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Exponential Diophantine equations and inequalities (**)

Let $\Sigma_{\mathbb{Q}}$ be the ring of *power sums*, i.e. of the functions of $\mathbb{N}$ of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n,$$

with $h \geq 1$, where the elements $c_1 > c_2 > \ldots > c_h$, called the *roots* of α, and b_i, called the *coefficients*, are nonzero rationals.

More generally, if $\mathbb{K} \subseteq \mathbb{C}$ is a field and $A \subseteq \mathbb{C}$ a ring, we will denote by $\mathbb{K}\Sigma_A$ the ring of power sums with coefficients in $\mathbb{K}$ and roots in A. The subring of power sums with only positive roots will be denoted by $\mathbb{K}\Sigma_A^+$.

Note that $\mathbb{Q} \subset \Sigma_{\mathbb{Z}}^+$, since the rationals can be considered constant power sums with just one root $c_1 = 1$.

A power sum is called *nondegenerate* if c_i/c_j is not a root of unity for all $i \neq j$.

Since long ago, Diophantine equations and inequalities involving power sums have been studied using the estimates for linear forms in logarithms due to A. Baker (see [**1**]). Here we state some results obtained with this method; they can be found in [**8**] and [**12**] respectively.

Pethö, 1982: Let $\alpha \in \Sigma_{\mathbb{Q}}$ be nondegenerate with $h = 2$. Under suitable conditions on the coefficients, if $\alpha(n) = sx^q$ for integers s, $x \neq 0$, $q \geq 2$ and $n > 0$, then $\max\{|x|, q, n\}$ is bounded by an effectively computable constant which depends on α and on the greatest prime factor of s.

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Shorey and Stewart, 1987: Let $\varepsilon > 0$ be fixed and suppose that $\alpha \in \Sigma_{\overline{\mathbb{Q}}}$ is a nondegenerate power sum with just one root c_1 with largest absolute value. Then the inequality

$$|Ex^q - \alpha(n)| > |c_1|^{n(1-\varepsilon)}$$

holds for all the nonzero integers E, x, for $n > 0$, and for every integer $q > q_0(\alpha, P)$, where P is the greatest prime factor of E, assuming that $Ex^q \neq b_1 c_1^n$.

Recently P. Corvaja and U. Zannier have found new results (see [**2**], [**3**] and [**4**]) on these problems by a new method, applying in this context the Subspace Theorem by W. M. Schmidt (see [**9**] and [**10**]) and its generalizations. The following result was obtained with this method and can be found in [**2**].

Corvaja and Zannier, 1998: For every $\varepsilon > 0$ fixed, $\alpha \in \Sigma_{\mathbb{Z}}$ and for every integer $d \geqslant 2$ there exist a finite set of power sums $\beta_1, \ldots, \beta_s \in \overline{\mathbb{Q}}\Sigma_{\overline{\mathbb{Q}}}$ such that all the solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality

$$|y^d - \alpha(n)| \ll |\alpha(n)|^{1 - \frac{1}{d} - \varepsilon},$$

satisfy $y = \beta_i(n)$ for a certain $i \in \{1, \ldots, s\}$.

As a corollary it follows that the equation $\alpha(n) = y^d$ has just finitely many solutions, if we suppose that in α the roots c_1 and c_2 are coprime.

Using the same method, the author has obtained the following results, which generalize some of the results by Corvaja and Zannier. They can be found in [**11**].

Theorem 1 (Scremin). *Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be absolutely irreducible, monic and of degree $d \geqslant 2$ in y; let $\alpha \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$, and let $\varepsilon > 0$ be fixed.*

Then there exists a finite set of power sums $\{\beta_1, \ldots, \beta_s\} \subset \Sigma_{\mathbb{Z}}^{+}$ such that every solution $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality

$$|F(\alpha(n), y)| < \left| \frac{\partial F}{\partial y}(\alpha(n), y) \right| \cdot |\alpha(n)|^{-\varepsilon} \tag{4}$$

satisfies $y = \beta_i(n)$, for a certain $i = 1, \ldots, s$.

The number of non constant power sums in the set $\{\beta_1, \ldots, \beta_s\}$ is at most d^2.

Moreover, the set of natural numbers n such that (n, y) is a solution of (4) *is the union of a finite set and a finite number of arithmetic progressions.*

From the theorem above it follows

Corollary 1. *Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be monic in y, absolutely irreducible and of degree $d \geq 2$ in y; let $f(x) \in \mathbb{Z}[x]$ be a non constant polynomial; let α be a non constant power sum with integral roots and algebraic coefficients.*

Then the equation

$$F(\alpha(n), y) = f(n) \tag{5}$$

has only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$.

The results above involve just one power sum in the equations that are considered; this gives rise to the following

Problem 1. *Generalize the above results to polynomial-exponential equations where several power sums are involved, i.e. equations of the form*

$$F(\alpha_0(n), \dots, \alpha_d(n), y) = 0,$$

where without losing generality F can be supposed a polynomial in y with coefficients in the ring of power sums.

Let $d \geq 2$, $F(x_0, \dots, x_d, y) \in \overline{\mathbb{Q}}[x_0, \dots, x_d, y]$ and $\alpha_0, \dots, \alpha_d \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}^+$. We want to consider the Diophantine equation

$$F(\alpha_0(n), \dots, \alpha_d(n), y) = 0.$$

Without any restriction, instead of the equation above we can study the equation

$$\alpha_0(n)\, y^d + \dots + \alpha_{d-1}(n)\, y + \alpha_d(n) = 0, \tag{6}$$

where $\alpha_0, \dots, \alpha_d \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}^+$, i.e. without losing generality we can consider polynomials in y with coefficients in the ring $\overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}^+$.

Let us show how can be associated to this equation another equation in some normal form.

Let $\alpha_0, \dots, \alpha_d \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}^+$ be defined by

$$\begin{aligned}
\alpha_0(n) &= a_1^{(0)} \alpha_1^{(0)^n} + a_2^{(0)} \alpha_2^{(0)^n} + \dots + a_{t^{(0)}}^{(0)} \alpha_{t^{(0)}}^{(0)^n}, \\
&\;\;\vdots \\
\alpha_d(n) &= a_1^{(d)} \alpha_1^{(d)^n} + a_2^{(d)} \alpha_2^{(d)^n} + \dots + a_{t^{(d)}}^{(d)} \alpha_{t^{(d)}}^{(d)^n},
\end{aligned}$$

where $a_i^{(j)}$ are algebraic and $\alpha_i^{(j)}$ are positive integers such that $\alpha_1^{(j)} > \alpha_2^{(j)} > \ldots > \alpha_{t^{(j)}}^{(j)}$ for $i = 1, \ldots t^{(j)}$ and $j = 0, \ldots, d$.

Set (for a positive real determination of the roots)

$$\alpha := \max_{i=1,\ldots,d} \left(\frac{\alpha_1^{(i)}}{\alpha_1^{(0)^{(d-i)/d}}} \right)^{1/i}.$$

Moreover, let

$$y = \frac{\alpha^n}{\alpha_1^{(0)^{n/d}}} z .$$

Set $f(x_0, \ldots, x_d, y) := x_0 y^d + \ldots + x_{d-1} y + x_d$, and consider the polynomial

$$\frac{1}{\alpha^{dn}} f\left(\alpha_0(n), \ldots, \alpha_d(n), \frac{\alpha^n}{\alpha_1^{(0)^{n/d}}} z\right).$$

This is a polynomial in z with coefficients in $\overline{\mathbb{Q}}\Sigma_{\overline{\mathbb{Q}}}$, and all the roots of the power sums appearing in its coefficients are $\leqslant 1$.

Let $\gamma_1, \ldots, \gamma_r$ be the distinct roots strictly less than 1 of these power sums. Identifying the expressions γ_i^n with new variables x_i we get a polynomial

$$g(x_1, \ldots, x_r, z) \in \overline{\mathbb{Q}}[x_1, \ldots, x_r, z]$$

such that

$$g(\gamma_1^n, \ldots, \gamma_r^n, z) = \frac{1}{\alpha^{dn}} f\left(\alpha_0(n), \ldots, \alpha_d(n), \frac{\alpha^n}{\alpha_1^{(0)^{n/d}}} z\right).$$

This polynomial is some normal form for equation (6).

We denote by $D(\alpha_0, \ldots, \alpha_d)$ the discriminant of g with respect to z evaluated at $(0, \ldots, 0)$, i.e.

$$D(\alpha_0, \ldots, \alpha_d) = \mathrm{disc}_z(g)(0, \ldots, 0).$$

We are now in the position to formulate the following result on Diophantine equations involving several power sums. It can be found in [**5**].

Theorem 2 (Fuchs and Scremin). *Let $d \geqslant 2$ and let $\alpha_0, \ldots, \alpha_d \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$. Assume that*

$$D(\alpha_0, \ldots, \alpha_d) \neq 0.$$

Then there exist finitely many power sums $\beta_1, \ldots, \beta_s \in \overline{\mathbb{Q}}\Sigma_{\overline{\mathbb{Q}}}$, *arithmetic progressions* $\mathcal{P}_1, \ldots, \mathcal{P}_s$, *and a finite set* $\mathcal{N}$ *of integers, such that for the set* S *of solutions* $(n, y) \in \mathbb{N} \times \mathbb{Z}$ *of the equation*

$$\alpha_0(n)\, y^d + \ldots + \alpha_{d-1}(n)\, y + \alpha_d(n) = 0 \tag{7}$$

we have

$$S = \bigcup_{i=1}^{s} \{(n, \beta_i(n)) : n \in \mathcal{P}_i\} \cup \{(n, y) : n \in \mathcal{N},\, y \in \mathbb{Z}\} \cup M,$$

where M *is a finite set.*

From Theorem 2 we can see that the solutions of (7) are generally finite, apart from two «trivial» infinite families of solutions which can appear in some particular cases easy to identify. In these cases the infinite families of solutions can be easily parametrized. Let us show it with some examples.

First infinite family of solutions:

Consider the equation

$$y^2 + (2^n + 3^n)y + 6^n = 0.$$

It has infinitely many solutions, namely $(n, -2^n)$ and $(n, -3^n)$ for arbitrary $n \in \mathbb{N}$, which are parametrized by power sums.

Second infinite family of solutions:

Consider the equation

$$(2^n - 2)\, y^2 + 3^n - 3 = 0.$$

It has infinitely many solutions, namely $(1, y)$ for arbitrary $y \in \mathbb{Z}$. In this case n satisfies $\alpha_0(n) = \alpha_1(n) = \ldots = \alpha_d(n) = 0$. Let us notice that assuming α_0 to be a nonzero constant, this family of solutions is excluded.

Main ideas of the proof:

From the construction of the polynomial g and from the definition of z, it can be shown that the sequence of z_n associated to the solutions (n, y_n) of (7) is bounded, i.e. it lies in the union of arbitrarily small neighbourhoods of the solutions of

$$g(0, \ldots, 0, z) = 0,$$

at least if n is large enough. Applying to the polynomial g a version of the Implicit

Function Theorem that can be found in [**7**] (recall that we have $D(\alpha_0, \dots, \alpha_d) \neq 0$), we obtain

$$z = z_0 + \sum_{|i_1 + \dots + i_r| > 0} c_i x_1^{i_1} \dots x_r^{i_r},$$

where $c_i \in \overline{\mathbb{Q}}$ and z_0 satisfies $g(0, \dots, 0, z_0) = 0$. This means that for every solution (n, z_n) with n large enough we have

$$z_n = z_0 + \sum_{|i_1 + \dots + i_r| > 0} c_i \gamma_1^{i_1 n} \dots \gamma_r^{i_r n},$$

for some suitable z_0 and some coefficients c_i. We approximate z_n with the power sum

$$V_n := z_0 + \sum_{0 < |i_1 + \dots + i_r| < H} c_i \gamma_1^{i_1 n} \dots \gamma_r^{i_r n},$$

choosing H large enough. We are now able to apply the Subspace Theorem as Corvaja and Zannier did in their works, concluding the proof. ■

A problem similar to Problem 1 can rise if we consider exponential Diophantine inequalities where several power sums are involved.

Problem 2. *Find lower bounds for the quantity*

$$|y^d + \alpha_1(n) y^{d-1} + \dots + \alpha_{d-1}(n) y + \alpha_d(n)|$$

with respect to a power of α^n.

An answer to Problem 2 is given by the following Theorem 3, which can be found in [**6**].

Here we define as above the quantity $D(\alpha_1, \dots, \alpha_d)$ for the polynomial g associated to the polynomial $y^d + \alpha_1(n) y^{d-1} + \dots + \alpha_{d-1}(n) y + \alpha_d(n)$, monic in y.

Theorem 3 (Fuchs and Scremin). *Let* $d \geq 2$, $\varepsilon > 0$, *and let* $\alpha_1, \dots, \alpha_d \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}^+$. *Assume that*

$$D(\alpha_1, \dots, \alpha_d) \neq 0.$$

Then there exist finitely many power sums $\beta_1, \dots, \beta_s \in \overline{\mathbb{Q}}\Sigma_{\overline{\mathbb{Q}}}$ *such that all the solutions* $(n, y) \in \mathbb{N} \times \mathbb{Z}$ *of the Diophantine inequality*

$$|y^d + \alpha_1(n) y^{d-1} + \dots + \alpha_d(n)| < \alpha^{n(d-1-\varepsilon)}, \tag{8}$$

except finitely many, have $y = \beta_i(n)$ *for some* $i = 1, \dots, s$.

Moreover, the set of natural numbers n such that (n, y) *is a solution of the* *inequality is the union of a finite set and a finite number of arithmetic progressions.*

Observe that for $\alpha_1 = \ldots = \alpha_{d-1} = 0$ and $\alpha_d \in \Sigma_{\mathbb{Z}}^+$ we get the conclusion of Theorem 3 for the inequality

$$|y^d - \alpha_d(n)| < \alpha^{n(d-1-\varepsilon)} = \alpha_1^{(d)^{n(1-(1/d)-\varepsilon')}},$$

where $\alpha_1^{(d)}$ is the dominant root of α_d, i.e. the result of Corvaja and Zannier in [**2**]. This also shows (cf. Remark 2 in [**2**]) that the exponent $d-1-\varepsilon$ in Theorem 3 is best possible.

From Theorem 3 can be derived the following Corollary, which states, under suitable assumptions, the finiteness of the solutions of a polynomial-exponential Diophantine equation involving several power sums, generalizing the result of Corollary 1.

Corollary 2. *Let $d \geq 2$, $\alpha_1, \ldots, \alpha_d \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}^+$ not all constant. Moreover, let $h(x) \in \mathbb{Z}[x]$ be a non constant polynomial. Assume that*

$$D(\alpha_1, \ldots, \alpha_d) \neq 0.$$

Then the Diophantine equation

$$y^d + \alpha_1(n)\, y^{d-1} + \ldots + \alpha_d(n) = h(n)$$

has only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$.

Let us remark that the results of Theorems 1, 2 and 3 and of their corollaries still hold by allowing the coefficients of the power sums to be polynomial functions of n, with the restriction that the coefficients of the roots of maximum modulus are constant.

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Abstract

Let us consider the ring of power sums with algebraic coefficients and positive integral roots, i.e. of functions of $\mathbb{N}$ *of the form*

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n, \tag{1}$$

with $b_i \in \overline{\mathbb{Q}}$ *and* $c_1 > c_2 > \ldots > c_h \in \mathbb{Z}^+$.

Since long ago, Diophantine equations and inequalities involving power sums have been studied using the estimates for linear forms in logarithms due to A. Baker, but many problems remained unsolved. Recently P. Corvaja and U. Zannier have found new results on these problems by a different method, applying in this context the Subspace Theorem by W. M. Schmidt.

Here we will first have a review on some of such results, then we will show some recent results obtained by the author, partially with C. Fuchs. We will first deal with the finiteness of the solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ *of the equation*

$$F(\alpha(n), y) = f(n),$$

where $F \in \overline{\mathbb{Q}}[X, Y]$ *is monic, absolutely irreducible of degree at least* 2, *and* $f \in \mathbb{Z}[X]$ *and the power sum* α *are not constant.*

Then we will consider equations and inequalities where several power sums are involved as, for example, the equation

$$\alpha_0(n)\, y^d + \alpha_1(n)\, y^{d-1} + \ldots + \alpha_{d-1}(n)\, y + \alpha_d(n) = 0 \tag{2}$$

and the inequality

$$|F(n, y)| < \alpha^{n(d-1-\varepsilon)}, \tag{3}$$

where $\alpha_0, \ldots, \alpha_d$ *are power sums,* $\varepsilon > 0$, F *is monic in* y *and* α *is a quantity depending on the dominant roots of the power sums appearing as coefficients in* F. *We will show that, under suitable assumptions, for all the solutions of* (2), y *can be parametrized by some power sum in a finite set. We will reach a similar conclusion also for* (3).

All these results generalize some results by P. Corvaja and U. Zannier on such problems.

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