1$ be odd and $D | s$. Let*

$$\kappa(D, s) := \sum_{d : \substack{d|s \\ D|d}} \mu\left(\frac{s}{d}\right) \frac{\phi(s)}{\phi(d)} d ,$$

then

$$\kappa(D, s) = D \left| \mu\left(\frac{s}{D}\right) \right| \delta_{(D, s/D) = 1} .$$

Proposition 2. *Let k, n be coprime odd integers. Then*

$$G(k, n) := \sum_{l=1}^{n} e^{2\pi i \frac{kl^2}{n}} = \left(\frac{k}{n}\right) \sqrt{n^*}, \tag{6}$$

$$R(k, n) := \sum_{\substack{l=1 \\ (l, n) = 1}}^{n} e^{2\pi i \frac{kl^2}{n}} = \begin{cases} \sum_{d|n} \mu\left(\frac{n}{d}\right) \left(\frac{kn/d}{d}\right) \sqrt{d^*} & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

In this formula $\left(\frac{k}{n}\right)$ is the Jacobi symbol, $n^* = n$ if $n = 1 \bmod (4)$ and $n^* = -n$ if $n = -1 \bmod (4)$ and $\sqrt{-n} = i\sqrt{n}$ where i is the same square root of -1 occurring in the definition of $G(k, n)$. The result in (6) is due to Gauss and its original proof is reproduced in Rademacher [**11**]. A different proof due to Dirichlet is reproduced in Davenport [**3**], other proofs can be found in [**1**] and in [**6**]. The second result can be deduced by (6) using the inclusion-exclusion principle.

For every integer D let V_D be the $\mathbb{C}$-vector space which is generated by the *primitive* characters modulo D. Moreover, let $E_D \subsetneq V_D$ and $O_D \subsetneq V_D$ be the subspaces which are generated by even and odd characters, respectively. The following proposition gives the dimensions of O_D and E_D.

Proposition 3. *For every pair of coprime odd integers m, n we have the isomorphisms*

$$O_{mn} \simeq (E_m \otimes O_n) \oplus (O_m \otimes E_n), \qquad E_{mn} \simeq (E_m \otimes E_n) \oplus (O_m \otimes O_n).$$

Therefore, when $D > 1$ *is odd, we have,*

$$\dim O_D = \frac{1}{2}(\psi(D) - \mu(D)), \qquad \dim E_D = \frac{1}{2}(\psi(D) + \mu(D)).$$

4 - Eigenvalues of $M_{\cdot,\, s}$

Theorem 1. *The characteristic polynomials of the matrices* $M_{\cdot,\, s}$ *are*

$$\det(x\mathbb{I} - M_{r,\, s}) = (x^2 - s)^{\frac{s-1}{2}},$$

and

$$\det(x\mathbb{I} - M_{2r,\, s}) = x^{\frac{s-1}{2}}(x - \sqrt{s})^{m_+}(x + \sqrt{s})^{m_-},$$

where

$$\begin{cases} m_+ + m_- = \dfrac{s-1}{2} \\ m_+ - m_- = \left(\dfrac{r}{s}\right) \delta_{s \equiv 3(4)}. \end{cases}$$

Proof. *The first claim.* We already remarked that $M_{r,\, s}^2 = s\mathbb{I}$, therefore $x^2 - s$ is the minimal polynomial of $M_{r,\, s}$ and its characteristic polynomial must be

$$\det(x\mathbb{I} - M_{r,\, s}) = (x - \sqrt{s})^{m_+}(x + \sqrt{s})^{m_-}, \tag{8}$$

for some $m_+, m_- \geqslant 1$ with $m_+ + m_- = s - 1$. Let us consider the trace of $M_{r,\, s}$. From (8) we get

$$\begin{aligned}(m_+ - m_-)\sqrt{s} = \mathrm{Tr}(M_{r,\, s}) &= \sqrt{2}\sum_{n=1}^{s-1} \sin\left(\frac{rn^2\pi}{s}\right) \\ &= \sqrt{2}\sum_{n=1}^{(s-1)/2}\left(\sin\left(\frac{rn^2\pi}{s}\right) + \sin\left(\frac{r(s-n)^2\pi}{s}\right)\right) = 0,\end{aligned}$$

hence $m_+ = m_-$ and the claim follows.

The second claim. An explicit computation shows that $(M_{2r,\, s}^2)_{n,\, m} = \frac{s}{2}(\delta_{n=m} - \delta_{n=s-m})$ so that by induction on s it is possible to prove that the characteristic polynomial of $M_{2r,s}$ is

$$\det(x\mathbb{I} - M_{2r,s}^2) = x^{(s-1)/2}(x-s)^{(s-1)/2}.$$

As a consequence the characteristic polynomial of $M_{2r,s}$ must be

$$\det(x\mathbb{I} - M_{2r,s}) = x^{(s-1)/2}(x-\sqrt{s})^{m_+}(x+\sqrt{s})^{m_-},$$

for some m_+, m_- with $m_+ + m_- = (s-1)/2$. Let us consider the trace of $M_{2r,s}$. By Proposition 2 we get

$$(m_+ - m_-)\sqrt{s} = \mathrm{Tr}(M_{2r,s}) = \sum_{n=1}^{s-1} \sin\left(\frac{2rn^2\pi}{s}\right) = \mathrm{Im}\, G(r,s) = \binom{r}{s}\sqrt{s}\,\delta_{s\equiv 3(4)}$$

and the claim is proved. ■

5 - Eigenvalues of $M'_{\cdot,s}$

Theorem 2. *The characteristic polynomial of $M'_{r,s}$ is*

$$\det(x\mathbb{I} - M'_{r,s}) = x^{d_0} \prod_{\substack{d|s \\ (d,s/d)=1 \\ \mu(s/d)\neq 0}} (x^2-d)^{(\psi(d)-\mu(d))/2},$$

with $d_0 := \phi(s) - \sum_{\substack{d|s \\ (d,s/d)=1 \\ \mu(s/d)\neq 0}} (\psi(d)-\mu(d))$.

Analogously,

Theorem 3. *The characteristic polynomial of $M'_{2r,s}$ is*

$$\det(x\mathbb{I} - M'_{2r,s}) = x^{d_0} \prod_{\substack{d|s \\ (d,s/d)=1 \\ \mu(s/d)\neq 0}} (x-\sqrt{d})^{m_{d,+}}(x+\sqrt{d})^{m_{d,-}},$$

where $d_0 := \phi(s) - \frac{1}{2}\sum_{\substack{d|s \\ (d,s/d)=1 \\ \mu(s/d)\neq 0}} (\psi(d)-\mu(d))$ and $m_{d,\pm}$ are the solutions of

$$\begin{cases} m_{d,+} + m_{d,-} = \dfrac{1}{2}(\psi(d)-\mu(d)) \\ m_{d,+} - m_{d,-} = c_{r,s,d}, \end{cases} \tag{9}$$

with

$$c_{r,s,d} = \begin{cases} \mu\left(\dfrac{s}{d}\right)\left(\dfrac{rs/d}{d}\right) & \text{if } s \text{ is squarefree}, \ d \equiv 3(4) \\ 0 & \text{otherwise}. \end{cases}$$

Remark. Since $\psi(1) - \mu(1) = 0$, 1 is never eigenvalue of $M'_{\cdot,s}$.

Remark. Using the multiplicativity of the function ψ it is easy to verify that when n is odd

$$\psi(n) - \mu(n) = \mu(n)(n-1) \bmod (4).$$

This fact shows that a corrective term $c_{r,s,d}$ which is not zero when d is squarefree and congruent to 3 modulo 4 is necessary in order to (9) has integer solutions.

Remark. When s is an *odd* prime $M_{\cdot,s} = M'_{\cdot,s}$, therefore the conclusions of Theorems 2 and 3 have to accord to Theorem 1, as a simple check shows.

At last, we come back to the original problem of the invertibility of matrices $M_{\cdot,s}$ and $M'_{\cdot,s}$. Theorem 1 and the following corollary of Theorems 2-3 show that only $M_{r,s}$ is invertible for every s and that (3) can be used to recover $H^a_{m,s}$ only when s is squarefree.

Corollary. $\det(M'_{2r,s}) = 0$ *and*

$$\det(M'_{r,s}) = \begin{cases} 0 & \text{if } s \text{ is not squarefree} \\ (-s)^{\frac{1}{2}(s-1)} & \text{if } s \text{ is prime} \\ \prod_{p|s} p^{\frac{1}{2}(p-2)\phi(s/p)} & \text{if } s \text{ is squarefree and not prime}, \end{cases}$$

where p runs on primes dividing s.

We come now to the proof of Theorems 2-3. As first step we compute the matrix $M'^{2}_{\cdot,s}$.

Proposition 4. *Let r, s be coprime odd integers, $s > 1$. For every pair m, n coprime with s, $0 < m, n < s$, let $(M'^{2}_{\cdot,s})_{m,n}$ be the m-th, n-th entry of the* *matrix* $M'^2_{\cdot,s}$, *then*

$$(M'^2_{r,s})_{m,n} = \sum_{d|s}\mu\left(\frac{s}{d}\right)(d(\delta_{m\equiv n(2d)} - \delta_{m\equiv -n(2d)}) + \delta_{m\not\equiv n(2)}(\delta_{m\not\equiv n(2d)} - \delta_{m\not\equiv -n(2d)}))\,,$$

$$(M'^2_{2r,s})_{m,n} = \sum_{d|s}\mu\left(\frac{s}{d}\right)\frac{d}{2}(\delta_{m\equiv n(d)} - \delta_{m\equiv -n(d)})\,,$$

both independent of r.

By Proposition 4 the entry $(M'^2_{r,s})_{m,n}$ is zero when $m\not\equiv n(2)$. As a consequence, there exists a permutation J such that

$$JM'^2_{r,s}J^{-1} = \begin{pmatrix} N_s & 0 \\ 0 & N_s \end{pmatrix}, \tag{10}$$

where N_s is a matrix of order $\frac{1}{2}\phi(s) \times \frac{1}{2}\phi(s)$ whose entries are

$$(N_s)_{m,n} := \sum_{d|s}\mu\left(\frac{s}{d}\right)d(\delta_{m\equiv n(d)} - \delta_{m\equiv -n(d)}) \quad \text{with} \quad 1\le m,\, n\le s,\ (mn,\, 2s) = 1\,.$$

The following two propositions provide a family of eigenvectors for N_s and $M'^2_{2r,s}$.

Proposition 5. *Let* $s>1$ *be an odd integer, let* $D|s$ *and let* $f\in O_D$. *Let* v^f *be the vector of* $\mathbb{C}^{\phi(s)/2}$ *whose entries are* v^f_m *with* $(m, 2s) = 1$, $1\le m\le s$ *and whose value is* $v^f_m = f(m)$. *Then* v^f *is an eigenvector of* N_s *with eigenvalue* $\kappa(D, s)$.

Proposition 6. *Let* $s>1$ *be an odd integer let* $D|s$ *and let* $f\in V_D$. *Let* v^f *be the vector of* $\mathbb{C}^{\phi(s)}$ *whose entries are* v^f_m *with* $(m, s) = 1$, $1\le m\le s$ *and whose value is* $v^f_m = f(m)$. *When* $f\in E_D$, $v^f\in\ker M'^2_{2r,s}$ *and when* $f\in O_D$ *then* v^f *belongs to the eigenspace of* $M'^2_{2r,s}$ *with eigenvalue* $\kappa(D, s)$.

From Propositions 1, 3, 5 and 6 we get the following characterization of the eigenspaces.

Proposition 7. *Let* T_d *be the d-eigenspace of* N_s. *Then* $\dim T_d$ *is*

$$\begin{cases} \frac{1}{2}(\psi(d)-\mu(d)) & \text{for } d\in\mathbb{N},\ d\,|\,s,\ |\mu(s/d)|=1 \\ & \text{and } (d,\ s/d)=1, \\ \frac{1}{2}\sum\limits_{\substack{d|s \\ \mu(s/d)=0 \\ (d,s/d)=1}}(\psi(d)-\mu(d))+\frac{1}{2}\sum\limits_{\substack{d|s \\ (d,s/d)>1}}(\psi(d)-\mu(d)) & \text{for } d=0, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, $\ker N_s=\{0\}$ *if and only if s is squarefree.*

Let S_d *be the d-eigenspace of* $M'^2_{2r,s}$. *Then* $\dim S_d$ *is*

$$\begin{cases} \frac{1}{2}(\psi(d)-\mu(d)) & \text{for } d\in\mathbb{N}, d\,|\,s,\ |\mu(s/d)|=1 \\ & \text{and } (d,\ s/d)=1, \\ \frac{\phi(s)}{2}+\frac{1}{2}\sum\limits_{\substack{d|s \\ \mu(s/d)=0 \\ (d,s/d)=1}}(\psi(d)-\mu(d))+\frac{1}{2}\sum\limits_{\substack{d|s \\ (d,s/d)>1}}(\psi(d)-\mu(d)) & \text{for } d=0, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, $\dim\ker M'^2_{2r,s}\geqslant\phi(s)/2$, *with* $\dim\ker M'^2_{2r,s}=\phi(s)/2$ *if and only if s is squarefree.*

A simple argument concludes the proof of Theorem 2. In fact, by (10) and the previous proposition we get that the eigenvalues of $M'^2_{r,s}$ are 0 and the integers d dividing s such that s/d is squarefree and coprime with d, with multiplicities d_0 (whose value is defined in the statement of Theorem 2) and $\psi(d)-\mu(d)$, respectively. It is important to recall that $d=1$ is not an eigenvalue: its multiplicity is $\psi(1)-\mu(1)=0$. As a consequence, the eigenvalues of $M'_{r,s}$ are 0 and $\pm\sqrt{d}$, where d is chosen as before. Let m_0 be the multiplicity of 0 and let $m_{d,+}$ and $m_{d,-}$ be those ones of $\sqrt{d}$ and $-\sqrt{d}$, respectively. From Proposition 7 (and (10)) we have $m_{+,d}+m_{d,-}=\psi(d)-\mu(d)$, so that

$$m_0=\phi(s)-\sum_{\substack{d|s \\ (d,\ s/d)=1 \\ |\mu(s/d)|=1 \\ d>1}}(m_{+,d}+m_{d,-})=\phi(s)-\sum_{\substack{d|s \\ (d,\ s/d)=1 \\ |\mu(s/d)=1 \\ d>1}}(\psi(d)-\mu(d))=d_0.$$

Moreover, the same argument we use to prove that $\operatorname{Tr}M_{r,s}=0$ can be repeated here to prove that also $\mathrm{Tr}\, M'_{r,s} = 0$, therefore we get

$$0 = \mathrm{Tr}\,(M'_{r,s}) = \sum_{\substack{d \mid s \\ (d,\, s/d) = 1 \\ |\mu(s/d)| = 1 \\ d > 1}} (m_{d,+} - m_{d,-}) \sqrt{d}. \tag{11}$$

Let us consider the integers d appearing in this equation. There is only one d which is a square, at most, and this fact happens if and only if s is not squarefree. For every other d appearing in (11) it is possible to find a prime p_d such that $p_d \mid d$ with odd order and $p_d \nmid d'$ if $d' \neq d$. As a consequence the numbers $\sqrt{d}$ are $\mathbb{Q}$-linearly independent so that (11) implies $m_{d,+} = m_{d,-}$. Since we know that $m_{d,+} + m_{d,-} = \psi(d) - \mu(d)$, we conclude that $m_{d,+} = m_{d,-} = \frac{1}{2}(\psi(d) - \mu(d))$ and the proof of Theorem 2 is completed.

The proof of Theorem 3 can be completed in similar way if we note that by (7)

$$\mathrm{Tr}\,(M'_{2r,s}) = \mathrm{Im}\, R(r, s) = \sum_{\substack{d \mid s \\ (d,\, s/d) = 1 \\ |\mu(s/d)| = 1 \\ d > 1}} c_{r,s,d} \sqrt{d}.$$

Remark. The anonymous referee pointed at our attention a formula due to Dedekind whose existence we were not aware (see Theorem 6.1 in [7]). An immediate consequence of such result is the identity

$$\det(x\mathbb{I} - M'_{r,s}) = \prod_{\chi \bmod s} \left(x - \sum_{a=1}^{s} \overline{\chi}(a) \sin\left(\frac{ar\pi}{s} \right) \right)$$

giving an interesting representation of the eigenvalues in terms of sums involving the characters modulo s. By this way it is possible to recover the exact value of every eigenvalue, multiplicity included, but a lot of tedious computations is required. We think that the approach we give here keeps its interest.

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Abstract

Let r, s be coprime integers, $s > 1$ and odd. The characteristic polynomials of the matrices

$$\left[\sin\left(\frac{rmn\pi}{s}\right)\right]_{0<m,\,n<s} \quad \text{and} \quad \left[\sin\left(\frac{rmn\pi}{s}\right)\right]_{\substack{0<m,\,n<s\\(mn,\,s)=1}}$$

are determined.

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