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A construction of quasiconvex functions (**)

Contents

1 - Introduction

In this paper we outline an elementary method for constructing quasiconvex functions. The method is based on the observation that given any suitably rank-1 convex function R and any strongly quasiconvex function P the function $R + tP$ is quasiconvex for sufficiently large numbers t. We regard the function R as the function that we ideally would like to show is quasiconvex and the term tP as a perturbation.

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The method is illustrated on a remarkable family of functions that was first considered by Dacorogna and Marcellini in [**9**], namely

$$|\xi|^4 - 2\gamma|\xi|^2 \det \xi, \tag{1.1}$$

which is defined for real two-by-two matrices $\xi \in \mathbb{R}^{2\times 2}$, and where $\gamma \in \mathbb{R}$ is a parameter. Here $|\xi|$ designates the Euclidean norm of the matrix ξ, and the above function is therefore in particular a homogeneous polynomial of degree 4. It is polyconvex precisely when $|\gamma| \leq 1$ and rank-1 convex precisely when $|\gamma| \leq 2/\sqrt{3}$ (see [**9**], [**11**]). The precise range of γ for which it is quasiconvex is still unknown. A result of Alibert and Dacorogna [**1**] states that there exists a positive number $\varepsilon > 0$ such that the function is quasiconvex whenever $|\gamma| \leq 1 + \varepsilon$. We shall recover this result as an application of our construction.

If we replace the Euclidean norm $|\cdot|$ in (1.1) by the spectral norm $\|\cdot\|$, we obtain the function

$$\|\xi\|^4 - 2\gamma\|\xi\|^2 \det \xi. \tag{1.2}$$

This function (up to a multiplicative constant) was considered in an apparently different context by Burkholder in [**5**]. Quasiconvexity of the function (1.2) and its generalizations turn out to be closely linked to some deep questions in harmonic analysis and geometric function theory (see [**2**] and [**18**]). The construction as we present it here does not apply to the function (1.2), which in this connection has some quite bad features, nondifferentiability being one of them. It is however possible to modify the construction and thereby show quasiconvexity of an interesting function related to (1.2). These results, that rely on a local version of polyconvexity (see [**6**] and [**24**]), will be published elsewhere.

The paper is organized as follows. The definitions of poly-, quasi- and rank-1 convexity are recalled in Section 2. The construction is given in an abstract set-up in Section 3, and Section 4 provides some useful examples of strongly quasiconvex functions that can be used as perturbation functions. The proofs for strong quasiconvexity rely on elementary results from harmonic analysis. Sections 5 and 6 contain examples with explicit construction of nontrivial quasiconvex functions.

2 - Three definitions

A continuous function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is *polyconvex* (see [**3**], [**6**] and [**25**]) if it can be written as a convex function of minors. In particular, when $n = N = 2$ this means that there exists a convex function $\hat{F}\colon \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $F(\xi) = \hat{F}(\xi, \det \xi)$ for all $\xi \in \mathbb{R}^{2\times 2}$.

A continuous function $F: \mathbb{R}^{N\times n} \to \mathbb{R}$ is *quasiconvex* (see [**25**]) if for all $\xi \in \mathbb{R}^{N\times n}$ and all smooth compactly supported maps $\varphi: \mathbb{R}^n \to \mathbb{R}^N$ the inequality

$$\int_{\mathbb{R}^n} \big(F(\xi + \nabla\varphi(x)) - F(\xi)\big)\, dx \geq 0$$

holds. Here $\nabla\varphi(x)$ denotes the usual Jacobi matrix of φ at x.

A continuous function $F: \mathbb{R}^{N\times n} \to \mathbb{R}$ is *rank*-1 *convex* if for any ξ, $\eta \in \mathbb{R}^{N\times n}$ where $\mathrm{rank}(\eta) = 1$ the function $\mathbb{R} \ni t \mapsto F(\xi + t\eta)$ is convex.

We refer the reader to [**10**] and [**26**] for the elementary properties and interrelations of these convexity notions. More recent developments related to this paper include [**4**], [**7**], [**8**], [**14**], [**21**], [**22**], [**27**], [**29**], [**30**] and [**33**].

3 - The construction

Throughout this section we let $R: \mathbb{R}^{N\times n} \to \mathbb{R}$ denote a C^3-smooth function which is positively homogeneous of degree $p > 3$, that is,

$$R(t\xi) = t^p R(\xi) \tag{3.1}$$

for all $\xi \in \mathbb{R}^{N\times n}$ and all $t \geq 0$.

We also assume that for some positive $\delta > 0$ the following inequality holds

$$R''(\xi)[\eta,\eta] \geq \delta |\xi|^{p-2} |\eta|^2 \tag{3.2}$$

for all $\xi \in \mathbb{R}^{N\times n}$ and $\eta \in \mathbb{R}^{N\times n}$ with $\mathrm{rank}(\eta) \leq 1$. The left-hand side stands for the second differential of R and is defined by

$$R''(\xi)[\eta,\eta] \equiv \frac{d^2}{dt^2} R(\xi + t\eta)|_{t=0} = \sum_{i,j=1}^{n} \sum_{a,\beta=1}^{N} \frac{\partial^2 R(\xi)}{\partial \xi_i^a \partial \xi_j^\beta}\, \eta_i^a \eta_j^\beta .$$

The inequality (3.2) is a strict form of the well-known Legendre-Hadamard condition. Observe that this inequality extends to complex rank-one matrices of the form $\eta = A \otimes a$, where $A \in \mathbb{C}^N$ and $a \in \mathbb{R}^n$. It then reads as

$$R''(\xi)[\eta,\overline{\eta}] \geq \delta |\xi|^{p-2} |\eta|^2, \tag{3.3}$$

where $\overline{\eta}$ denotes the (component-wise) complex conjugate of η and $|\eta|^2 = \langle \eta, \overline{\eta} \rangle$.

We say that a continuous function $P: \mathbb{R}^{N\times n} \to \mathbb{R}$ is strongly quasiconvex of degree p if for some positive $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^n} \big(P(\xi + \nabla\varphi(x)) - P(\xi)\big)\, dx \geq \varepsilon \int_{\mathbb{R}^n} |\nabla\varphi(x)|^p\, dx \tag{3.4}$$

whenever $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in \mathfrak{D}$. Here, and throughout the paper, we denote by $\mathfrak{D}$ the space of maps $\varphi = (\varphi_1, \dots, \varphi_N)^T \colon \mathbb{R}^n \to \mathbb{R}^N$ for which each coordinate function φ_j vanishes outside a bounded set and has continuous partial derivatives of any order. Observe that there are functions P satisfying (3.4) for an $\varepsilon > 0$, but for which $P - \rho|\cdot|^p$ is not quasiconvex for any $\rho > 0$. In particular, for $p > 3$, the condition (3.4) at a matrix $\xi \neq 0$ is a strictly weaker condition than strict, uniform quasiconvexity as defined by Evans in [**13**].

Theorem 3.1. *Suppose that R is C^3 and that* (3.1), (3.2) *hold. Then for a function P satisfying* (3.4) *there exists a constant t_0 such that the function*

$$R + tP$$

is quasiconvex for each $t \geq t_0$.

Remark. It follows from the proof that we may take

$$t_0 \equiv \max\{1, 2^{(p-4)(p-2)}\}\,\frac{\delta^{3-p}}{2\varepsilon}\sup_{|\xi|=1}\|R'''(\xi)\|^{p-2}.$$

Here and in the remainder of the paper $\|\cdot\|$ denotes the operator norm, so that for the tri-linear form $R'''(\xi)$ we have

$$\|R'''(\xi)\| \equiv \sup R'''(\xi)[\eta_1, \eta_2, \eta_3],$$

where the supremum is taken over all matrices η_1, η_2, η_3 of (Euclidean) norm one.

Proof. Fix an arbitrary $\xi \in \mathbb{R}^{N\times n}$ and a test map $\varphi \in \mathfrak{D}$. To simplify notation let $\Phi \equiv \nabla\varphi$. By Taylor expansion

$$R(\xi + \Phi) - R(\xi) = R'(\xi)[\Phi] + \frac{1}{2}R''(\xi)[\Phi, \Phi] + I, \tag{3.5}$$

where

$$I \equiv \int_0^1 (1-s)(R''(\xi + s\Phi) - R''(\xi))[\Phi, \Phi]\,ds.$$

The last term can be estimated with the aid of the following elementary lemma.

Lemma 3.1. *For each $\delta > 0$ define*

$$\gamma_0 \equiv \max\{1, 2^{(p-4)(p-2)}\}\,\frac{\delta^{3-p}}{2}\sup_{|\xi|=1}\|R'''(\xi)\|^{p-2}.$$

Then

$$\|R''(\xi+\zeta)-R''(\xi)\| \le \delta|\xi|^{p-2}+2\gamma_0|\zeta|^{p-2} \tag{3.6}$$

for $\xi, \zeta \in \mathbb{R}^{N\times n}$.

We omit the standard proof of this inequality, and only remark that it follows from the homogeneity and the smoothness of R. Therefore, for $0\le s\le 1$, we can write

$$\|R''(\xi+s\Phi)-R''(\xi)\| \le \delta|\xi|^{p-2}+2\gamma_0|\Phi|^{p-2},$$

which yields the inequality

$$I \ge -\frac{\delta}{2}|\xi|^{p-2}|\Phi|^2-\gamma_0|\Phi|^p. \tag{3.7}$$

Next, we integrate identity (3.5) over the entire space $\mathbb{R}^n$. Integration by parts shows that the integral of $R'(\xi)[\Phi(x)]$ vanishes. The integral of the last term at (3.5) is estimated from below by use of (3.7) as

$$-\frac{\delta}{2}|\xi|^{p-2}\int_{\mathbb{R}^n}|\Phi(x)|^2\,dx-\gamma_0\int_{\mathbb{R}^n}|\Phi(x)|^p\,dx.$$

We are left with the task of estimating the integral

$$\frac{1}{2}\int_{\mathbb{R}^n}R''(\xi)[\Phi(x),\Phi(x)]\,dx=\frac{1}{2}\sum_{i,j=1}^{n}\sum_{a,\beta=1}^{N}\frac{\partial^2R(\xi)}{\partial\xi_i^a\partial\xi_j^\beta}\int_{\mathbb{R}^n}\frac{\partial\varphi^a}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\,dx.$$

To this effect we follow a classical idea (see [**16**]). Accordingly, by orthogonality properties of the Fourier transformation,

$$\int_{\mathbb{R}^n}\frac{\partial\varphi^a}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\,dx=\int_{\mathbb{R}^n}y_iy_j[\mathcal{F}\varphi^a]\overline{[\mathcal{F}\varphi^\beta]}\,dy.$$

By virtue of the Legendre-Hadamard condition (3.3), applied to the complex rank-one matrix $\eta=\mathcal{F}\varphi\otimes y=\{y_i\mathcal{F}\varphi^a\}$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n}R''(\xi)[\Phi(x),\Phi(x)]\,dx &\ge \frac{\delta}{2}|\xi|^{p-2}\int_{\mathbb{R}^n}|y|^2|\mathcal{F}\varphi(y)|^2\,dy\\ &=\frac{\delta}{2}|\xi|^{p-2}\int_{\mathbb{R}^n}|\Phi(y)|^2\,dy.\end{aligned}$$

Collecting the above estimates we arrive at

$$\int_{\mathbb{R}^n}\big(R(\xi+\Phi(x))-R(\xi)\big)\,dx \ge -\gamma_0\int_{\mathbb{R}^n}|\Phi(x)|^p\,dx.$$

Finally, if we choose $t_0 \equiv \gamma_0/\varepsilon$, then in view of (3.4) we conclude with the desired estimate

$$\int_{\mathbb{R}^n} \big((R+tP)(\xi+\Phi(x)) - (R+tP)(\xi)\big)\, dx \geq (t\varepsilon - \gamma_0) \int_{\mathbb{R}^n} |\Phi(x)|^p\, dx \geq 0$$

for all $t \geq t_0$. □

4 - Some strongly quasiconvex functions

Let $\mathcal{A}\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ be a linear transformation. We assume that the kernel of $\mathcal{A}$ contains no matrices of rank one. Using the L^p-theory of the Riesz transforms (see [**31**]) one obtains for each $p \in (1,\infty)$ and any map $\varphi \in \mathfrak{D}$ the bound

$$\|\mathcal{A}[\nabla\varphi]\|_{L^p} \geq k_p \|\nabla\varphi\|_{L^p}, \tag{4.1}$$

where $k_p = k_p(\mathcal{A})$ is a positive constant depending on p and $\mathcal{A}$ only.

Lemma 4.1. *The function $P\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ defined by $P(\xi) = |\mathcal{A}\xi|^p$ is strongly quasiconvex of degree p for each $2 \leq p < \infty$.*

Proof. It is well-known that for $p \geq 2$ and for vectors X, Y in an arbitrary inner product space the inequality

$$|X|^p - |Y|^p \geq p|Y|^{p-2}\langle Y, X-Y\rangle + 2^{2-p}|X-Y|^p \tag{4.2}$$

holds (see for instance [**28**], Prop. A.1). In particular,

$$\begin{aligned}\int_{\mathbb{R}^n} \big(P(\xi+\nabla\varphi(x)) - P(\xi)\big)\, dx &\geq p|\mathcal{A}\xi|^{p-2}\left\langle \mathcal{A}\xi, \mathcal{A}\int_{\mathbb{R}^n} \nabla\varphi(x)\, dx\right\rangle \\ &\quad + 2^{2-p}\int_{\mathbb{R}^n} |\mathcal{A}\nabla\varphi(x)|^p\, dx \\ &= 2^{2-p}\|\mathcal{A}\nabla\varphi\|_{L^p}^p,\end{aligned}$$

and invoking (4.1) we obtain (3.4) with $\varepsilon = 2^{2-p}k_p^p$. □

The functions of Lemma 4.1 are convex, but not strictly convex since they are constant on translates of the kernel of $\mathcal{A}$.

Two examples of particular relevance to Lemma 4.1 are the functions

$$\xi \mapsto |\xi^-|^p \quad \text{and} \quad \xi \mapsto |\xi^+|^p, \qquad (1 < p < \infty) \tag{4.3}$$

defined for square matrices

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

Recall that the conformal part ξ^+ and the anticonformal part ξ^- of ξ are given by

$$\xi^{\pm} = \frac{1}{2} \begin{pmatrix} \xi_{11} \pm \xi_{22} & \xi_{12} \mp \xi_{21} \\ \xi_{21} \mp \xi_{12} & \xi_{22} \pm \xi_{11} \end{pmatrix}. \tag{4.4}$$

Lemma 4.1 applies to $\xi \mapsto |\xi^-|^p$ because the kernel of the linear transformation $\xi \mapsto \xi^-$ is precisely the conformal matrices that, apart from the zero matrix, all have rank two. A similar remark applies to $\xi \mapsto |\xi^+|^p$.

Using complex notation inequality (4.1) reduces to the familiar Beurling-Ahlfors inequality for the Cauchy-Riemann operators

$$\left\| \frac{\partial f}{\partial z} \right\|_{L^p} \le A_p \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^p}. \tag{4.5}$$

More precisely, the complex notation is facilitated via the isomorphism $\mathfrak{i}\colon \mathbb{R}^{2\times 2} \to \mathbb{C}^2$ defined as

$$\mathfrak{i}\left(\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \right) \equiv (z_1, z_2),$$

where

$$z_1 \equiv \frac{1}{2}((\xi_{11} + \xi_{22}) + i(\xi_{21} - \xi_{12})),$$

$$z_2 \equiv \frac{1}{2}((\xi_{11} - \xi_{22}) + i(\xi_{21} + \xi_{12})).$$

With the usual identification $\mathbb{C} \approx \mathbb{R}^2$ we have for $f \equiv u + iv\colon \mathbb{C} \to \mathbb{C}$ the real Jacobi matrix

$$\nabla f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

and $\mathfrak{i}(\nabla f) = (\partial f/\partial z, \partial f/\partial \bar{z})$.

We will derive (4.5) for $p > 2$ by means of elementary properties of harmonic functions and well-known inequalities for the sharp function. (Note that it follows by partial integration that $\|\partial f/\partial z\|_{L^2} = \|\partial f/\partial \bar{z}\|_{L^2}$.) We do not suggest that this method is easier than the approach based on Riesz transforms mentioned above, however, it has the virtue of also giving a pointwise version of the inequality.

Recall that for a square integrable function $f\colon \mathbb{C} \to \mathbb{C}$ the (centred, quadratic) sharp function is defined as

$$f^{\#}(z) \equiv \sup_{r>0} \left(\fint_{B(z,r)} |f(x) - f_{B(z,r)}|^2 \, dx \right)^{\frac{1}{2}},$$

where $f_{B(z,r)}$ denotes the average of f over $B(z,r)$ and the bar on the integral means average. The Hardy-Littlewood-Wiener maximal inequality and the Fefferman-Stein sharp inequality imply that for each $p > 2$ there exist constants a_p, β_p such that

$$a_p \|f\|_{L^p} \le \|f^{\#}\|_{L^p} \le \beta_p \|f\|_{L^p} \tag{4.6}$$

holds for all $f \in L^p(\mathbb{C}, \mathbb{C})$. These inequalities can be obtained by means of [**31**], pp. 146-148. In the statement of the next result we adopt the shorthand notation

$$\partial f \equiv \frac{\partial f}{\partial z} \quad \text{and} \quad \bar{\partial} f \equiv \frac{\partial f}{\partial \bar{z}}.$$

Lemma 4.2. *For smooth and compactly supported functions $f\colon \mathbb{C} \to \mathbb{C}$ the inequality*

$$(\bar{\partial} f)^{\#}(z) \le 8 (\partial f)^{\#}(z) \tag{4.7}$$

holds for all $z \in \mathbb{C}$.

Proof. Fix $z \in \mathbb{C}$ and a disk $B = B(z,r)$ centred at z. Let $h\colon B \to \mathbb{C}$ denote the Poisson integral of $f|_{\partial B}$. Then

$$\int_B |\partial h|^2 = \inf_{\psi \in f + W_0^{1,2}} \int_B |\partial \psi|^2 \quad \text{and} \quad \int_B |\bar{\partial} h|^2 = \inf_{\psi \in f + W_0^{1,2}} \int_Q |\bar{\partial} \psi|^2,$$

where $W_0^{1,2} = W_0^{1,2}(B, \mathbb{C})$ designates the L^2-Sobolev space of complex valued functions that vanish on the boundary of B. For $0 < \sigma < 1$, denote $\sigma B \equiv B(z, \sigma r)$. Then by standard properties of harmonic functions (in particular, [**17**], (6.2) p. 610),

$$\fint_{\sigma B} |\bar{\partial} h - (\bar{\partial} h)_{\sigma B}|^2 \le \sigma^2 \fint_B |\bar{\partial} h - (\bar{\partial} h)_B|^2 \le \sigma^2 \fint_B |\bar{\partial} f - (\bar{\partial} f)_B|^2.$$

Integration by parts yields $\int_B |\bar\partial f - \bar\partial h|^2 = \int_B |\partial f - \partial h|^2$, and since $|\partial f - \partial h| \le |\partial h - (\partial h)_B| + |\partial f - (\partial f)_B|$ we get

$$\fint_{\sigma B} |\bar\partial f - \bar\partial h|^2 \le \frac{4}{\sigma^2} \fint_B |\partial f - (\partial f)_B|^2.$$

Now for any $v \in \mathbb{C}$,

$$\left(\fint_{\sigma B} |\bar\partial f - (\bar\partial f)_{\sigma B}|^2 \right)^{\frac12} \le \left(\fint_{\sigma B} |\bar\partial f - v|^2 \right)^{\frac12},$$

so with $v = (\bar\partial h)_{\sigma B}$, the triangle inequality and the above inequalities we find

$$\begin{aligned} \left(\fint_{\sigma B} |\bar\partial f - (\bar\partial f)_{\sigma B}|^2 \right)^{\frac12} &\le \left(\fint_{\sigma B} |\bar\partial f - (\bar\partial h)_{\sigma B}|^2 \right)^{\frac12} \\ &\le \frac{2}{\sigma} \left(\fint_B |\partial f - (\partial f)_B|^2 \right)^{\frac12} \\ &\qquad + \sigma \left(\fint_B |\bar\partial f - (\bar\partial f)_B|^2 \right)^{\frac12} \\ &\le \frac{2}{\sigma} (\partial f)^\#(x) + \sigma (\bar\partial f)^\#(x). \end{aligned}$$

The proof is concluded if we let $\sigma = 1/2$ and take supremum over B. $\square$

Finally, the Beurling-Ahlfors inequality (4.5) is an immediate consequence of Lemma 4.2 and the inequalities (4.6).

5 - Examples

We return to the function (1.1):

$$R(\xi; \gamma) \equiv |\xi|^2 (|\xi|^2 - 2\gamma \det \xi)$$

defined for $\xi \in \mathbb{R}^{2\times 2}$ and where $\gamma \in \mathbb{R}$ is a parameter. In view of the identities: $|\xi|^2 = |\xi^+|^2 + |\xi^-|^2$ and $2 \det\xi = |\xi^+|^2 - |\xi^-|^2$, it takes the form

$$R(\xi; \gamma) = |\xi|^2 [(1-\gamma)|\xi^+|^2 + (1+\gamma)|\xi^-|^2].$$

We know that $R(\cdot;\gamma)$ is polyconvex if and only if $|\gamma| \leq 1$ and rank-1 convex if and only if $|\gamma| \leq 2/\sqrt{3}$, see also [**19**], [**20**]. Observe that for $|\gamma| < 2/\sqrt{3}$ the function $R(\cdot;\gamma)$ satisfies condition (3.2) with $p = 4$. This follows from the decomposition

$$R(\xi;\gamma) = \left(1 - \frac{\gamma\sqrt{3}}{2}\right)|\xi|^4 + \frac{\gamma\sqrt{3}}{2} R\left(\xi; \frac{2}{\sqrt{3}}\right).$$

However as previously mentioned, the question whether $R(\cdot;\gamma)$ is quasiconvex for the above range of the parameter γ remains open. We fix $1 < \gamma < 2/\sqrt{3}$ and consider the following perturbation of $R(\cdot;\gamma)$:

$$R(\xi;\gamma) + t|\xi^-|^4.$$

For sufficiently large t this new function becomes quasiconvex. It is not polyconvex, as it changes sign and is 4-homogeneous.

Another interesting example (announced in the Introduction) follows by perturbing $R(\cdot;\gamma)$ with the polyconvex function

$$P(\xi) = R(\xi;1) = 2|\xi|^2|\xi^-|^2.$$

It will be shown in Section 6 that P is also strongly quasiconvex of degree 4. Observe that

$$R(\xi;\gamma) + tP(\xi) = (1+t)|\xi|^2\left(|\xi|^2 - 2\frac{\gamma+t}{1+t}\ \det\xi\right).$$

If we take t sufficiently large we recover a result of Alibert and Dacorogna [**1**] saying that the Alibert-Dacorogna-Marcellini function remains quasiconvex for some parameters larger than 1, namely

$$\gamma' \equiv \frac{\gamma+t}{1+t} > 1.$$

6 - A polyconvex function

In order to establish strong quasiconvexity of $P(\xi) = 2|\xi|^2|\xi^-|^2$ we shall prove the following.

Proposition 6.1. *The polynomial* $Q\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ *given by*

$$Q(\xi) = 4P(\xi) - |\xi^-|^4 \tag{6.1}$$

is polyconvex on $\mathbb{R}^{2\times 2}$.

The polyconvexity of P has previously been established in [**9**] (see also [**1**], [**12**], [**15**], [**18**], [**19**], [**20**]). To prove Proposition 6.1 it is sufficient to establish the inequality

$$Q(\xi+\eta)-Q(\xi)\geq 16|\xi^-|^2\langle\xi,\eta\rangle+16|\xi|^2\langle\xi^-,\eta^-\rangle - 4|\xi^-|^2\langle\xi^-,\eta^-\rangle-8|\xi|^2\det\eta \tag{6.2}$$

for all $\xi,\eta\in\mathbb{R}^{2\times 2}$. The proof of (6.2) is divided into two steps. Let us remark that since Q is rotationally invariant the result of [**12**] implies that it suffices to prove polyconvexity on diagonal matrices. We do not use this result here and the proof below is self-contained; however, at a stage we use the rotational invariance to simplify to the case of diagonal matrices.

Lemma 6.2. *For all matrices ξ, $\eta\in\mathbb{R}^{2\times 2}$ we have*

$$P(\xi+\eta)-P(\xi)\geq 4|\xi^-|^2\langle\xi,\eta\rangle+4|\xi|^2\langle\xi^-,\eta\rangle-2|\xi|^2\det\eta + (|\eta^-|^2+2\langle\xi^-,\eta^-\rangle+2|\xi^-|^2)|\eta|^2. \tag{6.3}$$

Lemma 6.3. *With the above notation*

$$|\xi^-+\eta^-|^4-|\xi^-|^4\leq 4|\xi^-|^2\langle\xi^-,\eta^-\rangle+(4|\eta^-|^2+8\langle\xi^-,\eta^-\rangle+8|\xi^-|^2)|\eta|^2. \tag{6.4}$$

We begin with the easy estimate (6.4). The left-hand side minus $4|\xi^-|^2\langle\xi^-,\eta^-\rangle$ takes the form:

$$4\langle\xi^-,\eta^-\rangle^2+2|\xi^-|^2|\eta^-|^2+4|\eta^-|^2\langle\xi^-,\eta^-\rangle+|\eta^-|^4.$$

By Cauchy-Schwarz inequality this latter expression is not larger than

$$(6|\xi^-|^2+4\langle\xi^-,\eta^-\rangle+|\eta^-|^2)|\eta^-|^2.$$

The factor in the parentheses is nonnegative and not larger than

$$8|\xi^-|^2+8\langle\xi^-,\eta^-\rangle+4|\eta^-|^2.$$

Hence inequality (6.4) is immediate. The first lemma is more involved.

Proof of Lemma 6.2. After computing $P(\xi+\eta)$ inequality (6.3) reduces to

$$2|\eta|^2|\eta^-|^2+4|\eta^-|^2\langle\xi,\eta\rangle+4|\eta|^2\langle\xi,\eta^-\rangle+2|\eta|^2|\xi^-|^2 +|\xi|^2|\eta|^2+8\langle\xi,\eta\rangle\langle\xi,\eta^-\rangle-(|\eta^-|^2+2\langle\xi^-,\eta^-\rangle+2|\xi^-|^2)|\eta|^2\geq 0. \tag{6.5}$$

By use of the commutation rules $(O\eta)^{\pm} = O\eta^{\pm}$ and $(\eta O)^{\pm} = \eta^{\pm}O$ for $O \in SO(2)$, it follows that it suffices to prove (6.5) when η is diagonal.

The left-hand side of (6.5), denoted by $L(\xi, \eta)$, is a quadratic polynomial with respect to the variable $\xi \in \mathbb{R}^{2\times 2}$. If we decompose this variable as $\xi = \xi_1 + \xi_2$, where ξ_1 and ξ_2 are respectively the diagonal and the anti-diagonal part of ξ, i.e.

$$\xi_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } \xi_2 = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix},$$

then

$$L(\xi, \eta) = L(\xi_1, \eta) + |\xi_2|^2 |\eta|^2 \geq L(\xi_1, \eta).$$

Let us remark that this inequality would fail if η was not diagonal. In this way we are reduced to showing the inequality $L(\xi, \eta) \geq 0$ for diagonal matrices ξ and η, say

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } \eta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

This inequality takes the form

$$\begin{aligned} &(x^2+y^2)(x-y)^2 + 2(x-y)^2(ax+by) + 2(x^2+y^2)(x-y)(a-b) \\ &+(x^2+y^2)(a-b)^2 + (x^2+y^2)(a^2+b^2) + 4(ax+by)(x-y)(a-b) \\ &\geq \left[\frac{1}{2}(x-y)^2 + (x-y)(a-b) + (a-b)^2\right](x^2+y^2). \end{aligned} \tag{6.6}$$

When the terms are grouped properly we look at the left-hand side as a quadratic form

$$Ax^2 - 2Bxy + Cy^2. \tag{6.7}$$

Here the coefficients A, B, C still depend on x and y:

$$\begin{aligned} A &= (x-y+2a-b)^2 + a^2 + (a-b)^2, \\ B &= (a-b)(x-y+2a-2b), \\ C &= (y-x+2b-a)^2 + b^2 + (b-a)^2. \end{aligned}$$

Lengthy, but elementary verification of these formulas is left to the interested reader. The following estimate for quadratic forms is known.

$$\frac{Ax^2 - 2Bxy + Cy^2}{x^2+y^2} \geq \frac{A+C}{2} - \sqrt{\left(\frac{A-C}{2}\right)^2 + B^2}. \tag{6.8}$$

We compute the terms in the right-hand side as follows:

$$\frac{A+C}{2} = a^2 + b^2 + 3(a-b)^2 + 3(a-b)(x-y) + (x-y)^2,$$
$$\frac{A-C}{2} = (a+b)(x-y+2a-2b),$$
$$\sqrt{\left(\frac{A-C}{2}\right)^2 + B^2} = \sqrt{2(a^2+b^2)(x-y+2a-2b)^2}$$
$$\leq a^2 + b^2 + \frac{1}{2}(x-y+2a-2b)^2.$$

Hence we find that the right-hand side of (6.8) is not smaller than

$$\frac{1}{2}(x-y)^2 + (a-b)(x-y) + (a-b)^2,$$

establishing inequality (6.6). This also completes the proof of Lemma 6.2. □

Inequality (6.2) follows by subtracting (6.4) from (6.3), completing the proof of Proposition 6.1. In view of Lemma 4.1 it follows in particular that the polynomial

$$P(\xi) = 2|\xi|^2|\xi^-|^2 = \frac{1}{4}Q(\xi) + \frac{1}{4}|\xi^-|^4$$

is strongly quasiconvex of degree 4 on $\mathbb{R}^{2\times 2}$.

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Abstract

*We present an elementary method for constructing quasiconvex functions. It is based on the observation that given any suitably rank-*1 *convex function R and any strongly quasiconvex function P the function* $R + tP$ *is quasiconvex for sufficiently large numbers t. The method is illustrated on a remarkable family of functions defined on real two-by-two matrices.*

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