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From Jacobian to Hessian: distributional form and relaxation (**)

Contents

1 - Introduction

Recently higher order variational problems have attracted a great deal of attention due in part to their relevance in the study of problems emerging from materials science and engineering, including the Blake-Zisserman model for image segmentation in computer vision, singular perturbation approaches for phase

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transitions in elastic materials, the treatment of ferromagnetic and micromagnetic materials, and thin structure models leading to membrane, shell, and plate theories (e.g., see Conti, Fonseca & Leoni [**21**], Choksi, Kohn & Otto [**18**], Carriero, Leaci & Tomarelli [**17**], DeSimone [**25**], Kohn & Müller [**45**], Müller [**52**], Owen & Paroni [**58**], Rivière & Serfaty [**60**]).

Several attempts have been made to reduce these problems to the first order setting, where nowadays there is a considerable mastery of techniques within the realm of the Calculus of Variations. Here we highlight the work in Dal Maso, Fonseca, Leoni & Morini [**24**] (see also a preceding work by Müller & Šverák [**56**]) where it is proved that, up to mild regularity conditions, a higher order quasiconvex function with superlinear growth is simply the restriction of a (one-)quasiconvex function to the appropriate linear subspace of matrices. The linear growth case remains open as the techniques used in [**24**] do not apply. Indeed, it is now clear that first order techniques often cannot be naturally extended to tackle certain higher order problems, and the analytical and geometrical constraints that higher order derivatives are endowed require new theoretical arguments, as it will be illustrated below.

In this paper we pursue this avenue of thought now directed to Jacobians of higher order. Here we will draw a parallel between the theory developed for the distributional generalization of the notion of determinant of a gradient, $\mathcal{J}u$, to the determinant of the matrix of second order derivatives, $\mathcal{H}u$. The need to search for the least integrability spaces where $\mathcal{J}u$ is well defined and where weak continuity and mild regularity properties still hold, is motivated in part by issues in nonlinear elasticity and vorticity effects in Ginzburg-Landau type models (e.g. see Alberti, Balso & Orlandi [**3**], Ball [**8**], Bethuel, Brézis & Heléin [**11**], Fonseca, Leoni, Malý & Paroni [**31**], James & Spector [**42**], Jerrard & Soner [**44**], Müller & Spector [**55**]). A considerable progress in this regard has been achieved merging ideas from partial differential equations, continuum mechanics, the calculus of variations and geometric measure theory (e.g. see Ball [**7**], Brézis, Fusco & Sbordone [**13**], Brézis & Nirenberg [**14**], [**15**], Coifman, Lions, Meyer & Semmes [**19**], Dacorogna & Murat [**23**], Fonseca, Fusco & Marcellini [**27**], [**28**], Fonseca, Leoni & Malý [**30**], Giaquinta, Modica & Souček [**35**], [**36**], Hajłasz [**38**], Iwaniec & Sbordone [**41**], Müller [**51**], [**52**], [**53**], Müller, Tang & Yan [**54**]). It all starts with the observation that (see Morrey [**49**] and Reshetnyak [**59**])

$$u_n \rightharpoonup u \text{ in } W^{1,N}(\Omega;\mathbb{R}^N) \Rightarrow \det\nabla u_n \overset{*}{\rightharpoonup} \det\nabla u \text{ in the sense of measures,}$$

where Ω is a bounded domain in $\mathbb{R}^N$. This is consequence of a simple integration by parts and of the fact that if $u \in W^{1,N}(\Omega;\mathbb{R}^N)$ then the Jacobian determinant

$\det\nabla u$ agrees with the distribution

$$\mathcal{J}_1 u = \operatorname{Det}\nabla u := \sum_{i=1}^{N}(-1)^{i+1}\frac{\partial}{\partial x_i}\left(u_1\frac{\partial(u_2,\ldots,u_N)}{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)}\right),$$

i.e.

$$\langle \operatorname{Det} u, \phi\rangle = -\int_{\Omega} u_1 \det(\nabla\phi, \nabla u_2, \ldots, \nabla u_N)\, dx \tag{1}$$

for $\phi \in C_c^\infty(\Omega)$. The treatment of such a distributional form of Jacobian was initiated by Ball [7]. The spaces $W^{1,N^2/(N+1)}(\Omega;\mathbb{R}^N)$ and $W^{1,N-1}(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ are commonly adopted as domains of definition of the operator $\mathcal{J}_1$.

We want to discuss and compare various choices of domains and weak forms of J, the classical pointwise Jacobian operator, well defined on $W^{1,N}(\Omega;\mathbb{R}^N)$. This leads us to the following general definition.

Definition 1.1. A topological or convergence space $\mathcal{X}$ of measurable functions $u:\Omega\to\mathbb{R}^N$ is said to be an *admissible domain for the weak Jacobian* $\mathcal{J}$ if

(i) $C^1(\Omega;\mathbb{R}^N)\cap\mathcal{X}$ is dense in $\mathcal{X}$;

(ii) the mapping $J\colon u\mapsto\det\nabla u$ from $C^1(\Omega;\mathbb{R}^N)\cap\mathcal{X}$ to $\mathcal{D}'$ has a unique continuous extension $\mathcal{J}\colon\mathcal{X}\to\mathcal{D}'$.

Here the framework of «convergence space» stands for a set X endowed with a notion of sequential convergence (e.g. strong convergence, weak convergence, weak* convergence or BV-strict convergence).

It turns out that $W^{1,N-1}(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N))$ is an admissible domain with the weak convergence on $W^{1,N-1}(\Omega;\mathbb{R}^N)$ and the strong convergence on $L^\infty(\Omega;\mathbb{R}^N)$. Another admissible domain is $W^{1,p}(\Omega;\mathbb{R}^N)$ for $p>N^2/(N+1)$ endowed with the weak convergence, and, more generally, we may consider $W^{1,p}(\Omega,\mathbb{R}^N)\cap L^s(\Omega,\mathbb{R}^N)$ with

$$\frac{N-1}{p}+\frac{1}{s}\leq 1,$$

equipped with the weak convergence on $W^{1,p}(\Omega,\mathbb{R}^N)$ and the strong convergence on $L^s(\Omega,\mathbb{R}^N)$. In all these cases it is easy to see that Hölder inequality (and Sobolev Embedding Theorem in the case $p\geq N^2/(N+1)$) ensures that the products involved in the definition of $\operatorname{Det}\nabla u$ are well defined in L^1, and for $p>N^2/(N+1)$ Rellich-Kondrachov Compactness Theorem yields strong convergence in L^s with s the conjugate exponent of $\frac{p}{N-1}$. Thus (1) is the formula for the weak Jacobian $\mathcal{J}$ in these spaces. For this approach to weak Jacobians see e.g. Ball [**7**], Ball & Murat [**10**], Ball, Currie & Olver [**9**], Dacorogna & Murat [**23**], Fonseca, Leoni & Malý [**30**], Olver [**57**].

Not all notions of weak Jacobian involve integration by parts. Alberti & Ambrosio [**2**] proved that $C^1 \cap \mathcal{M}on(\Omega)$ is an admissible domain for the weak Jacobian, where $\mathcal{M}on(\Omega)$ is the class of functions in Ω which agree almost everywhere with some (maximal) monotone function $\overline{u}$ whose domain includes Ω, and here it takes the form

$$\mathcal{J}(u)(B) = |\overline{u}(B)|$$

for every Borel set $B \subset\subset \Omega$.

But we must move beyond first order determinants. Indeed, several geometric problems involving Gaussian curvatures and issues in the theory of nonlinear partial differential equations ranging from the equations of gas dynamics for subsonic flows to minimal surfaces or the p-harmonic equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, call for the mastery of weak forms of the Hessian.

In this paper we search for a weak form, $\mathcal{H}u$, of the Hessian of a map $u : \Omega \to \mathbb{R}$ where Ω is bounded open set in $\mathbb{R}^N$, that will allow us to draw a parallel with the study of the weak Jacobian $\mathcal{J}$, and to make some progress toward answering the following questions:

– when can we ensure that $u_n \rightharpoonup u$ implies $\mathcal{H}u_n \rightharpoonup \mathcal{H}u$ in the sense of distributions, for an appropriate notion of weak convergence?

– when can we recover $\det\nabla^2 u$ as an «absolutely continuous part» of $\mathcal{H}u$ (see Müller [**51**]) for the Jacobian)?

– can we characterize the relaxed energy

$$\mathcal{F}(u, U) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \int_\Omega |\det \nabla^2 u_n(x)|\, dx : u_n \in C^\infty(\Omega), u_n \rightharpoonup u \right\}$$

for an appropriate notion of weak convergence (see Fonseca, Fusco & Marcellini [**27**], [**28**], for the Jacobian)?

There are many different ways to define a distributional Hessian, and their significance and properties depend greatly on the choice of domain. Among all these forms one stands out as capturing the good properties that we have come to expect by analogy with the study of $\operatorname{Det}\nabla u$. To illustrate this, and following Iwaniec [**39**], we consider the two dimensional case where $u : \Omega \to \mathbb{R}$, Ω is an open bounded subset of $\mathbb{R}^2$, and we arrive at four distributional notions of the Hessian:

– zero order Hessian $\mathcal{H}_0 : W^{2,2}_{\mathrm{loc}}(\Omega) \to L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'_0(\Omega)$,

$$\mathcal{H}_0(u) := u_{xx}u_{yy} - u^2_{xy},$$

– first order Hessian $\mathcal{H}_1 : W^{2,4/3}_{\rm loc}(\Omega) \to L^1_{\rm loc}(\Omega) \subset \mathcal{D}'_1(\Omega)$,

$$\mathcal{H}_1(u) := (u_x u_{yy})_x - (u_x u_{xy})_y (u_y u_{xx})_y - (u_y u_{xy})_x,$$

– second order Hessian $\mathcal{H}_2 : W^{2,1}_{\rm loc}(\Omega) \to \mathcal{D}'_2(\Omega)$,

$$\mathcal{H}_2(u) := \frac{1}{2}[(uu_{xx})_{yy} + (uu_{yy})_{xx} - (uu_{xy})_{xy}],$$

– very weak Hessian $\mathcal{H}^*_2 : W^{1,2}_{\rm loc}(\Omega) \to \mathcal{D}'_2(\Omega)$,

$$\mathcal{H}^*_2(u) := (u_x u_y)_{xy} - \frac{1}{2}(u_x u_x)_{yy} - \frac{1}{2}(u_y u_y)_{xx},$$

where $\mathcal{D}'_0(\Omega$ stands for the space of distributions in Ω of order k, $k = 0, 1, \dots$

Clearly, any two of these functionals coincide in the intersection of their domains.

Here we focus on $\mathcal{H}^*_2(u)$ as it is the weakest formulation among all four above.

In the three dimensional case, the very weak Hessian $\mathcal{H}^*_2 : W^{2,9/5}_{\rm loc}(\Omega) \to \mathcal{D}'_2(\Omega)$ is defined by

$$\begin{aligned}\mathcal{H}^*_2 u = &\frac{1}{6}[(2u_y u_z u_{yz} - u_y^2 u_{zz} - u_z^2 u_{yy})_{xx}\\ &+ (2u_z u_x u_{xz} - u_z^2 u_{xx} - u_x^2 u_{zz})_{yy}\\ &+ (2u_x u_y u_{xy} - u_x^2 u_{yy} - u_y^2 u_{xx})_{zz}\\ &+ 2(u_x u_y u_{zz} + u_z^2 u_{xy} - u_x u_z u_{yz} - u_z u_y u_{zx})_{xy}\\ &+ 2(u_y u_z u_{xx} + u_x^2 u_{yz} - u_x u_z u_{xy} - u_x u_y u_{zx})_{yz}\\ &+2(u_z u_x u_{yy} + u_y^2 u_{xz} - u_y u_x u_{yz} - u_y u_z u_{xy})_{yz}].\end{aligned} \tag{2}$$

It can be shown that, in contrast with the two dimensional case, it is not possible to find a clever way of using integration by parts that will lead to a formula involving no derivatives of order two.

All the forms of weak Hessian considered above in dimension two have their counterparts in higher dimension (we follow Iwaniec [**39**], see also Ball, Currie & Olver [**9**], Dacorogna & Murat [**23**], Olver [**57**], among others). The operator $\mathcal{H}_0$ assigns to $u \in W^{2,N}_{\rm loc}$ the pointwise Hessian Hu. The operator $\mathcal{H}_1$ is the weak Jacobian applied to the gradient, namely, $\mathcal{H}_1 u = \operatorname{Det}\nabla(\nabla u)$, $u \in W^{2,N^2/(N+1)}_{\rm loc}$. In the space $W^{2,N-1}_{\rm loc}$ the operator $\mathcal{H}$ takes the form

$$\langle \mathcal{H}_2 u, \phi\rangle = \frac{1}{N}\sum_{i=1}^{N}\int_\Omega u \det(\nabla D_1 u, \dots, \nabla D_{i-1}u, \nabla D_i\phi, \nabla D_{i+1}u, \nabla D_N u)\,dx.$$

The very weak distributional form of the Hessian is $\mathcal{H}^*_2 u : W^{2,N^2/(N+2)}_{\rm loc}(\Omega) \to \mathcal{D}'_2(\Omega)$

given by

$$(3)\qquad \begin{aligned}\langle \mathcal{H}_2^* u, \phi\rangle &= -\frac{1}{N!}\sum_{p,q\in P} \operatorname{sgn} p \operatorname{sgn} q \int_\Omega D_{p_1}uD_{q_1}uD_{p_2q_2}\phi D_{p_3q_3}u\dots \mathcal{D}_{p_Nq_N}u\,dx\\ &= -\frac{1}{N!}\sum_{p\in P}\operatorname{sgn} p\int_\Omega \det(D_{p_1}u\nabla u, \nabla D_{p_2}\phi, \nabla D_{p_3}u,\dots,\nabla D_{p_N}u)\,dx,\end{aligned}$$

where P be the set of all permutations of $\{1,\dots,N\}$ and $\phi\in C_c^\infty(\Omega)$.

As in the case of weak Jacobians, there are several choices of domains for weak forms of H, the pointwise Hessian operator $u\mapsto \det\nabla^2 u$, and we are led to the following definition.

Definition 1.2. A topological or convergence space $\mathcal{X}$ of measurable functions $u:\Omega\to\mathbb{R}$ is said to be an *admissible domain for the weak Hessian* $\mathcal{H}$ if

(i) $C^2(\Omega)\cap\mathcal{X}$ is dense in $\mathcal{X}$;

(ii) the mapping $H: u\mapsto \det\nabla^2 u$ from $C^2(\Omega)\cap\mathcal{X}$ to $\mathcal{D}'$ has a unique continuous extension $\mathcal{H}:\mathcal{X}\to\mathcal{D}'$.

We start by showing that

Theorem 1.3. *If $N=2$ then $W^{1,2}(\Omega)$ endowed with the strong convergence is an admissible domain for the weak Hessian, and the extension of H to $W^{1,2}(\Omega)$ is the operator $\mathcal{H}_2^*$.*

If $N\geq 3$ then $W^{2,p}(\Omega)\cap W^{1,s}(\Omega)$, with the weak convergence on $W^{2,p}(\Omega)$ and the strong convergence on $W^{1,s}(\Omega)$, and with

$$(4)\qquad \frac{N-2}{p}+\frac{2}{s}\leq 1,$$

is an admissible domain for the weak Hessian. Precisely, if $u_n\in W^{2,N}(\Omega)$ are such that $u_n\to u$ in $W^{1,s}(\Omega)$ and $u_n\rightharpoonup u$ in $W^{2,p}(\Omega)$ then

$$\det\nabla^2(u_n)\to\mathcal{H}u\quad\text{in } \mathcal{D}'.$$

Furthermore, $\mathcal{H}u$ acts on test functions as in (3).

We remark that, in view of Sobolev's Embedding Theorem, (4) is satisfied if $p=\dfrac{N^2}{N+2}$, $s=p^*$, and here $\mathcal{H}u$ is well defined as a distribution of order 2. Moreover, if $u_n\rightharpoonup u$ in $W^{2,p}$ with $p>\dfrac{N^2}{N+2}$, then by Rellich-Kondrachov Compactness Theorem we are in position to apply this theorem with $s:=2\Big(\dfrac{p}{N-2}\Big)'$, where in general r' stands for the conjugate exponent of r. This result was previously obtained by Dacorogna & Murat [**23**]. They also showed that $W^{N^2/(N+2)}$ is not an admissible domain when equipped with the weak convergence «only». Also, Iwaniec [**39**] proved that for $N = 2$ the functional $\mathcal{H}_2^*$ is not weakly continuous, precisely there exists a sequence $\{u_n\}$ converging weakly to zero in $W^{1,2}(\Omega)$ (even in $W^{1,p}(\Omega)$ weak with $p > 2$, so this phenomenon is not a «borderline»-type result) such that $H_2^*(u_n)$ does not go to zero in $\mathcal{D}'(\Omega)$, i.e. $W^{1,2}(\Omega)$ with the weak convergence is not an admissible domain for the weak Hessian.

Jerrard & Jung [**43**] showed that $BV^2(\Omega) \cap W^{1,\infty}(\Omega)$ with the strict convergence on BV^2 and weak* convergence on $W^{1,\infty}$ is an admissible domain for the weak Hessian, where $BV^2(\Omega) := \{u \in W^{1,1}(\Omega) : \nabla u \in BV(\Omega; \mathbb{R}^N)\}$.

Clearly, if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ with $p \geq \dfrac{N^2}{N+1}$, then $\mathcal{H}u = \mathrm{Det}\,\nabla(\nabla u)$. In Theorem 1.3 it was important to exploit the structure of second order derivatives. Had we looked only at Det $\nabla(\nabla u)$ we would have needed $W^{2,\frac{N^2}{N+1}}(\Omega)$ in order to draw a similar conclusion.

In connection with the Monge-Ampère equation, several authors have introduced measure-valued Hessian operators on the family of convex functions equipped mostly with the topology of locally uniform convergence, see e.g. Alexandrov [**4**], Bakelman [**6**], Trudinger & Wang [**61**], Alberti & Ambrosio [**2**] and references therein. In fact, the weak Hessian of a convex function is the weak Jacobian of its subgradient in the sense of [**2**].

Next we compare $\mathcal{H}(u)$ with the pointwise Hessian. We will regularize the weak Hessian by a standard family $\{\psi_\varepsilon\}_{\varepsilon>0}$ of mollifiers, i.e. ψ_ε are nonnegative functions from $C_c^\infty(\mathbb{R}^N)$ with support in $\overline{B}(0,\varepsilon)$ such that $\int \psi_\varepsilon \, dx = 1$, and they obey the scaling rule

$$\psi_\varepsilon(x) = \varepsilon^{-N}\psi_1(x/\varepsilon), \qquad x \in \mathbb{R}^N, \ \varepsilon > 0. \tag{5}$$

Theorem 1.4. *Let $N = 2$ and $u \in W^{1,2}(\Omega) \cap BV^2(\Omega)$, or let $N \geq 3$ and $u \in W^{2,p}(\Omega)$ with $p \geq \dfrac{N^2}{N+2}$. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a standard family of mollifiers. Then for a.e. $x \in \Omega$, $(\mathcal{H}u * \psi_\varepsilon)(x) \to \det \nabla^2 u(x)$, where $\nabla^2 u$ stands for the absolutely continuous part of $D^2 u$ with respect to the 2-dimensional Lebesgue measure when $N = 2$. In particular, if $\mathcal{H}u$ is a Radon measure then its absolutely continuous part with respect to the N-dimensional Lebesgue measure is $\det \nabla^2 u$.*

Iwaniec [**39**] established this result for $u \in W^{2,N-1}(\Omega)$; note that $W^{2,N-1}(\Omega) \subset W^{2,\frac{N^2}{N+2}}(\Omega)$. For the case of the Jacobian, we refer to Müller [**51**] (see also Iwaniec &

Martin [**40**], Iwaniec and Sbordone [**41**], Greco [**37**], Müller, Qi and Yan [**54**], Fonseca and Leoni [**29**] for other so called det = Det results).

It is interesting here to call the attention to the fact that different admissible domains do carry different properties. We know from Müller [**51**] that the analog of Theorem 1.4 holds for Det∇u provided $p \geq \dfrac{N^2}{N+1}$. This is false in the admissible space $W^{1,N-1}(\Omega;\mathbb{R}^N)\cap\mathbb{L}^\infty(\Omega;\mathbb{R}^N)$. We observe that if the admissibility of the space is based on the duality pairing between adj $\nabla u \in L^{p/(N-1)}$ and $u\in L^s$ and $W^{1,p}$ is not embedded to L^s (such is the case of $W^{1,N-1}\cap L^\infty$), then the weak Jacobian loses a part of its stability. Precisely,

Theorem 1.5. *Let $N-1\leq p<N^2/(N+1)$. Then there exist a bounded function $u\in W^{1,p}(Q(0,1),\mathbb{R}^N)$, a standard family $\{\psi_\varepsilon\}$ of mollifiers, and a set $E\subset\mathbb{R}^N$ of positive measure such that $u=0$ on E, with*

$$\limsup_{\varepsilon\to 0+}(\mathrm{Det}\nabla u * \psi_\varepsilon)(x)>0, \qquad x\in E.$$

Similarly to Theorem 1.5, we can prove

Theorem 1.6. *Let $N-2\leq p<N^2/(N+2)$. Then there exist a Lipschitz function $u\in W^{2,p}(Q(0,1),\mathbb{R}^N)$, a standard family $\{\psi_\varepsilon\}$ of mollifiers, and a set $E\subset\mathbb{R}^N$ of positive measure such that $u=0$ on E, with*

$$\limsup_{\varepsilon\to 0+}(\mathcal{H}u * \psi_\varepsilon)(x)>0, \qquad x\in E.$$

Theorems 1.5 and 1.6 will be proved in a subsequent paper [**33**].

At this point, we may ask what can we say about the dimensionality of the singular part of $\mathcal{H}u$ when this is a Radon measure. Müller in [**53**] proved that if $a\in(0,N)$ then there exists $u\in W^{1,p}(\Omega;\mathbb{R}^N)\cap\mathbb{C}(\overline{\Omega};\mathbb{R}^N)$ for all $1\leq p<N$ such that

$$\mathrm{Det}\,\nabla u = \det\nabla u\mathcal{L}^N\lfloor\Omega+\mu_s$$

where μ_s is a positive Radon measure, singular with respect to $\mathcal{L}^N$ and such that supp μ_s is a closed set of Hausdorff dimension a. The a-dimensionality of μ_s is supported by the fact that μ_s is absolutely continuous with respect to $\mathcal{H}^a$. The Hessian counterpart of this existence result is also true, and it was established by Alberti & Ambrosio [**2**].

Next, we move on to relaxation results for the weak Hessian. We first extend to the k-th order context the relaxation results obtained for first order problems (see Fonseca & Malý [**32**], Bouchitté, Fonseca & Malý [**12**]; see also Marcellini [**48**], Fonseca & Marcellini [**34**]). Precisely, let

$$F(u,U) = \int_U f(\nabla^k u)\,dx \tag{6}$$

where U is an open subset of the open bounded set $\Omega \subset \mathbb{R}^N$, $f : E_k^{N\times d} \to \mathbb{R}$ is a continuous function and $E_k^{N\times d}$ denotes the space of all symmetric k-linear maps from $\mathbb{R}^N$ to $\mathbb{R}^d$. Consider

$$\mathcal{F}(u,U,(W^{k,p},w)) := \inf_{\{u_n\}}\Big\{\liminf_n F(u_n,U) \colon u_n \in \mathcal{C}^\infty(U), \quad u_n \rightharpoonup u \text{ in } W^{k,p}(U)\Big\}.$$

The following theorem has been proven in the particular case where $k \geq 2$ and $q < \dfrac{Nk-N}{Nk-N-1}\,p$ by Esposito & Mingione [**26**]. We note $\dfrac{N}{N-1} > \dfrac{Nk-N}{Nk-N-1}$ if $k > 2$.

Theorem 1.7. *Let F satisfy*

$$\gamma|\xi|^p \leq f(\xi) \leq C(1+|\xi|^q). \tag{7}$$

with

$$1 < p \leq q < \frac{N}{N-1}\,p,\ \gamma \geq 0. \tag{8}$$

If $u \in W^{k,p}(\Omega,\mathbb{R}^d)$ and $\mathcal{F}(u,\Omega,(W^{k,p},w)) < \infty$ then there exists a Radon measure $\mathcal{R}(u,\cdot)$ on Ω such that

$$\mathcal{R}(u,U) = \mathcal{F}(u,U,(W^{k,p},w))$$

for every open set $U \subset \Omega$. Moreover, the absolutely continuous part of $\mathcal{R}(u,\cdot)$ is $Q_k f(\nabla^k u)$.

We recall that by $Q_k f$ we denote the k-quasiconvex envelope of f, namely

$$Q_k f(\xi) = \inf\Big\{ ⨍_B f(\xi + \nabla^k\varphi)\,dx,\ \varphi \in C_c^\infty(B)\Big\},$$

here B is the unit ball in $\mathbb{R}^N$.

The method of Bouchitté, Fonseca & Malý [**12**] is based on extension and trace preserving operators. Here we modify the method to avoid the use of extension operators and the argument is now based on trace preserving operators. Not only this renders it more transparent, even in the case where $k = 1$ thus simplifying earlier work in [**12**], but also for higher order problems the advantage is notable. In particular, we do not need to differentiate boundary traces when dealing with $k \geq 2$!

With this in hand, we take $d = 1$, $q = N$, $k = 2$, set

$$F(u, U) := \int_U |\det \nabla^2 u| \, dx,$$

and denote the corresponding relaxation and relaxation measure by $\mathcal{F}_H$, $\mathcal{R}_H$, respectively. We draw a parallel with the work of Fonseca, Fusco & Marcellini [**27**], [**28**], by establishing

Theorem 1.8. *Let $u \in W^{2,p}(\Omega)$. If $p > N-1$ and if $\mathcal{F}_H(u, \Omega, (W^{2,p}, w)) < \infty$ then there exists a Radon measure $\mathcal{R}_H(u, \cdot)$ such that*

$$\mathcal{R}_H(u, U) = \mathcal{F}_H(u, U, (W^{2,p}, w))$$

for every open set $U \subset \Omega$. If $p > \dfrac{N^2}{N+2}$ and if $\mathcal{F}_H(u, \Omega, (W^{2,p}, w)) < \infty$ then $\mathcal{H}u = \mathcal{H}_2^* u$ is a Radon measure and its total variation measure satisfies the inequality

$$|\mathcal{H}u| \le \mathcal{F}_H(u, \cdot, (W^{2,p}, w)).$$

Again here we remark that often we cannot reduce these second order problems to the first order setting. To illustrate this, we exhibit two examples on which

$$\mathcal{F}_J(\nabla u, B(0,1), (W^{1,p}, w)) = 0 < \mathcal{H}(u)(B(0,1)) \le \mathcal{F}_H(u, B(0,1), (W^{2,p}, w)),$$

where for $q = d = N$, $k = 1$, $\mathcal{F}_J$ stands for the relaxation of

$$u \mapsto \int_U |\det \nabla u| \, dx.$$

Theorem 1.9. *Suppose that $N - 1 \ge p > N^2/(N+2)$. Then there exists a function $u \in W^{2,p}(B(0,1))$ such that*

$$\mathcal{R}_J(\nabla u, \cdot) = 0 \ne \mathcal{H}u = |\mathcal{H}u| \le \mathcal{R}_H(u, \cdot)$$

If $p = N - 1$ then the first inequality in Theorem 1.9 was established by Acerbi & Dal Maso [**1**] with $u(x) := |x|$ $\left(\text{thus } \nabla u(x) = \dfrac{x}{|x|}\right)$, while if $p < N - 1$ then Malý [**47**] proved that the first inequality holds for the smooth function $u(x) = |x|^2$. In both cases the inequality $\mathcal{H}(u)(B(0,1)) > 0$ is obtained by direct inspection.

Finally, we recall the class of functions considered in Fonseca, Fusco & Marcellini [**28**]. Let $B(0,1) \subset \mathbb{R}^2$ be the unit ball, and suppose that $u : B \to \mathbb{R}^2$ is a zero homogenous function of type

$$u(re^{-it}) := \xi(t),$$

where $\xi : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function. Then it can be shown that

$$\operatorname{Det} u = c\delta_0, \text{ with } c = \int_{\mathbb{R}^2} \operatorname{Ind}_\xi z \, dz_1 \, dz_2, \tag{9}$$

whereas $\mathcal{F}_J(u, \cdot, (W^{1,p}, w)$ with $p > 1$ is a Radon measure $\mathcal{R}_J(u, \cdot)$ and

$$\mathcal{R}_J(u, \cdot) = c'\delta_0 \text{ with } c' \geq \int_{\mathbb{R}^2} |\operatorname{Ind}_\xi z| \, dz_1 \, dz_2. \tag{10}$$

It is clear, therefore, that cancellations may occur when computing $|\operatorname{Det} u|$ and an example of strict inequality

$$|\operatorname{Det} u|(B(0,1)) < \mathcal{R}_J(u, B(0,1))$$

may be easily found.

Notice that the inequality in (10) can be strict. The example is based on an «eight like curve» invented originally in connection with the theory of Cartesian currents, see Malý [**46**], Giaquinta, Modica and Souček [**35**], [**36**], Mucci [**50**].

There is a two-dimensional example of a function $u \in W^{2,p}(B)$, $p > 1$, the so-called «fish-like example» such that

$$\nabla u(re^{-it}) = \xi(t),$$

where ξ is a 2π-periodic curve,

$$\mathcal{H}u = c\delta_0, \qquad \mathcal{R}_H(u, \cdot) = c'\delta_0$$

and

$$c' > c \geq 0.$$

As a consequence we obtain

Theorem 1.10. *Suppose that $N = 2$ and $1 < p < 2$. Then there exists a function $u \in W^{2,p}(B(0,1))$ such that $|\mathcal{H}u| \neq \mathcal{R}_H(u, \cdot)$.*

The fish-like example will be explained in a subsequent paper [**33**].

2 - Continuity properties of the distributional Hessian. The Proofs of Theorems 1.3 and 1.4

The proof of Theorem 1.3 is identical to that of Dacorogna & Murat [**23**] for $p > N^2/(N+2)$. For completeness we recall it below.

Proof of Theorem 1.3 Suppose first that $N = 2$ and let $\{u_n\}$ be a sequence of functions converging to u strongly in $W^{1,2}(\Omega)$. If $\varphi \in C_c^\infty(\Omega)$ then clearly

$$\langle \mathcal{H}u_n, \varphi\rangle \to \langle \mathcal{H}u, \varphi\rangle.$$

If $N \geq 3$, then let $\{u_n\}$ be a sequence of functions converging to u strongly in $W^{1,s}(\Omega)$ and weakly in $W^{2,p}(\Omega)$. Fix $\varphi \in C_c^\infty(\Omega)$ and write

$$\langle \mathcal{H}u_n, \varphi\rangle = -\frac{1}{N!}\sum_{p,q\in P} \operatorname{sgn} p \operatorname{sgn} q \int_\Omega D_{p_1}u_n D_{q_1}u_n D_{p_2q_2}\varphi D_{p_3q_3}u_n \ldots D_{p_Nq_N}u_n\,dx.$$

We have

$$D_{p_1}u_n D_{q_1}u_n \to D_{p_1}u D_{q_1}u \text{ in } L^{s/2},$$

and, using the well know theory of weak convergence of minors, we obtain

$$dD_{p_3}u_n \wedge \ldots \wedge dD_{p_N}u_n \rightharpoonup dD_{p_3}u \wedge \ldots \wedge dD_{p_N}u$$

weakly in $L^{p/(N-2)}$ (or weakly* in the sense of measures if $p = N-2$). Thus we conclude immediately that

$$\langle \mathcal{H}u_n, \varphi\rangle \to \langle \mathcal{H}u, \varphi\rangle. \qquad \square$$

Next,

Proof of Theorem 1.4. If $p \geq N$ then the result is trivial as $\mathcal{H}(u)$ reduces to $\det\nabla^2 u \in L^{p/N}(\Omega)$. Suppose now that $p < N$.

Step 1. We treat first the case where $N = 2$. Let $x_0 \in \Omega$ be a Lebesgue point for u, for the absolutely continuous part $\nabla^2 u$ of D^2u and is such that $(D^2u)_s(B(x_0,\varepsilon)) = o(\varepsilon^N)$ as $\varepsilon \to 0$. We write $u(x) =: u(x_0) + \nabla u(x_0)(x-x_0) + \frac{1}{2}\nabla^2 u(x_0)(x-x_0)\cdot(x-x_0) + v(x)$, so that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon)} |\nabla v(x)|\,dx = 0 \quad \text{and} \lim_{\varepsilon\to 0}\frac{|D^2v(B(x_0,\varepsilon))|}{\varepsilon^2} = 0. \tag{11}$$

We abbreviate

$$A(x) := u(x_0) + \nabla u(x_0)(x-x_0),$$
$$Q(x) := \frac{1}{2}\nabla^2 u(x_0)(x-x_0)\cdot(x-x_0).$$

Then $u = A + Q + v$ and

$$\begin{aligned}(\mathcal{H}u*\psi_\varepsilon)(x_0) &= -\frac{1}{2}\sum_{p,q\in P}\operatorname{sgn} p \operatorname{sgn} q \int_\Omega D_{p_1}uD_{q_1}uD_{p_2q_2}\psi_\varepsilon(x_0-x)\,dx\\ &= (\mathcal{H}\,Q*\psi_\varepsilon)(x_0) + \mathcal{R}_\varepsilon\\ &= \det\nabla^2 u(x_0) + \mathcal{R}_\varepsilon,\end{aligned}$$

where

$$\mathcal{R}_\varepsilon = -\frac{1}{2}\sum_{p\in P} \operatorname{sgn} p\,\mathcal{R}_{\varepsilon,p} \quad \text{with} \quad R_{\varepsilon,p} = L^1_\varepsilon + L^2_\varepsilon + L^3_\varepsilon + L^4_\varepsilon + L^5_\varepsilon,$$

and for a fixed permutation p of $\{1,2\}$ we have

$$L^1_\varepsilon := \int_\Omega D_{p_1}A\,du \wedge dD_{p_2}\psi_\varepsilon,$$

$$L^2_\varepsilon := \int_\Omega D_{p_1}(Q+v)\,dA \wedge dD_{p_2}\psi_\varepsilon,$$

$$L^3_\varepsilon := \int_\Omega D_{p_1}Q\,dv \wedge dD_{p_2}\psi_\varepsilon,$$

$$L^4_\varepsilon := \int_\Omega D_{p_1}v\,dQ \wedge dD_{p_2}\psi_\varepsilon,$$

and

$$L^5_\varepsilon := \int_\Omega D_{p_1}v\,dv \wedge dD_{p_2}\psi_\varepsilon.$$

We claim that $\mathcal{R}_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed,

$$L^1_\varepsilon = 0$$

because

$$\int_\Omega du \wedge d\eta = 0 \quad \text{for all } \eta \in C^\infty_c(\Omega),$$

and since

$$L^2_\varepsilon := \int_\Omega D_{p_1}u\,dA \wedge dD_{p_2}\psi_\varepsilon,$$

collecting all terms of type I^2_ε together we notice that

$$\sum_{p,q\in P} \operatorname{sgn} p \operatorname{sgn} q \int_\Omega D_{p_1}uD_{q_1}AD_{p_2q_2}\psi_\varepsilon\,dx = \sum_{q\in P} \operatorname{sgn} q \int_\Omega D_{q_1}Adu \wedge dD_{q_2}\psi_\varepsilon = 0,$$

where in the last equality we used the same argument as for L^1_ε.

Using the fact that $\|\nabla^2\psi_\varepsilon\|_{L^\infty} \le C/\varepsilon^4$ and that $\|\nabla Q\|_{L^\infty} \le \varepsilon$, we have

$$\left|L_\varepsilon^3\right| + \left|L_\varepsilon^4\right| \le \frac{C}{\varepsilon} \fint_{B(x_0,\varepsilon)} |\nabla v(x)|\,dx \le C \lim_{\varepsilon\to 0} \frac{\left|D^2 v(B(x_0,\varepsilon))\right|}{\varepsilon^2} \to 0,$$

by (11), and where we used Poincaré's inequality in BV (see [**5**] (3.42)), and the fact that in L_ε^3 we could have written $v - \fint_{B(x_0,\varepsilon)} v(y)\,dy$ in place of v (the same applies to L_ε^4 and L_ε^5). Finally

$$\left|L_\varepsilon^5\right| \le \frac{C}{\varepsilon^2} \fint_{B(x_0,\varepsilon)} |\nabla v(x)|^2\,dx \le C \lim_{\varepsilon\to 0} \frac{\left|D^2 v(B(x_0,\varepsilon))\right|}{\varepsilon^2} \to 0,$$

by the Sobolev-Poincaré inequality since here $p^* = 1^* = 2$.

Step 2. Let $N \ge 3$ and assume that $\dfrac{N^2}{N+2} \le p < N$. Consider a point $x_0 \in \Omega$ which is a p-Lebesgue point for u, ∇u and $\nabla^2 u$. As in Step 1 we write $u(x) =: u(x_0) + \nabla u(x_0)(x - x_0) + \dfrac{1}{2}\nabla^2 u(x_0)(x-x_0)\cdot(x-x_0) + v(x) = A(x) + Q(x) + v(x)$, so that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{B(x_0,\varepsilon)} |\nabla^2 v(x)|^p\,dx = 0, \qquad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{B(x_0,\varepsilon)} |\nabla v(x)|^p\,dx = 0. \tag{12}$$

Just as above

$$\begin{aligned}(\mathcal{H}u * \psi_\varepsilon)(x_0) &= -\frac{1}{N!} \sum_{p,q\in P} \operatorname{sgn} p \operatorname{sgn} q \int_\Omega D_{p_1} u D_{q_1} u D_{p_2 q_2} \psi_\varepsilon(x_0 - x) D_{p_3 q_3} u \dots D_{p_N q_N} u\,dx \\ &= (\mathcal{H}Q * \psi_\varepsilon)(x_0) + \mathcal{R}_\varepsilon \\ &= \det \nabla^2 u(x_0) + \mathcal{R}_\varepsilon,\end{aligned}$$

where now

$$\mathcal{R}_\varepsilon = -\frac{1}{N!} \sum_{p\in P} \operatorname{sgn} p\ R_{\varepsilon,p} \quad \text{and } R_{\varepsilon,p} := I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3 + I_\varepsilon^4 + I_\varepsilon^5 + I_\varepsilon^6,$$

and for a fixed permutation p of $\{1,\dots,N\}$ we have

$$I_\varepsilon^1 := \int_\Omega D_{p_1} A\,du \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \dots \wedge dD_{p_N} u,$$

$$I_\varepsilon^2 := \int_\Omega D_{p_1}(Q + v)\,dA \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \dots \wedge dD_{p_N} u,$$

$$I_\varepsilon^3 := \int_\Omega D_{p_1} Q\, dQ \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u$$

$$- \int_\Omega D_{p_1} Q\, dQ \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} Q \wedge \ldots \wedge dD_{p_N} Q,$$

$$I_\varepsilon^4 := \int_\Omega D_{p_1} Q\, dv \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u,$$

$$I_\varepsilon^5 := \int_\Omega D_{p_1} v\, dQ \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u,$$

and

$$I_\varepsilon^6 := \int_\Omega D_{p_1} v\, dv \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u.$$

In order to prove that $\mathcal{R}_\varepsilon \to 0$ as $\varepsilon \to 0$, we observe first that

$$I_\varepsilon^1 = 0$$

because

$$\int_\Omega du \wedge d\eta \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u = 0 \quad \text{for all}\, \eta \in C_c^\infty(\Omega).$$

Also, since

$$\int_\Omega D_{p_1} A\, dA \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u = 0,$$

we have

$$I_\varepsilon^2 := \int_\Omega D_{p_1} u\, dA \wedge dD_{p_2}\psi_\varepsilon \wedge dD_{p_3} u \wedge \ldots \wedge dD_{p_N} u,$$

and collecting all terms of type I_ε^2 together we notice that

$$\sum_{p,q\in P} \operatorname{sgn} p \operatorname{sgn} q \int_\Omega D_{p_1} u D_{q_1} A D_{p_2 q_2}\psi_\varepsilon D_{p_3 q_3} u \ldots D_{p_N q_N} u\, dx$$

$$= \sum_{q\in P} \operatorname{sgn} q \int_\Omega D_{q_1} A\, du \wedge dD_{q_2}\psi_\varepsilon \wedge\ dD_{q_3} u \ldots \wedge dD_{q_N} u = 0,$$

where in the last equality we used the same argument as for I_ε^1.

Using the fact that $\|\nabla^2\psi_\varepsilon\|_{L^\infty} \le C/\varepsilon^{N+2}$ and that $\|\nabla Q\|_{L^\infty} \le \varepsilon$, we have

$$|I_\varepsilon^3| \le C \sum_{k=1}^{N-2} \fint_{B(x_0,\varepsilon)} |\nabla^2 v(x)|^k \, dx \to 0$$

by (12). Also, noting that in I_ε^4, I_ε^5 and I_ε^6 we could have written $v - \fint_{B(x_0,\varepsilon)} v(y)\,dy$ in place of v, using Poincaré's inequality and the fact that $\frac{1}{p*} + \frac{N-2}{p} \le 1$, with $p* := \frac{Np}{N-p}$, we have

$$\begin{aligned}|I_\varepsilon^4| &\le \frac{C}{\varepsilon} \fint_{B(x_0,\varepsilon)} |\nabla v||\nabla^2 u|^{N-2}\,dx \\ &\le \frac{C}{\varepsilon}\Big(\fint_{B(x_0,\varepsilon)} |\nabla v|^{p*}\,dx\Big)^{1/p*}\Big(\fint_{B(x_0,\varepsilon)} |\nabla^2 u|^p\,dx\Big)^{(N-2)/p} \\ &\le C\Big(\fint_{B(x_0,\varepsilon)} |\nabla^2 v|^p\,dx\Big)^{1/p}\Big(\fint_{B(x_0,\varepsilon)} |\nabla^2 u|^p\,dx\Big)^{(N-2)/p}.\end{aligned}$$

The boundedness of $\fint_{B(x_0,\varepsilon)} |\nabla^2 u|^p\,dx$ and (12) yield

$$\lim_{\varepsilon\to 0} I_\varepsilon^4 = 0.$$

It can be seen easily that I_ε^5 may be treated as I_ε^4, and since $\frac{2}{p*} + \frac{N-2}{p} \le 1$,

$$\begin{aligned}|I_\varepsilon^6| &\le \frac{C}{\varepsilon^2} \fint_{B(x_0,\varepsilon)} |\nabla v|^2|\nabla^2 u|^{N-2}\,dx \\ &\le C\Big[\frac{1}{\varepsilon}\Big(\fint_{B(x_0,\varepsilon)} |\nabla v|^{p*}\,dx\Big)^{1/p*}\Big]^2\Big(\fint_{B(x_0,\varepsilon)} |\nabla^2 u|^p\,dx\Big)^{(N-2)/p} \\ &\le C\Big(\fint_{B(x_0,\varepsilon)} |\nabla^2 v|^p\,dx\Big)^{2/p}\Big(\fint_{B(x_0,\varepsilon)} |\nabla^2 u|^p\,dx\Big)^{(N-2)/p} \to 0,\end{aligned}$$

and this concludes the proof.

3 - Relaxation of variational integrals. The Proofs of Theorems 1.7 and 1.8

In this section we follow the arguments introduced in Fonseca & Malý [**32**] and in Bouchitté, Fonseca & Malý [**12**], and we divide the proof into the two propositions below.

Proposition 3.1. *Let F satisfy* (7) *with* (8). *Suppose that $u \in W^{k,p}(\Omega; \mathbb{R}^d)$ and $\mathcal{F}(u, \Omega, (W^{k,p}, w)) < \infty$. Then there exists a Radon measure $\mathcal{R}(u, \cdot)$ on Ω such that*

$$\mathcal{R}(u, U) = \mathcal{F}(u, U, (W^{k,p}, w))$$

for each open set $U \subset \Omega$. Moreover, if $\gamma > 0$ then there exists a «minimizing» sequence $\{u_n\}$ of smooth functions such that $u_n \rightharpoonup u$ in $W^{k,p}(\Omega; \mathbb{R}^d)$ and

$$F(u_n, \cdot) \overset{*}{\rightharpoonup} \mathcal{R}(u, \cdot) \quad \text{in the sense of measures.}$$

Proposition 3.2. *Let F satisfy* (7) *with* (8). *Suppose that $u \in W^{k,p}(\Omega, \mathbb{R}^d)$ and $\mathcal{F}(u, \Omega, (W^{k,p}, w)) < \infty$. Then*

$$\frac{d\mathcal{R}(u, \cdot)}{d\mathcal{L}^N}(x) = Q_k f(x) \qquad \text{a.e. in } \Omega.$$

Proof of Proposition 3.1. There is no novelty here as compared with the results of [**32**], and so we limit ourselves to present a road map of the proof following that paper: Reproduce Theorem 3.2 in [**32**] with the obvious adaptations. This theorem uses the subbaditivity established in Lemma 3.4 in [**32**], and, in turn, this needs a delicate process of gluing to energy bounded sequences converging to the same target on a small layer with vanishing energy. This is accomplished in Lemma 2.4 in [**32**], and here is where we must replace the then used projection operator by our new trace preserving operator T. The positive exponent τ in Lemma 2.4 now reads $\tau := \dfrac{N}{q} - \dfrac{N-1}{p}$. □

Concerning now Proposition 3.2, and in order to prove the inequality

$$\frac{d\mathcal{R}(u, \cdot)}{d\mathcal{L}^N}(x) \geq Q_k f(x) \qquad \text{a.e. in } \Omega,$$

we could have tried to show directly that if $u_n \rightharpoonup u$ in $W^{k,p}(\Omega; \mathbb{R}^d)$ then

$$\int_\Omega Q_k f(\nabla^k u)\, dx \leq \liminf_{n\to\infty} \int_\Omega f(\nabla^k u_n)\, dx.$$

Here we are tempted to use the results in [**24**] to reduce this to the 1-quasiconvexity setting and then apply the lowersemicontinuity result of Fonseca & Malý [**32**] for 1-quasiconvex functions within the gap range of (8). However, this is not possible as (see Theorems 1.2 and 1.3 in [**24**]) in order to obtain the (approximate) extensions of the k-quasiconvex envelope of f to 1-quasiconvex functions, we would need to consider q-coercive perturbations of f, $f_\varepsilon := f + \varepsilon|\cdot|^q$, and we do not have an uniform bound for $\|\nabla^k u_n\|_{L^q}$.

An important tool to prove Proposition 3.2 will be the *trace preserving operator T*

that will play the role of the projection operator introduced in [**32**] and later used in [**12**]. This operator could be constructed via an argument similar to that of Th. 3.6.2 in [**62**], however here we opt for a method using Whitney balls. In what follows, we denote $\rho(x) = \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega)$. We recall that a countable collection $\mathcal{W} = \{B_i\}_i$ of open balls is called a *family of Whitney balls* for Ω, if (see e.g. [**20**])

(W1) if $B(x,r) \in \mathcal{W}$, then $10r = \rho(x)$.

(W2) the family $\mathcal{W}$ covers Ω;

(W3) the family $\{B(x,4r) : B(x,r) \in \mathcal{W}\}$ has a bounded overlap multiplicity by a constant depending only on N.

Let $\{\omega_i\}_i$ be a partition of unity subordinated to $\mathcal{W}$, i.e. ω_i are nonnegative smooth functions, $\sum_i \omega_i = 1$ in Ω, $B(x_i, r_i) \subset \operatorname{spt}\omega_i \subset B(x_i, 2r_i)$, and

$$|\nabla^j \omega_i| \le C_j / r_i^j, \qquad j = 1, 2 \dots. \tag{13}$$

The construction of ω_i is the following: Let $\tilde{\omega}$ be a smooth cutoff function between $B(0,1)$ and $B(0,2)$ and set

$$\tilde{\omega}_i(x) := \tilde{\omega}\Big(\frac{x - x_i}{r_i}\Big), \qquad i = 1, 2, \dots.$$

Define

$$\begin{aligned} \omega_1 &:= \tilde{\omega}_1, \\ \omega_i &:= \tilde{\omega}_i (1 - \tilde{\omega}_1) \dots (1 - \tilde{\omega}_{i-1}), \qquad i = 2, 3, \dots. \end{aligned}$$

The estimate (13) follows from the bounded overlap multiplicity of $\mathcal{W}$, so that the product defining ω_i has a bounded number of factors different from 1.

Denote $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, $\varepsilon > 0$, and consider a family $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ of mollifiers such that φ_ε are functions in $C_c^\infty(\mathbb{R}^N)$ with support in $\overline{B}(0,\varepsilon)$, as it is usual φ_ε obey the scaling rule

$$\varphi_\varepsilon(x) = \varepsilon^{-N} \varphi_1(\chi/\varepsilon), \qquad \chi \in \mathbb{R}^N, \ \varepsilon > 0,$$

and the property

$$\varphi_\varepsilon * P = P \tag{14}$$

is satisfied for every polynomial P of degree $\le k$ and all $\varepsilon > 0$ (for the existence of $\{\varphi_\varepsilon\}$ see Lemma 3.5.6. in [**62**]). Remark that, in view of (14), we cannot assume that these mollifiers are nonnegative. Given $u \in W^{1,p}(\Omega)$, for each $i \in \mathbb{N}$ let

$$T_i u := \begin{cases} \varphi_{r_i} * u & \text{in } \Omega_{r_i}, \\ 0 & \text{elsewhere,} \end{cases}$$

and set

$$Tu := \sum_i \omega_i \, T_i u.$$

Notice that the jump of $T_i u$ is far from the support of ω_i, so that Tu is smooth in Ω. Indeed,

$$\operatorname{supp}\omega_i \subset B(x_i, 2r_i) \subset B(x_i, 9r_i) \subset \Omega_{r_i}.$$

Theorem 3.3. *Let $1 < p \le q < \dfrac{N}{N-1}\,p$. Then the operator T has the following properties:*

(T1) $|\nabla^l Tu(x)| \le C \displaystyle\fint_{B(x_i,3r_i)} |\nabla^l u|\,dy$ if $u \in W^{k,p}(\Omega)$, $x \in B(x_i, r_i)$, $l = 0, 1 \ldots, k$;

(T2) $\Big(\displaystyle\fint_{B(x_i,r_i)} |\nabla^l Tu|^q\,dx\Big)^{1/q} \le C\Big(\displaystyle\fint_{B(x_i,3r_i)} |\nabla^l u|^p\,dx\Big)^{1/p}$ if $u \in W^{k,p}(\Omega)$, $l = 0, \ldots, k$;

(T3) *$T : W^{k,p}(\Omega) \to W^{k,p}(\Omega)$ is a bounded linear operator;*

(T4) *if $u \in W^{k,p}(\Omega)$ then $Tu - u \in W^{k,p}_0(\Omega)$;*

(T5) $\Big(\displaystyle\int_{\Omega\setminus\Omega_\varepsilon} |\nabla^k\, Tu|^q\,dx\Big)^{1/q} \le C\varepsilon^{\frac{N}{q}-\frac{N-1}{p}} \sup_{0<\delta\le 2\varepsilon} \delta^{-1/p}\Big(\displaystyle\int_{\Omega\setminus\Omega_\delta} |\nabla^k u|^p\,dx\Big)^{1/p}$ *for $u \in W^{k,p}(\Omega)$.*

Proof. *Step 1.* Suppose that $B(x_i, 2r_i) \cap B(x_j, 2r_j) \ne \emptyset$. Then

$$10r_j = \rho(x_j) \le \rho(x_i) + |x_i - x_j| \le 10r_i + 2r_i + 2r_j,$$

and it follows that

$$\begin{aligned} &r_j \le \frac{3}{2} r_i, \\ &B(x, r_j) \subset B(x_i, r_i + r_j) \subset B(x_i, 3r_i), \qquad x \in B(x_i, r_i). \end{aligned} \tag{15}$$

Interchanging the roles of i and j, we also have

$$r_i \le \frac{3}{2} r_j. \tag{16}$$

Suppose that $x \in B(x_i, r_i)$ and that

$$j \in I(i) := \{j :\ B(x_j, 2r_j) \cap B(x_i, r_i) \ne \emptyset\}.$$

Notice that by the property (W3) of the Whitney covering,

$$\#\, I(i) \le C. \tag{17}$$

Then for $l \in \{0,\dots,k\}$ we have

$$\begin{aligned}\nabla^l Tu(x) &= \sum_{m=0}^{l}\sum_j C_m \nabla^{l-m}\omega_j(x)(\varphi_{r_j} * \nabla^m u)(x)\\ &= \sum_{m=0}^{l}\sum_{j\in I(i)} C_m \nabla^{l-m}\omega_j(x)(\varphi_{r_j} * (\nabla^m u - P_m))(x)\end{aligned}$$

for some constants $C_m \in \mathbb{R}$ and certain polynomials P_m of order $l-m-1$ for which the Poincaré inequality

$$\int_{B(x_i,3r_i)} |\nabla^m u - P_m|\,dy \le C r_i^{l-m} \int_{B(x_i,3r_i)} |\nabla^l u|\,dy \tag{18}$$

holds (set $P=0$ if $m=l$). Note that here we used the fact that, by virtue of (14), we can write

$$\begin{aligned}\sum_{j\in I(i)} \nabla^{l-m}\omega_j(x)\ (\varphi_{r_i} * P_m)(x) &= \sum_{j\in I(i)} \nabla^{l-m}\omega_j(x)\ P_m(x)\\ &= P_m(x)\nabla^{l-m}\sum_j \omega_j(x) = P_m(x)\nabla^{l-m}1 = 0.\end{aligned}$$

Using (13), (16), (17) and (18) we now have

$$\begin{aligned}|\nabla^l Tu(x)| &\le C\sum_{m=0}^{l}\sum_{j\in I(i)} \big|\nabla^{l-m}\omega_j(\varphi_{r_i} * (\nabla^m u - P_m))\big|(x)\\ &\le C\sum_{m=0}^{l}\sum_{j\in I(i)} r_j^{m-l} \fint_{B(x,r_j)} |\nabla^m u - P_m|(y)\,dy\\ &\le C\sum_{m=0}^{l} r_i^{m-l} \fint_{B(x_i,3r_i)} |\nabla^m u - P_m|(y)\,dy\\ &\le C \fint_{B(x_i,3r_i)} |\nabla^l u|\,dy.\end{aligned}$$

Step 2. (T2) is an obvious consequence of (T1).

Step 3. A standard application of the properties of the Whitney covering and of (T2) with the choice $q=p$ yields (T3).

Step 4. Property (T4) follows from (T3) and from the fact that

$$Tu - u = \lim_{j\to\infty}\sum_{i=1}^{j} \omega_i(T_i u - u).$$

Step 5. To prove (T5), set

$$M := \sup_{0<\delta\le 2\varepsilon} \delta^{-1/p}\Big(\int_{\Omega\setminus\Omega_\delta} |\nabla^k u|^p \, dx\Big)^{1/p}.$$

If $x \in B(x_i, r_i) \setminus \Omega_\varepsilon$ then

$$10r_i = \operatorname{dist}(x_i, \partial\Omega) \le \operatorname{dist}(x, \partial\Omega) + |x_i - x| \le r_i + \varepsilon$$

and so $9r_i \le \varepsilon$. We have by (T2)

$$\begin{aligned}
\Big(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla^k\, Tu|^q \, dx\Big)^{1/q} &\le \Big(\sum_{j=3}^{\infty} \sum_{\{i:2^{-j-1}<\frac{r_i}{\varepsilon}<2^{-j}\}} \int_{B(x_i,r_i)} |\nabla^k\, Tu|^q \, dx\Big)^{1/q} \\
&\le C\Big(\sum_{j=3}^{\infty} \sum_{\{i:2^{-j-1}<\frac{r_i}{\varepsilon}<2^{-j}\}} r_i^{N(1-\frac{q}{p})} \Big(\int_{B(x_i,3r_i)} |\nabla^k\, u|^p \, dx)^{q/p}\Big)^{1/q} \\
&\le C\Big(\sum_{j=3}^{\infty} (2^{-j}\varepsilon)^{N(1-\frac{q}{p})} \Big(\sum_{\{i:2^{-j-1}<\frac{r_i}{\varepsilon}<2^{-j}\}} \int_{B(x_i,3r_i)} |\nabla^k\, u|^p \, dx)^{q/p}\Big)^{1/q} \\
&\le C\Big(\sum_{j=3}^{\infty} (2^{-j}\varepsilon)^{N(1-\frac{q}{p})} \Big(\int_{\Omega\setminus\Omega_{2^{4-j}\varepsilon}} |\nabla^k\, u|^p \, dx)^{q/p}\Big)^{1/q} \\
&\le C\Big(\sum_{j=1}^{\infty} (2^{-j}\varepsilon)^{N(1-\frac{q}{p})} (2^{-j}\varepsilon)^{\frac{q}{p}} M^{\frac{q}{p}}\Big)^{1/q} \\
&\le C\varepsilon^{\frac{N}{q}-\frac{N-1}{p}} M^{\frac{1}{p}},
\end{aligned}$$

where we used the facts that if $x \in B(x_i, 3r_i)$ for some r_i such that $2^{-j-1} < \dfrac{r_i}{\varepsilon} < 2^{-j}$ then

$$\operatorname{dist}(x, \partial\Omega) \le \operatorname{dist}(x_i, \partial\Omega) + |x_i - x| \le 10r_i + 3r_i < 2^{4-j}\varepsilon,$$

and

$$\sum_{j=1}^{\infty} 2^{-j(\frac{N}{q}-\frac{N-1}{p})} < +\infty$$

in view of (8). □

The proof of Proposition 3.2 follows closely the proof of Theorem 3.1 in [**12**] with the obvious modifications, where (3.17) is

$$\frac{d\mathcal{R}(u,\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0} \frac{\boldsymbol{m}(u, B(x_0, r))}{\mathcal{L}^N(B(x_0, r))}, \tag{19}$$

and where the limit is taken on a «good» sequence of radii. Here, when $B \subset\subset \Omega$ is an open ball, we set

$$\boldsymbol{m}(u,B) = \inf\{F(v,B) : v \in W^{k,q}(B;\mathbb{R}^d),\ v-u \in W_0^{k,q}(B;\mathbb{R}^d)\}.$$

Proposition 3.2, similarly to Theorem 3.1 in [**12**], now seats on Proposition 3.1, and on the analogs of Lemmas 3.6 and 3.7 in [**12**], that now read as, respectively

Lemma 3.4. *Suppose that* $u \in W^{k,p}(\Omega;\mathbb{R}^d)$ *and* $\mathcal{F}(u,\Omega,(W^{k,p},w)) < \infty$. *Let* $B(x_0,R_0) \subset \Omega$. *Then*

$$\boldsymbol{m}(u,B(x_0,r_0)) \le \mathcal{F}(u,B(x_0,r_0),(W^{k,p},w)) \tag{20}$$

holds for $\mathcal{L}^1$ *a.e.* $r_0 \in (0,R_0)$.

Lemma 3.5. *Suppose that* $u \in W^{k,p}(\Omega;\mathbb{R}^d)$ *and* $\mathcal{F}(u,\Omega,(W^{1,p},w)) < \infty$. *Let* $B(x_0,R_0) \subset \Omega$. *Then* $\mathcal{L}^1$ *a.e.* $r_0 \in (0,R_0)$ *has the following property: If* $v \in W^{k,q}(B(x_0,r_0);\mathbb{R}^d)$ *and* $u - v \in W_0^{k,p}(B(x_0,r_0);\mathbb{R}^d$ *then*

$$\mathcal{R}(w,\partial B(x_0,r_0)) = 0, \tag{21}$$

where

$$w = \begin{cases} v & \text{in } B(x_0,r_0), \\ u & \text{outside } B(x_0,r_0). \end{cases} \tag{22}$$

So we are left with the proofs of Lemmas 3.4 and 3.5, and they both rely on a careful choice of radii provided by the following result.

Lemma 3.6. *Suppose that* $u \in W^{k,p}(\Omega)$ *and* $u_n \in W^{k,q}_{\mathrm{loc}}(\Omega)$, $u_n \rightharpoonup u$ *in* $W^{k,p}(\Omega)$. *Let* $B(x_0,R_0) \subset \Omega$. *Then* $\mathcal{L}^1$ *a.e.* $r_0 \in (0,R_0)$ *has the following property: For* $j = 1,2,\ldots,$ *there exist integers* $n_j \to \infty$, *radii* $r_j \nearrow r_0$, $r_j' \searrow r_0$ *and functions* $v_j \in W^{k,q}_{\mathrm{loc}}(\Omega)$ *such that* $v_j = u_{n_j}$ *on* $\Omega \setminus \big(B(x,r_j') \setminus B(x_0,r_j)\big)$, $v_j - u \in W_0^{k,p}(B(x_0,r_0))$ *and*

$$\int\limits_{B(x_0,r_j')\setminus B(x_0,r_j)} |\nabla^k v_j|^q \, dx \to 0.$$

Proof. Using the Rellich-Kondrashev compact embedding theorem and passing if necessary to a subsequence (not relabeled), we may find $a_n \to \infty$ and a Radon measure υ on Ω such that

$$|\nabla^k u_n|^p + |\nabla^k u|^p + a_n|\nabla^{k-1}(u_n-u)|^p + \ldots + a_n|u_n-u|^p \overset{*}{\rightharpoonup} \upsilon$$

in the sense of measures.

We define a Radon measure ψ on $(0, R_0)$ by

$$\psi(E) := \upsilon(\{x: |x - x_0| \in E\}), \qquad E \subset (0, R_0),$$

and denote

$$\phi(r) := \int\limits_{\partial B(x_0,r)} |\nabla^k u|^p \, dx, \quad 0 < r < R_0.$$

Let $r_0 > 0$ be a radius such that the maximal functions of ϕ and ψ are finite at r_0, i.e.

$$M\phi(r_0) = \sup_{0<\delta<r_0} \frac{1}{\delta} \int\limits_{r_0-\delta}^{r_0+\delta} \phi(s) \, ds < +\infty,$$

$$M\psi(r_0) = \sup_{0<\delta<r_0} \frac{\psi([r_0 - \delta, r_0 + \delta])}{\delta} < +\infty.$$

We abbreviate

$$M := \max\{M\phi(r_0), M\psi(r_0)\}.$$

We first choose $R_j, R_j' \in (0, R_0) \setminus \{r_0\}$,

$$R_j \nearrow r_0, \quad R_j' \searrow r_0, \quad R_j' - r_0 = r_0 - R_j \, .$$

Define

$$\phi_{n,j}(r) := \begin{cases} \displaystyle\int\limits_{\partial B(x_0,r)} (|\nabla^k u_n|^p + |\nabla^k u|^p \\ \qquad + a_n |\nabla^{k-1}(u_n - u)|^p + \ldots + a_n |u_n - u|^p) \, dx & r \in [R_j, R_j'], \\ 0 & r \notin [R_j, R_j']. \end{cases}$$

Set

$$E_{n,j} := \{r \in (0, R_0): M\phi_{n,j}(r) > \lambda\}, \quad \text{where } \lambda := 25(M + 1).$$

Choose n_j so large that

$$a_{n_j} \geq (R_j' - R_j)^{-pk} \tag{23}$$

and

$$\begin{aligned} \int\limits_{R_j}^{R_j'} \phi_{n,j}(r) \, dr = \int\limits_{B(x_0,R_j') \setminus B(x_0,R_j)} & \big(|\nabla^k u_{n_j}|^p + |\nabla^k u|^p \\ & + a_{n_j} |\nabla^{k-1}(u_{n_j} - u)|^p + \ldots + a_{n_j} |u_{n_j} - u|^p\big) \, dx \\ \leq \psi([R_j, R_j']) + R_j' - R_j \leq (M + 1)(R_j' - R_j). & \end{aligned} \tag{24}$$

By the Hardy-Littlewood maximal theorem, we deduce

$$\mathcal{L}^1\big(E_{n,j}\big) \le \frac{5}{\lambda}\int_{R_j}^{R_j'} \phi_{n,j}(r)\,dr \le \frac{5}{\lambda}(M+1)(R_j'-R_j) = \frac{1}{5}(R_j'-R_j), \tag{25}$$

so that there exist

$$r_j \in \Big(R_j, \frac{1}{2}(R_j+r_0)\Big), \quad r_j' \in \Big(\frac{1}{2}(R_j'+r_0), R_j'\Big)$$

such that $r_j, r_j' \notin E_{n,j}$, and thus

$$\max\{M\phi_{n,j}(r_j), M\phi_{n,j}(r_j')\} \le \lambda. \tag{26}$$

With the choice $n = n_j$, define

$$\begin{aligned}
A_j &:= B(x_0, r_j') \setminus \overline{B(x_0, r_j)},\\
D_j &:= A_j \setminus \partial B(x_0, r_0),\\
v_j &:= \begin{cases} u_n & \text{on } \Omega\setminus A_j,\\ T^*u + (1-\eta_j)\, T(u_n-u) & \text{on } A_j,\end{cases}
\end{aligned}$$

where T is the trace preserving operator for A_j, T^* is the trace preserving operator for D_j and η_j is a smooth cutoff function such that

$$\begin{aligned}
\eta_j(x) = 1, &\qquad |x-x_0| \le r_0 + \frac{1}{8}(R_j'-R_j),\\
\eta_j(x) = 0, &\qquad |x-x_0| \ge r_0 + \frac{1}{4}(R_j'-R_j),\\
(r_j'-r_j)^i|\nabla^i\eta_j| \le C, &\quad i = 1,\dots,k.
\end{aligned}$$

Note that the support of η_j is contained in A_j and that nearby $\partial B(x_0, r_0)$ the function $\eta_j = 1$, so that $v_j \in W^{k,p}(\Omega)$ and $v_j - u \in W_0^{k,p}(B(x_0, r_0))$. We have

$$\begin{aligned}
\Big(\int_{A_j} |\nabla^k v|^q\,dx\Big)^{p/q} \le C\Big(&\int_{D_j} |\nabla^k T^*u|^q\,dx\Big)^{p/q}\\
&+ \Big(\int_{A_j} |\nabla^k((1-\eta_j)T(u_n-u))|^q\,dx\Big)^{p/q}.
\end{aligned}$$

We observe that $\operatorname{dist}(x, \partial D_j) \le \operatorname{dist}(x, \partial A_j) < r_j' - r_j$ for every $x \in D_j$. Appealing to the properties of the trace preserving operator, and using (26), we estimate

$$
\begin{aligned}
&\Big(\int_{D_j} |\nabla^k T^* u|^q \, dx\Big)^{p/q} \\
&\quad \le C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} \sup_{0<\delta<2(r_j'-r_j)} \delta^{-1} \int_{\{x\in D_j : \operatorname{dist}(x,\partial D_j)<\delta\}} |\nabla^k u|^p \, dx \\
&\quad \le C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} \sup_{0<\delta<2(r_j'-r_j)} [\delta^{-1} \int_{B(x_0,r_j')\setminus B(x_0,r_j'-\delta)} |\nabla^k u|^p \, dx \\
&\qquad + \delta^{-1} \int_{B(x_0,r_0+\delta)\setminus B(x_0,r_0-\delta)} |\nabla^k u|^p \, dx + \delta^{-1} \int_{B(x_0,r_j+\delta)\setminus B(x_0,r_j)} |\nabla^k u|^p \, dx] \\
&\quad \le C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} (M\phi_{n,j}(r_j') + M\phi(r_0) + M\phi_{n,j}(r_j)) \\
&\quad \le C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} (2\lambda + M) \to 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\Big(\int_{A_j} \nabla^k((1-\eta_j)T(u_n-u))|^q \, dx\Big)^{p/q} \\
&\quad \le C\Big(\int_{A_j} (|\nabla^k T u_n|^q + |\nabla^k T u|^q) \, dx\Big)^{p/q} \\
&\qquad + C\Big(\sum_{i=1}^k (r_j'-r_j)^{-qi} \int_{A_j} (|\nabla^{k-i} T(u_n-u)|^q) \, dx\Big)^{p/q} \\
&\quad \le C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} \sup_{0<\delta<2(r_j'-r_j)} \delta^{-1} \int_{A_j\setminus(A_j)_\delta} (|\nabla^k u_n| + |\nabla^k u|^p) \, dx \\
&\qquad + C\sum_{i=1}^k (r_j'-r_j)^{-pi+N\frac{p}{q}-(N-1)} \sup_{0<\delta<2(r_j'-r_j)} \delta^{-1} \int_{A_j\setminus(A_j)_\delta} (|\nabla^{k-i}(u_n-u)|^p) \, dx.
\end{aligned}
$$

Since $A_j \setminus (A_j)_\delta = B(x_0, r_j') \setminus B(x_0, r_j' - \delta) \cup B(x_0, r_j + \delta) \setminus B(x_0, r_j)$, using (23) we deduce that the last sum may be estimated from above by

$$
\begin{aligned}
&C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} \Bigg(M\phi_{n,j}(r_j) + M\phi_{n,j}(r_j')) + \sum_{i=1}^k \frac{M\phi_{n,j}(r_j) + M\phi_{n,j}(r_j')}{a_n (r_j'-r_j)^{pi}} \Bigg) \\
&\le C(r_j' - r_j)^{N\frac{p}{q}-(N-1)} \lambda \to 0,
\end{aligned}
$$

and this concludes the proof. □

Proof of Lemma 3.4 and Lemma 3.5. As in Lemma 3.6 in [**12**], without loss of generality we may assume that f is p-coercive, i.e. $\gamma > 0$ in (7). Appealing to Proposition 3.1, there exists a «minimizing sequence» $\{u_n\}$ of smooth functions such that

$$u_n \rightharpoonup u \text{ in } W^{k,p}(\Omega;\mathbb{R}^d)$$

and

$$F(u_n,\cdot) \overset{*}{\rightharpoonup} \mathcal{R}(u,\cdot) \text{ in the sense of measures.}$$

Let r_0 be a radius which satisfies the properties described in Lemma 3.6 in this paper. Then for $j = 1, 2, \ldots$, there exist integers $n_j \to \infty$, radii $r_j \nearrow r_0$, $r'_j \searrow r_0$ and functions $v_j \in W^{k,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ such that $v_j = u_{n_j}$ on $\Omega \setminus (B(x_, r'_j) \setminus B(x_0, r_j))$, $v_j - u \in W^{k,p}_0(B(x_0,r_0);\mathbb{R}^d)$ and

$$\int\limits_{B(x_0,r'_j)\setminus B(x_0,r_j)} |\nabla^k v_j|^q\,dx \to 0. \tag{27}$$

We may also assume that $\mathcal{R}(u, \partial B(x_0, r_0)) = 0)$. The functions v_j are legitimate test function for $\boldsymbol{m}(u, B(x_0, r_0))$, and thus, using (7) and (27),

$$\begin{aligned}\boldsymbol{m}(u,B(x_0,r_0)) &\le \inf_j F(v_j, B(x_0,r_0))\\ &\le \limsup_n F(u_n, B(x_0,r_0))\\ &\le \mathcal{R}(u,\overline{B}(x_0,r_0)) = \mathcal{R}(u,B(x_0,r_0)).\end{aligned}$$

This proves Lemma 3.4. Now, given a function $v \in W^{k,q}(B(x_0,r_0);\mathbb{R}^d)$ with $u - v \in W^{k,p}_0(B(x_0,r_0);\mathbb{R}^d)$, we set

$$w_j =: \begin{cases} v & \text{on } B(x_0,r_0),\\ v_j & \text{outside } B(x_0,r_0).\end{cases}$$

Choose $\delta > 0$. Since $w_j \rightharpoonup w$ in $W^{k,p}(B(x_0, r_0+\delta) \setminus \overline{B}(x_0, r_0-\delta))$, we have

$$\begin{aligned}\mathcal{R}(w,\partial B(x_0,r_0)) &\le \liminf_j F(w_j, B(x_0,r_0+\delta)\setminus \overline{B}(x_0,r_0-\delta))\\ &\le \limsup_n F(u_n, B(x_0,r_0+\delta)\setminus \overline{B}(x_0,r_0))\\ &\qquad + F(v, B(x_0,r_0)\setminus \overline{B}(x_0,r_0-\delta))\\ &\le \mathcal{R}(u,\overline{B}(x_0,r_0+\delta)\setminus B(x_0,r_0))\\ &\qquad + F(v, B(x_0,r_0)\setminus \overline{B}(x_0,r_0-\delta)).\end{aligned}$$

Letting $\delta \to 0+$ we easily observe that $\mathcal{R}\big(w, \partial B(x_0,r_0)\big) = 0$, and thus Lemma 3.5 is proved. □

Proof of Theorem 1.8. The first part of the statement when $p > N - 1$ follows immediately from Theorem 1.7 with $q = N$.

If now $p > \dfrac{N^2}{N+2}$, we choose $\varepsilon > 0$ and consider a sequence $u_n \in W^{2,N}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{2,p}(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega |\det \nabla^2 u_n(x)|\, dx < \mathcal{F}_H(u, \Omega, (W^{2,p}, w) + \varepsilon. \tag{28}$$

By Theorem 1.3 we have that $\mathcal{H}u = \mathcal{H}_2^* u$, and for all $\varphi \in C_c^\infty(\Omega)$ and by (28)

$$\begin{aligned} |\langle \mathcal{H}u, \varphi\rangle| &= \left| \lim_{n\to\infty} \int_\Omega \varphi(x) \det \nabla^2 u_n(x)\, dx \right| \\ &\le \|\varphi\|_{L^\infty} (\mathcal{F}_H(u, \Omega, (W^{2,p}, w) + \varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the assertion. □

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References

[1] E. ACERBI and G. DAL MASO, *New lower semicontinuity results for polyconvex integrals*, Calc. Var. Partial Differential Equations, **2** (1994), 329-371.

[2] G. ALBERTI and L. AMBROSIO, *A geometrical approach to monotone functions in* R^n, Math. Z. **230** (1999), 259-316.

[3] G. ALBERTI, S. BALDO and G. ORLANDI, *Functions with prescribed singularities*, J. Eur. Math. Soc. (JEMS) **5** (2003), 275-311.

[4] A. D. ALEKSANDROV, *Dirichlet's problem for the equation* Det $||z_{ij}|| = \varphi(z_1, \cdots, z_n, z, x_1, \cdots, x_n)$, I. (Russian), Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1958), 5-24.

[5] L. AMBROSIO, N. FUSCO, and D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, New York 2000.

[6] I. J. BAKELMAN, *Convex analysis and nonlinear geometric elliptic equations*, Springer-Verlag, Berlin 1994.

[7] J. M. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal. **63** (1977), 337-403.

[8] J. M. BALL, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Philos. Trans. Roy. Soc. London, A **306** (1982), 557-611.

[9] J. M. BALL, J. C. CURRIE and P. J. OLVER, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal. **41** (1981), 135-174.

[10] J. M. BALL and F. MURAT, $W^{1,p}$*-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225-253.

[11] F. BETHUEL, H. BRÉZIS and F. HELÉIN, *Ginzburg-Landau vortices*, Birkhäuser, Boston 1994.

[12] G. BOUCHITTÉ, I. FONSECA and J. MALÝ, *The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 463-479.

[13] H. BRÉZIS, N. FUSCO and C. SBORDONE, *Integrability for the Jacobian of orientation preserving mappings*, J. Funct. Anal. **115** (1993), 425-431.

[14] H. BRÉZIS and L. NIRENBERG, *Degree theory and* BMO: I, Selecta Math. **1** (1995), 197-263.

[15] H. BRÉZIS and L. NIRENBERG, *Degree theory and* BMO: II, Selecta Math. **2** (1996), 309-368.

[16] A. P. CALDERÓN and A. ZYGMUND, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171-225.

[17] M CARRIERO, A. LEACI and F. TOMARELLI, *Strong minimizers of Blake & Zisserman functional*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **15** (1997), 257-285.

[18] R. CHOKSI, R. V. KOHN and F. OTTO, *Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy*, Comm. Math. Phys. **201** (1999), 61-79.

[19] R. COIFMAN, P. L. LIONS, Y. MEYER and S. SEMMES, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247-286.

[20] R. R. COIFMAN and G. WEISS, *Analyse harmonique non-commutative sur certain espaces homogenés*, Lecture Notes in Mathematics No. 242, Springer-Verlag, Berlin-New York 1971.

[21] S. CONTI, I. FONSECA and G. LEONI, *A* Γ*-convergence result for the two-gradient theory of phase transitions*, Comm. Pure Appl. Math. **55** (2002), 857-936.

[22] B. DACOROGNA, *Direct methods in the calculus of variations*, Appl. Math. Sciences 78, Springer-Verlag, Berlin 1989.

[23] B. DACOROGNA and F. MURAT, *On the optimality of certain Sobolev exponents for the weak continuity of determinants*, J. Funct. Anal. **105** (1992), 42-62.

[24] G. DAL MASO, I, FONSECA, G. LEONI and M. MORINI, *Higher order quasiconvexity reduces to quasiconvexity*, Arch. Ration. Mech. Anal. **171** (2004), 55-81.

[25] A. DESIMONE, *Energy minimizers for large ferromagnetic bodies*, Arch. Ration. Mech. Anal. **125** (1993), 99-143.

[26] L. ESPOSITO and G. MINGIONE, *Relaxation results for higher order integrals below the natural growth exponent*, Differential Integral Equations **15** (2002), 671-696.

[27] I. FONSECA, N. FUSCO and P. MARCELLINI, *On the total variation of the Jacobian*, J. Funct. Anal. **207** (2004), 1-32.

[28] I. FONSECA, N. FUSCO and P. MARCELLINI, *Topological degree, Jacobian determinant and relaxation*, Boll. Un. Mat. Ital. Sez. B **8** (2005), 187-250.

[29] I. FONSECA and G. LEONI, *Modern methods in the calculus of variations*, Book in preparation.

[30] I. FONSECA, G. LEONI and J. MALÝ, *Weak continuity and lower semicontinuity results for determinants*, Arch. Ration. Mech. Anal. (to appear).

[31] I. FONSECA, G. LEONI, J. MALÝ and R. PARONI, *A note on Meyers' theorem in* $W^{k,1}$, Trans. Amer. Math. Soc. **354** (2002), 3723-3741.

[32] I. FONSECA and J. MALÝ, *Relaxation of multiple integrals below the growth exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 309-338.

[33] I. FONSECA, J. MALÝ and V. POSPÍŠIL, *Examples concerning weak Hessian*, in preparation.

[34] I FONSECA and P. MARCELLINI, *Relaxation of multiple integrals in subcritical Sobolev spaces*, J. Geom. Anal. **7** (1997), 57-81.

[35] M. GIAQUINTA, G. MODICA and J. SOUČEK, *Graphs of finite mass which cannot be approximated in area by smooth graphs*, Manuscripta Math. **78** (1993), 259-271.

[36] M. GIAQUINTA, G. MODICA and J. SOUČEK, *Cartesian currents in the calculus of variations, I., II.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 37. Springer-Verlag, Berlin 1998.

[37] L. GRECO, *A remark on the equality* $\det Df = \operatorname{Det} Df$, Differential Integral Equations **6** (1993), 1089-1100.

[38] P. HAJŁASZ, *A note on weak approximation of minors*, Ann. Inst. H. Poincaré **12** (1995), 415-424.

[39] T. IWANIEC, *On the concept of the weak Jacobian and Hessian*, Papers on analysis, 181-205, Rep. Univ. Jyväskylä Dep. Math. Stat. 83, Univ. Jyväskylä, Jyväskylä 2001.

[40] T. IWANIEC and G. MARTIN, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York 2001.

[41] T. IWANIEC and C. SBORDONE, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Ration. Mech. Anal. **119** (1992), 129-143.

[42] R. D. JAMES and S. J. SPECTOR, *The formation of filamentary voids in solids*, J. Mech. Phys. Solids **39** (1991), 783-813.

[43] R. L. JERRARD and N. JUNG, *Strict convergence and minimal liftings in BV*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 1163-1176.

[44] R. L. JERRARD and H. M. SONER, *Functions of bounded higher variation*, Indiana Univ. Math. J. **51** (2002), 645-677.

[45] R. V. KOHN and S. MÜLLER, *Surface energy and microstructure in coherent phase transitions*, Comm. Pure Appl. Math. **47** (1994), 405-435.

[46] J. MALÝ, L^p*-approximation of Jacobians*, Comment. Math. Univ. Carolin. **32** (1991), 659-666.

[47] J. MALÝ, *Weak lower semicontinuity of polyconvex integrals*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 681-691.

[48] P. MARCELLINI, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), 391-409.

[49] C. B. MORREY, *Multiple Integrals in the calculus of variations*, Springer-Verlag, New York 1966.

[50] D. MUCCI, *Graphs of finite mass which cannot be approximated by smooth graphs with equibounded area*, J. Funct. Anal. **152** (1998), 467-480.

[51] S. MÜLLER, Det = det. *A remark on the distributional determinant*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 13-17.

[52] S. MÜLLER, *Variational models for microstructures and phase transitions*, Lecture Notes, MPI Leipzig 1998.

[53] S. MÜLLER, *On the singular support of the distributional determinant*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 657-696.

[54] S. MÜLLER, TANG QI and B. S. YAN, *On a new class of elastic deformations not allowing for cavitation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 217-243.

[55] S. MÜLLER and S. J. SPECTOR, *An existence theory for nonlinear elasticity that allows for cavitation*, Arch. Ration. Mech. Anal. **131** (1995), 1-66.

[56] S. MÜLLER, and V. ŠVERÁK, *Convex integration for Lipschitz mappings and counterexamples to regularity*, Ann. of Math. (2) **157** (2003), 715-742.

[57] P. J. OLVER, *Hyper-Jacobians, determinantal ideals and weak solutions to variational problems*, Proc. Roy. Soc. Edinburgh Sect. A **95** (1983), 317-340.

[58] D. R. OWEN and R. PARONI, *Second-order structured deformations*, Arch. Ration. Mech. Anal. **155** (2000), 215-235.

[59] Y. RESHETNYAK, *Weak convergence and completely additive vector functions on a set*, Sibirsk. Mat. Zh. **9** (1968), 1039-1045.

[60] R. RIVIÈRE and S. SERFATY *Limiting domain wall energy for a problem related to micromagnetics*, Comm. Pure Appl. Math. **54** (2001), 294-338.

[61] N. TRUDINGER and X.-J. WANG, *Hessian measures. I.*, Topol. Methods Nonlinear Anal. **10** (1997). 225-239.

[62] W. P. ZIEMER, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York 1989.

Abstract

A weak formulation of the determinant of the matrix of second order derivatives is introduced and several of its properties are explored in analogy with the theory developed for the weak Jacobian.

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