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Optimal regularity for second order parabolic systems via A-caloric approximation (**)

Contents

1 - Introduction

In the regularity theory for solutions to second order non-linear parobolic systems a crucial step is the comparison of a given weak solution to a solution of a linear homogeneous second order parabolic system of partial differential equations with constant coefficients. For $\mathbb{R}^N$-valued functions $h(x, t)$ of n spatial variables $x \in \mathbb{R}^n$ and the time t, such systems are determined by a bilinear form A on $\mathbb{R}^{nN}$ which is strongly elliptic with ellipticity constant $\lambda > 0$ and upper bound $\Lambda \in (0, \infty)$; this

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means that

$$A(p,p) \geq \lambda |p|^2\,, \qquad |A(p,q)| \leq \Lambda |p||q| \quad \forall\, q,p \in \mathbb{R}^{nN}\,.$$

The constant coefficient parabolic system may be written, in a weak formulation on a space-time cylinder $Q_T = \Omega \times (-T, 0)$, as follows

$$\text{(1)} \qquad \int_{Q_T} \Big(h\varphi_t - A(Dh, D\varphi) \Big)\, dz = 0 \quad \text{for all } \varphi \in C_c^\infty(Q_T, \mathbb{R}^N).$$

The subscript «c» indicates compact support of the test functions φ in Q_T; Ω denotes a bounded domain in $\mathbb{R}^n$ and $z = (x,t)$. The solutions h to (1) will be called A-**caloric functions**, because in the case that A is the canonical inner product on $\mathbb{R}^{nN}$ they are simply the classical $\mathbb{R}^N$-valued caloric functions on Q_T.

In the context of second order non-linear parabolic systems

$$u_t = \operatorname{div} a(x,t,u(x,t), Du(x,t))$$

for $\mathbb{R}^N$-valued functions $u(x,t)$ defined on a space-time cylinder Q_T the form A is usually obtained by computing the derivative of the coefficients $a(x,t,u,p)$ with respect to the variable p at suitable points (x_0, t_0, u_0, p_0), i.e. $A = D_p a(x_0, t_0, u_0, p_0)$. If u is a solution of the non-linear parabolic system, then one can deduce estimates for u near (x_0, t_0) by comparing u to suitable A-caloric functions on space-time cylinders $Q_\rho(x_0, t_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ centered a (x_0, t_0).

Following the techniques which have been utilized in the regularity theory for non-linear second order elliptic systems there are different methods to define the A-caloric functions suitable for the comparison argument in the regularity proof. One method is the **blow-up technique**. Here one considers a sequence of non-degenerate parabolic cylinders $Q_{\rho_k}(z_k) \Subset Q_T$, $z_k = (x_k, t_k)$, and considers the rescaled functions

$$u_k(y,\tau) = \frac{u(x_k + \rho_k y, t_k + \rho_k^2 \tau) - (u)_{z_k,\rho_k} - \rho_k (Du)_{z_k,\rho_k} y}{\rho_k \varepsilon_k}$$

where $(y,\tau) \in Q_1 = B_1 \times (-1, 0)$ and $\varepsilon \downarrow 0$. (Here $(v)_{z,\sigma}$ denotes the mean value of v on $Q_\sigma(z)$.) Each of the rescaled functions u_k is a solution of the associated rescaled parabolic system. Then, under appropriate hypothesis, one should be able to show that the rescaled functions u_k converge, in some sense, to an A-caloric function. From this convergence and standard estimates for solutions to constant coefficient linear parabolic systems one might deduce, by a contradiction argument, corresponding estimates for the u_k for k sufficiently large. For the original solution these estimates then should imply a decay property for the excess. This excess decay estimate finally should lead to a partial regularity result. In the setting of geometric measure theory the blow-up method was introduced by DeGiorgi [8] (see also Almgren [**3**]).

Another method to define an A-caloric comparison function h for the solution of our non-linear parabolic system u near $z_0 = (x_0, t_0)$ is the **solution of an initial Dirichlet problem** for (1) on a small parabolic cylinder $Q_\rho(z_0)$ centered at z_0 with boundary condition $h = u$ on the parabolic boundary $\partial_p Q_\rho(z_0)$. In order to compare the solution u with the A-caloric function h, the non-linear parabolic system solved by u must been exploited to obtain further information about u; to be precise, higher integrability $|Du| \in L^{2(1+\sigma)}_{\mathrm{loc}}$ for some $\sigma > 0$ (when a is of linear growth with respect to p, i.e. $|a(z, u, p)| \leq L(1 + |p|)$) is needed to deduce excess decay for u at z_0 from an argument comparing u with h. Such higher integrability can be derived by use of a (parabolic) Gehring type lemma from a reverse Hölder inequality for u which itself is based on a suitable Caccioppoli inequality which can be derived by testing the parabolic system with a suitable testfunction obtained from a direct construction. This approach has been introduced in the calculus of variations by Giaquinta and Modica [**21**].

In the present note we want to give an exposition of a third method of constructing A-caloric comparison functions. This method is based on the fact that one is able to obtain a good approximation of certain function $v \in L^2(t_0 - \rho^2, t_0; W^{1,2}(B_\rho(x_0, \mathbb{R}^N))$, where $Q_\rho(z_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ is a parabolic cylinder, by A-caloric functions $h \in L^2(t_0 - \rho^2, t_0; W^{1,2}(B_\rho(x_0, \mathbb{R}^N))$ in the L^2-topology. The requirement is that the functions v are in a certain sense approximately A-caloric, i.e. that

$$\left| \int_{Q_\rho(z_0)} (v\varphi_t - A(Dv, D\varphi))\, dz \right|$$

is sufficiently small for all testfunctions $\varphi \in C^1_c(Q_T, \mathbb{R}^N)$. The precise statement is given below in Lemma 1. In the applications of this lemma one chooses for v a suitably rescaled version of the solution u of the parabolic system. The required approximate A-caloricity of v is then a direct consequence of the fact that u solves the parabolic system. The comparison argument relating v and h yields in combination with a Caccioppoli type inequality the desired excess decay estimate. This is, of course, a very rough description of the method. In Section 2 we concentrate on giving an exposition of the results obtained in [**16**] via this method.. We will formulate the optimal regularity result (see [**16**], Theorem 8.1) and give a detailed outline of its proof.

The current approach, which we call **method of A-caloric approximation** has some conceptual and technical advantages. The method is direct and all the steps can be carried out with complete control of the sensitivity of the estimates on the structural data. For example, the method yields an explicit modulus of continuity for the spatial derivative Du with respect to x and t. The method requires minimal smoothness assumptions for the coefficients a and the obtained modulus of continuity for the spatial derivative Du is optimal with respect to required smoothness assumptions. Moreover, the method avoids the technical difficulties associated with applying «Reverse Hölder inequalities». This will be a crucial point in the treatment of non-linear parabolic systems with super-linear growth, i.e. $|a(z,u,p)| \leq L(1+|p|^{s-1})$ for some $s>2$, by a further variant of the method of A-caloric approximation (see [**19**]). For such parabolic systems, «Higher integrability» of solutions has been proved by Kinnunen and Lewis [**30**] (see also Misawa [**35**] for a refinement); however, the proofs do not yield a «Reverse Hölder inequality» for Du. Therefore the application of the second method to the partial regularity problem for non-linear parabolic systems with super-linear growth seems to be not immediately possible.

Harmonic approximation has been used for a long time in the regularity theory for elliptic problems. The technique finds its origin in Simon's proofs [**42**], [**43**] – via DeGiorgi's harmonic approximation – of the Allard regularity theorem [**2**] and the regularity theorem for minimizing harmonic maps of Schoen-Uhlenbeck [**39**]. This has been later generalized to degenerate elliptic problems by the author and Mingione, see [**17**], [**18**]. The first application of the A-harmonic approximation lemma was given in [**20**], where new and optimal interior and boundary regularity theorems were proved for almost minimizing rectifiable integer multiplicity currents to general elliptic parametric variational integrals. Subsequently it was shown in [**12**], [**14**], [**11**], [**13**], [**24**], [**25**], [**32**], [**15**] that the method can also be used to derive new and optimal regularity theorems for almost minimizing functions to quasi-convex integrals in the calculus of variations and for weak solutions to non-linear elliptic systems.

We conclude the introduction with the precise statement of the A-caloric approximation lemma.

Lemma 1 (**A-caloric approximation lemma**). *There exists a positive function $\delta(n,N,\lambda,\Lambda,\varepsilon) \leq 1$ with the following property: Whenever A is a bilinear form on $\mathbb{R}^{nN}$ which is strongly elliptic with ellipticity constant $\lambda>0$ and upper bound Λ, ε is a positive number, and $u \in L^2(-1,0;W^{1,2}(B,\mathbb{R}^N))$ with*

$$\int_Q \Big(|u|^2 + |Du|^2\Big)\,dz \leq 1$$

is approximatively A-caloric in the sense that

$$\left|\int_Q \Big(u\varphi_t - A(Du,D\varphi)\Big)dz\right| \leq \delta \sup_Q |D\varphi| \qquad \text{for all } \varphi \in C_c^\infty(Q,\mathbb{R}^N)$$

then there exists an A-caloric function h such that

$$\int_Q \Big(|h|^2 + |Dh|^2\Big)dz \le 1 \qquad \text{and} \qquad \int_Q |u-h|^2 dz \le \varepsilon\,.$$

Proof. Supposing the lemma to be false we have existence of a sequence of quadratic forms (A_k) on $\mathbb{R}^{nN}$, with fixed ellipticity constant $\lambda > 0$ and uniform upper bound $\Lambda < \infty$, and a sequence of functions $(v_k)_{k\in\mathbb{N}}$ with $v_k \in L^2(-1,0;W^{1,2}(B,\mathbb{R}^N))$ satisfying

$$\int_Q \Big(|v_k|^2 + |Dv_k|^2\Big)dz \le 1 \tag{2}$$

and

$$\left|\int_Q \Big(v_k\varphi_t - A_k(Dv_k, D\varphi)\Big)dz\right| \le \frac{1}{k}\sup_Q |D\varphi| \tag{3}$$

for all $\varphi \in C^1_c(B,\mathbb{R}^N)$ and $k \in \mathbb{N}$, such that for some $\varepsilon > 0$ the inequality

$$\int_Q |v_k - h|^2 dz > \varepsilon \tag{4}$$

is valid for all k and $h \in \mathcal{H}_k$, where here

$$\mathcal{H}_k = \left\{ f \in L^2(-1,0;W^{1,2}(B,\mathbb{R}^N)) : \begin{array}{c} f \quad \text{is an } A_k\text{-caloric} \\ \text{function on } Q, \displaystyle\int_Q (|f|^2 + |Df|^2)dz \le 1 \end{array} \right\}.$$

Passing to a subsequence (also labelled with k) we obtain the existence of $v \in L^2(-1,0;W^{1,2}(B,\mathbb{R}^N))$ and A such that there holds

$$\begin{cases} v_k \rightharpoonup v & \text{weakly in } L^2(Q,\mathbb{R}^N), \\ Dv_k \rightharpoonup Dv & \text{weakly in } L^2(Q,\mathbb{R}^{nN}), \\ A_k \to A & \text{as bilinear forms on } \mathbb{R}^{nN}. \end{cases} \tag{5}$$

Then

$$\int_Q \Big(|v|^2 + |Dv|^2 dz\Big) \le 1\,, \tag{6}$$

and by rewriting

$$\int\limits_Q \Big(v\varphi_t - A(Dv, D\varphi)\Big)\, dz = \int\limits_Q \Big((v - v_k)\varphi_t - A(Dv - Dv_k, D\varphi)\Big)\, dz$$

$$- \int\limits_Q (A - A_k)(Dv_k, D\varphi)\, dz + \int\limits_Q \Big(v_k\varphi_t - A_k(Dv_k, D\varphi)\Big)\, dz$$

and using (5), (2) and (3) we see that the weak limit v is an A-caloric function on Q.

In order to get compactness in $L^2(Q, \mathbb{R}^N)$, i.e. $v_k \to v$ in $L^2(Q, \mathbb{R}^N)$, we estimate the time derivatives of v_k. For this we let $\varphi \in C_c^\infty(Q, \mathbb{R}^N)$ and compute, using in turn (3), the Cauchy-Schwartz inequality and (2):

$$\Big|\int\limits_Q v_k\varphi_t dz\Big| \le |A_k| \int\limits_{-1}^{0} \|D\varphi(\cdot, t)\|^2_{L^2(B)} dt + \frac{1}{k} \sup_{-1\le t\le 0} \|D\varphi\|_{L^\infty(B)}\,. \tag{7}$$

Now, for $-1 < s_1 < s_2 < 0$ and $\mu > 0$ small enough we choose

$$\zeta_\mu(t) = \begin{cases} 0, & \text{for } -1 \le t \le s_1 - \mu\,, \\ \frac{1}{\mu}(t - s_1 + \mu) & \text{for } s_1 - \mu \le t \le s_1\,, \\ 1 & \text{for } s_1 \le t \le s_2\,, \\ -\frac{1}{\mu}(t - s_2 - \mu) & \text{for } s_2 \le t \le s_2 + \mu\,, \\ 0 & \text{for } s_2 + \mu \le t \le 1\,, \end{cases}$$

and let $\varphi(x, t) = \zeta_\mu(t)\psi(x)$ for $\psi \in C_c^\infty(B, \mathbb{R}^N)$. Testing (7) with φ we obtain

$$\begin{aligned} &\left|\int\limits_B \Bigg(\frac{1}{\mu} \int\limits_{s_1-\mu}^{s_1} v_k(x, t)\, dt - \frac{1}{\mu} \int\limits_{s_2}^{s_2+\mu} v_k(x, t)\, dt \Bigg) \cdot \psi(x)\, dx \right| \\ &\quad \le |A_k| \Bigg(\int\limits_{-1}^{0} \zeta_\mu(t)^2 dt \Bigg)^{\frac{1}{2}} \|D\psi\|_{L^2(B)} + \frac{1}{k} \|D\psi\|_{L^\infty(B)} \sup_{-1\le t\le 0} \zeta_\mu(t) \\ &\quad \le \Bigg(|A_k| \sqrt{s_2 - s_1 + 2\mu} + \frac{1}{k} \Bigg) \|D\psi\|_{L^\infty(B)}\,. \end{aligned}$$

By the Sobolev embedding theorem we have

$$\|D\psi\|_{L^\infty(B)} \le c(n, \ell)\|\psi\|_{W_0^{\ell,2}(B)}, \quad \ell > \frac{n+2}{2},$$

which implies

$$\left|\int_B \left(\frac{1}{\mu}\int_{s_1-\mu}^{s_1} v_k(x,t)\,dt - \frac{1}{\mu}\int_{s_2}^{s_2+\mu} v_k(x,t)\,dt\right)\cdot\psi(x)\,dx\right|$$
$$\le c(n,\ell)\left(|A_k|\sqrt{s_2-s_1+2\mu}+\frac{1}{k}\right)\|\psi\|_{W_0^{\ell,2}(B)}\,.$$

Passing to the limit $\mu\downarrow 0$ we obtain for a.e. $-1<s_1<s_2<0$

$$\left|\int_B (v_k(\cdot,s_2)-v_k(\cdot,s_1))\cdot\psi\,dx\right|\le c(n,\ell)\left(|A_k|\sqrt{s_2-s_1}+\frac{1}{k}\right)\|\psi\|_{W_0^{\ell,2}(B)}$$

for any $\psi\in C_c^\infty(B,\mathbb{R}^N)$. By the density of $C_c^\infty(B,\mathbb{R}^N)$ in $W_0^{\ell,2}(B,\mathbb{R}^N)$ the last inequality is also valid for any $\psi\in W_0^{\ell,2}(B,\mathbb{R}^N)$. Taking the supremum over all $\psi\in W_0^{\ell,2}(B,\mathbb{R}^N)$ with $\|\psi\|_{W_0^{\ell,2}(B,\mathbb{R}^N)}\le 1$ we infer

$$\|v_k(\cdot,s_2)-v_k(\cdot,s_1)\|_{W^{-\ell,2}(B,\mathbb{R}^N)}\le c(\ell,n)\left(a\sqrt{s_2-s_1}+\frac{1}{k}\right),$$

where $\sup_{k\ge1}|A_k|\le a$. From the previous estimate we easily obtain

$$\int_{-1}^{-h}\|v_k(\cdot,t+h)-v_k(\cdot,t)\|_{W^{-\ell,2}(B,\mathbb{R}^N)}\,dt\le c\left(a\sqrt{h}+\frac{1}{k}\right),$$

yielding

$$\lim_{h\downarrow0}\int_{-1}^{-h}\|v_k(\cdot,t+h)-v_k(\cdot,t)\|_{W^{-\ell,2}(B,\mathbb{R}^N)}\,dt\to0\qquad\text{uniformly in } k\in\mathbb{N}.$$

Therefore we are in a position to apply Theorem 5 of [**41**] with the choice $X=W^{1,2}(B,\mathbb{R}^N)$, $B=L^2(B,\mathbb{R}^N)$, $Y=W^{-\ell,2}(B,\mathbb{R}^N)$, $F=(v_k)_{k\in\mathbb{N}}$, $p=2$ to conclude that $(v_k)_{k\in\mathbb{N}}$ is relatively compact in $L^2(Q,\mathbb{R}^N)=L^2(-1,0;L^2(B,\mathbb{R}^N))$, i.e. there exists a subsequence $(v_k)_{k\in\mathbb{N}}$ (again labelled by k) such that

$$v_k\to v\quad\text{strongly in } L^2(Q,\mathbb{R}^N).$$

To deduce the desired contradiction we denote by $w_k\colon Q\to\mathbb{R}^N$ the unique solution to the following initial-Dirichlet problem and possessing the properties listed below; the existence of w_k can be shown using standard arguments from [**33**], [**34**]:

$$\begin{cases} w_k\in C^0([-1,0];L^2(B,\mathbb{R}^N))\cap L^2(-1,0;W_0^{1,2}(B,\mathbb{R}^N)),\\ \partial_t w_k\in L^2(-1,0;W^{-1,2}(B,\mathbb{R}^N))\,,\\ w_k(\cdot,-1)=0\,; \end{cases}$$

$$\int_Q \Big(w_k \varphi_t - A_k(Dw_k, D\varphi) \Big)\, dz = \int_Q (A - A_k)(Dv, D\varphi)\, dz \quad \forall\ \varphi \in C_0^\infty(Q, \mathbb{R}^N)\,;$$

$$(8)\qquad \begin{cases} \tfrac{1}{2}\|w_k(\cdot, t)\|^2_{L^2(B)} + \displaystyle\int_{B\times(-1,t)} A_k(Dw_k, Dw_k)\, dz \\[2ex] \qquad\qquad = \displaystyle\int_{B\times(-1,t)} (A_k - A)(Dv, Dw_k)\, dz \quad \text{for a.e. } t \in (\,-1, 0). \end{cases}$$

By ellipticity of the A_k's the left-hand side of (8) is bounded from below by $\lambda \int_{B\times(-1,t)} |Dw_k|^2\, dz$. Moreover, using Cauchy-Schwarz inequality, the bound $\int_Q |Dv|^2 dz \le 1$ from (6) and Young's inequality the right-hand side of (8) is estimated from above by $\frac{2}{\lambda}|A - A_k|^2 + \frac{\lambda}{2}\int_{B\times(-1,t)} |Dw_k|^2\, dz$. This implies in particular for a.e. $t \in [\,-1, 0]$ and $k \in \mathbb{N}$ that:

$$\frac{1}{2}\int_B |w_k(\cdot, t)|^2 dx + \frac{\lambda}{2}\int_{B\times(-1,t)} |Dw_k|^2 dz \le \frac{2}{\lambda}|\,A_k - A|^2\,.$$

Taking the supremum over $t \in (\,-1, 0)$ we arrive at

$$(9)\qquad \sup_{t\in(-1,0)} \tfrac{1}{2}\int_B |w_k(\cdot, t)|^2\, dx + \tfrac{\lambda}{2}\int_Q |Dw_k|^2\, dz \to 0 \qquad \text{as >k→∞>.}$$

Letting $g_k := v - w_k \in L^2(\,-1, 0, W^{1,2}(B, \mathbb{R}^N))$ we easily see that g_k agrees with v on the parabolic boundary $\partial_p Q$ of Q and satisfies

$$\int_Q \Big(g_k \varphi_t - A_k(Dg_k, D\varphi) \Big)\, dz = 0\ , \qquad \forall\ \ \varphi \in C_c^\infty(Q, \mathbb{R}^N)\,.$$

From (9) and the definition of g_k we see that

$$\int_Q \Big(|g_k - v|^2 + |Dg_k - Dv|^2 \Big)\, dz \to 0 \qquad \text{as } k \to \infty,$$

and the same assertion is true when we replace g_k by the scaled function $\tilde{g}_k := \dfrac{g_k}{b_k} \in \mathcal{H}_k$ where $b_k := \max\left\{ 1, \left(\displaystyle\int_Q |g_k|^2 + |Dg_k|^2\, dz \right)^{\frac{1}{2}} \right\}$, contradicting (4). □

2 - Second order parabolic systems

In this section we discuss how the A-caloric approximation lemma can be used to prove a partial regularity theorem for weak solutions of non-linear, second order parabolic systems. We give an outline of the results contained in the paper [**16**] where new and optimal results have been obtained. We do not intend to give full proofs, but we will give precise formulations of the results. Moreover we want to explain how the A-caloric approximation lemma is applied in the proof.

In the following Ω will denote a bounded domain in $\mathbb{R}^n$, and Q_T will denote the parabolic cylinder $\Omega \times (-T, 0)$, $T > 0$. We consider non-linear, second order parabolic systems of the type

$$u_t - \operatorname{div} a(x, t, u, Du) = 0\,, \tag{10}$$

for $z = (x, t) \in Q_T$, u mapping Q_T into $\mathbb{R}^N$ and coefficients $a(z, u, p) \in \mathbb{R}^{nN}$ for $(z, u, p) \in Q_T \times \mathbb{R}^N \times \mathbb{R}^{nN}$.

Our **structure conditions** for the coefficients $a(z, u, p)$ and its first derivative $D_p a(z, u, p)$ with respect to the variable $p \in \mathbb{R}^{nN}$ are the following: We assume that the functions

$$(z, u, p) \mapsto a(z, u, p), \qquad (z, u, p) \mapsto D_p a(z, u, p)$$

are continuous on $Q_T \times \mathbb{R}^N \times \mathbb{R}^{nN}$. Moreover, there exist constants $\lambda > 0$ and $L < \infty$ such that for all $z \in Q_T$, $u \in \mathbb{R}^N$ and p, $\tilde{p} \in \mathbb{R}^{nN}$ the following inequalities hold:

$$|a(z, u, p)| \le L(1 + |p|)\,, \tag{11}$$

$$|D_p a(z, u, p)| \le L\,, \tag{12}$$

$$\langle D_p a(z, u, p)\tilde{p}, \tilde{p}\rangle \ge \lambda |\tilde{p}|^2\,. \tag{13}$$

With respect to the variables (z, u) we shall assume that the function $(z, u) \mapsto (1 + |p|)^{-1} a(z, u, p)$ is Hölder continuous with respect to the metric $d_p(z, z_0) + |u - u_0|$ (where $d_p((x, t), (y, s)) := \sqrt{|x - y|^2 + |t - s|}$ stands for the parabolic metric) with Hölder exponent $\beta \in (0, 1)$; i.e. we shall assume that we have a modulus of continuity of the form $\theta(t, s) := \min\{1, \tilde{K}(t)s^\beta\}$, where $\tilde{K}: [0, \infty) \to [1, \infty)$ is a given non-decreasing function, such that

$$|a(z, u, p) - a(z_0, u_0, p_0)| \le L\theta(|u| + |u_0|, d_p(z, z_0) + |u - u_0|)(1 + |p|) \tag{14}$$

for all z, $z_0 \in Q_T$, u, $u_0 \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$. Since (14) is a bit difficult to handle, we replace (14) by the following weaker requirement that

$$|a(z, u, p) - a(z_0, u_0, p_0)| \le K(|u|)(d_p(z, z_0) + |u - u_0|)^\beta (1 + |p|) \tag{15}$$

for all z, $z_0 \in Q_T$, u, $u_0 \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$. Here $K\colon [0,\infty) \to [L,\infty)$ is a given non-decreasing function.

Finally, from the fact that $D_p a(z,u,p)$ is continuous we can deduce the existence of a family of non-decreasing functions $\omega\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ with $\omega(t,0)=0$ for all t such that $t \mapsto \omega(t,s)$ is non-decreasing for fixed s and $s \mapsto \omega(t,s)^2$ is concave for fixed t, and such that

$$\big|D_p a(z,u,p) - D_p a(z_0,u_0,p_0)\big| \le \omega\Big(M, d_p(z,z_0)^2 + |u-u_0|^2 + |p-p_0|^2\Big) \tag{16}$$

for all z, $z_0 \in Q_T$, u, $u_0 \in \mathbb{R}^N$ and p, $p_0 \in \mathbb{R}^{nN}$.

Definition 1. *For given coefficients a satisfying* (11) *we say that a function $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ is a weak solution of the non-linear second order parabolic system* (10), *if*

$$\int_{Q_T} \Big(u\varphi_t - a(z,u,Du)D\varphi\Big)\,dz = 0 \tag{17}$$

holds for all $\varphi \in C_c^\infty(Q_T,\mathbb{R}^N)$. □

Consider now a weak solution $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ of the non-linear parabolic system (10), where the coefficients satisfy the structural assumptions (11)–(13) and (15). The aim of the **interior regularity theory** is to prove that, for a large set of points $z \in Q_T$ – **the regular set** – the spatial derivative Du is locally represented by a continuous function; i.e. we have

$$\operatorname{Reg}(u) = \{z \in Q_T : Du \text{ is continuous on a neighbourhood of } z\},$$

and the **singular set** is the defined by

$$\operatorname{Sing}(u) = Q_T \setminus \operatorname{Reg}(u).$$

Moreover, one wants to establish the best possible modulus of continuity for Du with respect to parabolic metric, which is allowed by the structural assumptions for the coefficients a. Simple examples show that $Du \in C^{\beta,\beta/2}$-regularity locally at regular points is best possible if the coefficients are Hölder continuous in the the variables (z,u) with respect to the parabolic metric with Hölder-exponent $\beta \in (0,1)$. Here $C^{\beta,\beta/2}(U)$ denotes the space of functions which are Hölder continuous with exponent β with respect to the space variable x and with exponent $\beta/2$ with respect to the time variable t, i.e. they are Hölder continuous with exponent β with respect to the parabolic metric d_p.

We are now in a position to state the main **partial regularity theorem**, which is taken from Theorem 8.1 of [**16**]. We shall denote the mean value of a function v on a parabolic cylinder $Q_\rho(z_0)$ by $v_{z_0,\rho}$ or $(v)_{z_0,\rho}$.

Theorem 1. *Let* $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ *be a weak solution of the non-linear parabolic system* (10), *where the coefficients a satisfy the structural assumptions* (11)–(13) *and* (14) *and denote by* Sing (u) *the singular set of u. Then*

$$\mathrm{Sing}\,(u) \subset \Sigma_1 \cup \Sigma_2$$

where

$$\Sigma_1 := \left\{ z_0 \in Q_T : \liminf_{\rho\downarrow 0} \fint_{Q_\rho(z_0)} \left|Du - (Du)_{z_0,\rho}\right|^2 dz > 0 \right\}$$
$$\cup \left\{ z_0 \in Q_T : \liminf_{\rho\downarrow 0} \fint_{Q_\rho(z_0)} \left|u - u_{z_0,\rho}\right|^2 dz > 0 \right\},$$

and

$$\Sigma_2 := \left\{ z_0 \in Q_T : \limsup_{\rho\downarrow 0} \left(|(u)_{z_0,\rho}| + |(Du)_{z_0,\rho}| \right) < \infty \right\};$$

in particular $|\mathrm{Sing}\,(u)| = 0$.

In addition, in a neighbourhood of any $z_0 \in Q_T \setminus \mathrm{Sing}\,(u)$ *Du is Hölder continuous with exponent* β *with respect to the parabolic metric* d_p, *i.e.* $Du \in C^{\beta,\beta/2}$ *on a neighbourhood of* z_0. □

Before discussing an outline of the proof of Theorem 1 we want to make some remarks concerning related results. Partial regularity of solutions to non-linear parabolic systems have been established for quasi-linear systems – i.e. for system of the type $u_t - \mathrm{div}(a(x,u)Du) = 0$ in Q_T – in [**45**], [**22**], [**23**], [**4**], [**31**]. Non-Linear systems with p-Lapalcian structure have been considered in [**35**]. Moreover, in low-dimensions the problem has been investigated in [**38**], [**28**], [**29**], [**37**]. The general case was treated in [**48**] assuming that the solution is a priori bounded or even Hölder continuous. Everywhere regularity was shown under special (diagonal type) structures for the coefficients; for example the case of p-Laplacian systems was treated in [**10**], [**36**]. Counterexamples to everwhere regularity are constructed in [**46**], [**44**], [**27**] (in the case of elliptic systems we refer to [**9**], [**26**], [**47**]). Finally, we want to mention [**40**], [**1**] for partial regularity results for weak solutions to Navier-Stokes type systems resp. parabolic systems with coefficients satisfying a non-standard growth condition of $p(x,t)$-type. In these papers the coefficients $a(x,t,Du)$ are assumed to be more regular then Hölder continuous with respect to (x,t) and therefore the methods are not suitable to treat the low regularity assumptions we consider. In any case the optimal regularity – i.e. the assertion that the modulus of continuity of the spatial derivative Du is exactly the one of the coefficients $a(x,u,p)$ with respect to (x,u) – was never achieved even under stronger hypothesis due to the different techniques used before this paper.

We are now going to give an **outline of the regularity proof** describing the various steps. In the rest of the paper $C \in [1,\infty)$ will denote a constant that may vary from line to line. The relevant dependencies will be indicated by writing $C = C(n, N, \dots)$.

The first step of the proof is to establish a suitable **Caccioppoli inequality**. The following lemma is a version taken [**16**], Lemma 5.1. We define $H(s) = K(s)(1+s)$ where K is from (15).

Lemma 2. *Let a satisfy* (11)–(13) *and* (15) *and let* $M > 0$*. Then there exists a constant* $C_{Cacc} = C(\lambda, L, H(M))$ *such that for every* $Q_\rho(z_0) \Subset Q_T$ *with* $\rho \le 1$*, every affine fuction* $\ell(z) = \ell(x)$ *independent of t satisfying* $|\ell(z_0)| + |D\ell| \le M$ *and every weak solution* $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^N))$ *to (10) we have*

$$\fint_{Q_{\rho/2}(z_0)} |Du - D\ell|^2 \, dz \le C_{Cacc} \left(\fint_{Q_\rho(z_0)} \left| \frac{u-\ell}{\rho} \right|^2 dz + \rho^{2\beta} \right). \tag{18}$$

Proof. Since the proof is close to the proof of similar Caccioppoli inequalities given in the elliptic setting we only explain the choice of the suitable test-function which yields the result after a lengthy but straightforwad computation.

In (17) we take $\varphi = \eta^2\zeta^2(u-\ell)$ where $\eta \in C_0^1(B_\rho(x_0))$ is a cut-off function in space such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\rho/2}(x_0)$ and $|D\eta| \le 4\rho^{-1}$ while $\zeta \in C^1(\mathbb{R})$ is a cut-off function in time such that, with $0 < \varepsilon < \rho^2/4$ being arbitrary, $\zeta \equiv 1$ on $(t_0 - \rho^2/4, t_0 - \varepsilon^2)$, $\zeta \equiv 0$ on $(-\infty, t_0 - \rho^2) \cup (t_0, \infty)$, $0 \le \zeta \le 1$ on $\mathbb{R}$, $\zeta_t \le 0$ on $(t_0 - \rho^2/4, \infty)$, $|\zeta_t| \le 3/\rho^2$ on $(t_0 - \rho^2, t_0 - \rho^2/4)$. To proceed in a rigorous way, one should also use a smoothing procedure in time via a family of non-negative mollifying functions or via Steklov averages. □

A crucial role in the regularity theory of solutions u to non-linear parabolic systems is played by an **excess functional** measuring the mean square deviation of Du from its mean value $(Du)_{z_0,\rho}$ on small parabolic cylinders $Q_\rho(z_0) \Subset Q_T$. For an affine function $\ell(z) = \ell(x)$ independent of t we define:

$$\Phi_2(z_0, \rho, D\ell) := \fint_{Q_\rho(z_0)} |Du - D\ell|^2 \, dz \,,$$

$$\Psi_2(z_0,\rho,\ell) := \fint_{Q_\rho(z_0)} \left|\frac{u-\ell}{\rho}\right|^2 dz\,.$$

An essential next step in our method of A-caloric approximation is to establish approximative A-caloricity of $u-\ell$ on small parabolic cylinders $Q_\rho(z_0)$, where $\ell(z)=\ell(x)$ is a suitable affine function independent of t. Here

$$A(p,\tilde{p}) := \langle D_p a(z_0,\ell(z_0),D\ell)p,\tilde{p}\rangle \qquad p,\tilde{p}\in\mathbb{R}^{nN}\,,$$

and the deviation of $u-\ell$ from being exactly A-caloric on $Q_\rho(z_0)$ is measured in terms of the excess-functionals Φ_2 and Ψ_2. We note that A is strongly elliptic with ellipticity constant λ by (13) and bounded by L by (12). The following lemma is the version of approximative A-caloricity taken from [**16**], Lemma 6.1.

Lemma 3. *Under the same conditions as Lemma 2 there exists a constant $C_{Eu}=C(H(M),L)$ such that for every weak solution $u\in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ to* (10)*, every parabolic cylinder $Q_\rho(z_0)\Subset Q_T$ with $\rho\le 1$ and every affine function $\ell(z)=\ell(x)$ independent of time, satisfying $|\ell(z_0)|+|D\ell|\le M$ we have*

$$\left|\fint_{Q_\rho(z_0)} \Big((u-\ell)\varphi_t - A(Du-D\ell,D\varphi)\Big)\,dz\right|$$
$$\le c_{Eu}\Big(\omega(M+1,\Phi_2)\sqrt{\Phi_2}+\Phi_2+\Psi_2+\rho^\beta\Big)\sup_{Q_\rho(z_0)}|D\varphi|\,,$$

for all $\varphi\in C_c^\infty(Q_\rho(z_0),\mathbb{R}^N)$. Here we have abbreviated $\Phi_2=\Phi_2(z_0,\rho,D\ell)$, $\Psi_2=\Psi_2(z_0,\rho,\ell)$ and $A(\cdot,\cdot)=\langle D_p a(z_0,\ell(z_0),D\ell)\cdot,\cdot\rangle$. □

The **idea of the proof** of Lemma 3 is to use the first order Taylor expansion

$$a(z_0,\ell(z_0),Du) = a(z_0,\ell(z_0),Du) - \int_0^1 \langle D_p a(z_0,\ell(z_0),D\ell+\tau(Du-D\ell))(Du-\ell,\cdot\rangle d\tau$$

and to compare $\int_0^1 \langle D_p a(z_0,\ell(z_0),D\ell+\tau(Du-D\ell))(Du-\ell,D\varphi\rangle d\tau$ with $A(Du-D\ell,$ $D\varphi)$ using the modulus of continuity ω of D_pa from (16). This gives a connection of A to the non-linear parabolic system.

As is well known from partial regularity theory the crucial step in the regularity proof is **excess improvement**. If u is a weak solution to non-linear parabolic system (17) with small excess $\Psi_2(u,z_0,\ell)$ in some parabolic cylinder $Q_\rho(z_0)\Subset Q_T$ for some affine function $\ell(z)=\ell(x)$ independent of t, then one shows that the excess of u in some small parabolic cylinder $Q_{\vartheta\rho}(z_0)$, $0 < \vartheta \ll 1$, with respect to some suitable chosen affine function close to ℓ, is substantially smaller than $\Psi_2(u, z_0, \ell)$. Iterating this one obtains a decay rate of the excess in concentric parabolic cylinders of radius r with respect to some fixed affine function independent of t as $r \downarrow 0$. From this the regularity of u in a small parabolic cylinder eventually follows.

The idea here is to apply Lemma 3 on $Q_{\rho/2}(z_0)$ to $v := u - \ell$, where $Q_\rho(z_0) \Subset Q_T$ is a fixed parabolic cylinder with radius $\rho \le 1$ and $\ell(z) = \ell(x)$ is an affine function independent of t satisfying $|\ell(z_0)| + |D\ell| \le M$ ($M \ge 1$ given). From Caccioppoli's inequality (18) we infer

$$\Phi_2(z_0, \rho/2, D\ell) \le C_{Cacc}\big(\Psi_2(z_0, \rho, \ell) + \rho^{2\beta}\big) =: C_{Cacc}\widetilde{\Psi}_2(z_0, \rho, \ell)\,. \tag{19}$$

From an application of Lemma 3 we therefore get for any $\varphi \in C_c^\infty(Q_{\rho/2}(z_0), \mathbb{R}^N)$ that

$$\begin{aligned} &\left| \fint_{Q_{\rho/2}(z_0)} \Big(v\varphi_t - A(Dv, D\varphi)\Big)\, dz \right| \\ &\qquad \le C\Big[\omega\Big(M+1, \widetilde{\Psi}_2\Big)\sqrt{\widetilde{\Psi}_2} + \widetilde{\Psi}_2 + \rho^\beta\Big] \sup_{Q_{\rho/2}(z_0)} |D\varphi|\,, \end{aligned} \tag{20}$$

where $C = C(\lambda, L, \beta, H(M))$. Here we have set $A(\cdot,\cdot) = \langle D_p a(z_0, \ell(z_0), D\ell)\cdot, \cdot\rangle$ and $\widetilde{\Psi}_2 = \widetilde{\Psi}_2(z_0, \ell(z_0), D\ell)$. We are now in a position to apply the A-caloric approximation lemma. More precisely, keeping in mind the notation and hypothesis of Lemma 1 we consider

$$w = \frac{u - \ell}{C\sqrt{\Psi_2 + \delta^{-2}\rho^2}}$$

with $C \ge 1$ suitably chosen to obtain with the help of (18) that

$$\Big(\frac{\rho}{2}\Big)^{-2} \fint_{Q_{\rho/2}(z_0)} |w|^2\, dz + \fint_{Q_{\rho/2}(z_0)} |Dw|^2\, dz \le 1\,.$$

Using (20) we then apply Lemma 1 (in a suitably scaled version) to obtain for a given $\varepsilon > 0$ an A-caloric function $h\colon Q_{\rho/2}(z_0) \to \mathbb{R}^N$ satisfying

$$\Big(\frac{\rho}{2}\Big)^{-2} \fint_{Q_{\rho/2}(z_0)} |h|^2\, dz + \fint_{Q_{\rho/2}(z_0)} |Dh|^2\, dz \le 1 \tag{21}$$

and

$$\Big(\frac{\rho}{2}\Big)^{-2} \fint_{Q_{\rho/2}(z_0)} |w - h|^2\, dz \le \varepsilon\,. \tag{22}$$

provided the **smallness condition**

$$\omega^2(M+1,\widetilde{\Psi}_2(z_0,\rho,\ell))+\widetilde{\Psi}_2(z_0,\rho,\ell)\le \tfrac{1}{2}\delta^2(n,N,\lambda,L,\varepsilon)$$

is satisfied. Denoting by $\ell_{z_0,\sigma}$ the unique affine function (in space), i.e. $\ell_{z_0,\sigma}(z) = \ell_{z_0,\sigma}(x)$, minimizing $m \mapsto \fint_{Q_\sigma(z_0)} |u-m|^2\,dz$ amongst all affine functions $m(z)=m(x)$ independent of t, one infers from (22), (21) and the standard estimate [**5**], Lemma 5.1, for the A-caloric function h a bound

$$(\vartheta\rho)^{-2}\fint_{Q_{\vartheta\rho}(z_0)} |u-\ell_{z_0,var\theta\rho}|^2dz \le C(\vartheta^{-n-4}\varepsilon+\vartheta^2)\Big(\rho^{-2}\fint_{Q_\rho(z_0)}|u-\ell|^2dz+\rho^{2\beta}\Big)$$

for $0<\vartheta\le\dfrac{1}{2}$. This means that we have estimated the excess Ψ_2 over $Q_{\theta\rho}(z_0)$ with respect to the affine function $\ell_{z_0,\theta\rho}$ by a small quantity times $\Psi_2(z_0,\rho,\ell)+\rho^{2\beta}$. This is the desired **excess improvement**. The result, given as Lemma 7.1 in [**16**] is the following:

Lemma 4. *Consider u satisfying the assumptions of Theorem 1, $M\ge 1$ and $a\in(\beta,1)$. Then we can find positive constants $\vartheta\in\left(0,\dfrac{1}{2}\right)$ and $\delta\in(0,1]$ depending only on n, N, λ, L, a, β and $H(M)$ such that the smallness condition*

$$\omega^2\Big(M+1,\widetilde{\Psi}_2(z_0,\rho,\ell_{z_0,\rho})\Big)+\widetilde{\Psi}_2(z_0,\rho,\ell_{z_0,\rho})\le\frac{1}{2}\delta^2 \tag{23}$$

on $Q_\rho(z_0)\Subset Q_T$ for some $0<\rho\le 1$ and

$$|\ell_{z_0,\rho}(z_0)|+|D\ell_{z_0,\rho}|\le M\,, \tag{24}$$

together imply the excess improvement

$$\widetilde{\Psi}_2(z_0,\vartheta\rho,\ell_{z_0,\vartheta\rho})\le\vartheta^{2a}\widetilde{\Psi}_2(z_0,\rho,\ell_{z_0,\rho})+C\rho^{2\beta}\,. \tag{25}$$

Here $C:=1+\delta^{-2}$. □

The last step to prove **excess decay** is to iterate (25). This is standard. In order to make the iteration work one has to check that conditions (23) and (24) are satisfied on each scale $Q_{\vartheta^j\rho}(z_0)$, $j\in\mathbb{N}$ once they are fullfilled on $Q_\rho(z_0)$. The result, given as Lemma 7.3 in [**16**], is the following:

Lemma 5. *For $M\ge 1$ there exist $\rho_0(M)>0$ and $\widetilde{\Psi}_0(M)$ such that if the con-*

ditions

$$\begin{array}{rrcl} \text{(i)} & |\ell_{z_0,\rho}(z_0)| + |D\ell_{z_0,\rho}| & \le & M\,, \\ \text{(ii)} & \rho & \le & \rho_0(M)\,, \\ \text{(iii)} & \widetilde{\Psi}_2(z_0,\rho,\ell_{z_0,\rho}) & \le & \widetilde{\Psi}_0(M)\,, \end{array}$$

are satisfied on $Q_\rho(z_0) \Subset Q_T$*, then for every* $j \in \mathbb{N}$ *we have*

$$\widetilde{\Psi}_2(z_0, \vartheta^j\rho, \ell_{z_0,\vartheta^j\rho}) \le \vartheta^{2aj}\widetilde{\Psi}_2(z_0,\rho,\ell_{z_0,\rho}) + c_4(M)(\vartheta^j\rho)^{2\beta}$$

and

$$|\ell_{z_0,\vartheta^j\rho}(z_0)| + |D\ell_{z_0,\vartheta^j\rho}| \le 2M\,.$$

Furthermore, the limit

$$\Upsilon_{z_0} := \lim_{j\to\infty} (Du)_{z_0,\vartheta^j\rho}$$

exists, and there exists a constant $C(a,\beta,M) < \infty$ *such that for all* $0 < r \le \rho/2$ *there holds:*

$$\fint_{Q_r(z_0)} |Du - \Upsilon_{z_0}|^2\, dz \le c\Big[\Big(\frac{r}{\rho/2}\Big)^{2a}\Psi_2(z_0,\rho,\ell_{z_0,\rho}) + r^{2\beta}\Big]. \qquad \Box$$

An immediate consequence of Lemma 5, the integral characterization of Hölder continuity (with respect to the parabolic metric d_p) of Campanato-Da Prato [**7**] and the elementary estimate

$$|(Du)_{z_0,\sigma} - D\ell_{z_0,\sigma}|^2 \le n(n+2)\sigma^{-2} \fint_{Q_\sigma(z_0)} |u - (u)_{z_0,\sigma} - (Du)_{z_0,\sigma}|^2 dz$$

from [**32**], we immediately get a first regularity result which is formulated in Theorem 7.5 of [**16**].

Theorem 2. *Let* $u \in L^2(-T,0; W^{1,2}(\Omega,\mathbb{R}^N))$ *be a weak solution of the non-linear parabolic system* (10)*, where the coefficients a satisfy the structural assumptions* (11)–(13) *and* (14) *and denote by* $\operatorname{Sing}(u)$ *the singular set of u. Then* $Du \in C^{\beta,\beta/2}(Q_T \setminus \operatorname{Sing}(u))$ *and*

$$\operatorname{Sing}(u) \subset \Sigma_0 \cup \Sigma_2$$

where

$$\Sigma_0 := \left\{ z_0 \in Q_T : \liminf_{\rho\downarrow 0} \rho^{-2} \fint_{Q_\rho(z_0)} |u - (u)_{z_0,\rho} - (Du)_{z_0,\rho})(x - x_0)|^2\, dz > 0 \right\},$$

and

$$\Sigma_2 := \left\{ z_0 \in Q_T : \limsup_{\rho \downarrow 0} \Big(|(u)_{z_0,\rho}| + |(Du)_{z_0,\rho}| \Big) < \infty \right\}. \qquad \square$$

While it is clear from Lebesgue's theorem that $|\Sigma_2| = 0$ it is far from being obvious that the same is also true for Σ_0. In the case of elliptic systems this would follow via Poincaré's inequality, i.e.

$$\rho^{-2} \fint_{B\rho(x_0)} |u - (u)_{x_0,\rho} - (Du)_{x_0,\rho}(x - x_0)|^2 dx \le C \fint_{B\rho(x_0)} |Du(u) - (Du)_{x_0,\rho}|^2 dx \,.$$

However, such a Poincaré inequality is not known in the parabolic case since the solution u is not assumed to be differentiable with respect to time; to be precise, we do not a-priori assume that $\partial_t u \in L^2_{\mathrm{loc}}(Q_T, \mathbb{R}^N)$. Therefore Theorem 2 is not a partial regularity theorem in the sense that it does not imply that $|\mathrm{Sing}\,(u)| = 0$. It only gives a characterization of the points contained in the regular set $\mathrm{Reg}\,(u)$.

The last step in the proof of the partial regularity theorem is to establish that $\mathrm{Sing}\,(u) \subset \Sigma_1 \cup \Sigma_2$; see Thereom 1 for the definition of Σ_1. This would follow if we could show that in points $z_0 \in Q_T$ satisfying

$$\liminf_{\rho \downarrow 0} \fint_{Q_\rho(z_0)} |Du - (Du)_{z_0,\rho}|^2 dz = 0\,, \quad \liminf_{\rho \downarrow 0} \fint_{Q_\rho(z_0)} |u - (u)_{z_0,\rho}|^2 dz = 0 \tag{26}$$

and

$$\sup_{\rho > 0} \Big(|(u)_{z_0,\rho}| + |(Du)_{z_0,\rho}| \Big) \le M < \infty \tag{27}$$

we would have

$$\liminf_{\rho \downarrow 0} \rho^{-2} \fint_{Q_\rho(z_0)} |u - (u)_{z_0,\rho} - (Du)_{z_0,\rho}(x - x_0)|^2 dz = 0\,. \tag{28}$$

To obtain the desired conclusion we need a suitable **Poincaré type estimate** for the solution u at points z_0 satisfying (26) and (27). The proof is carried out in two steps. In the following we fix a parabolic cylinder $Q_{2\rho}(z_0) \Subset Q_T$ around z_0 and we denote by $v \in L^2(t_0 - 4\rho^2, t_0; W^{1,2}(B_{2\rho}(x_0), \mathbb{R}^n))$ the unique weak solution of the initial boundary value problem:

$$\int_{Q_{2\rho}(z_0)} \Big(v\varphi_t - a(z_0, (u)_{z_0,2\rho}, Dv) D\varphi \Big)\, dz = 0 \tag{29}$$

for every $\varphi \in C_c^\infty(Q_{2\rho}(z_0), \mathbb{R}^N)$, and

$$v = u \text{ on } \partial Q_{2\rho}(z_0). \tag{30}$$

For $\sigma > 0$ we define $\psi(\sigma) := \fint_{Q_\sigma(z_0)} |u - (u)_{z_0,\sigma}| dz$. Moreover, keeping in mind (14) we abbreviate:

$$\mu_M(\sigma) := \tilde{K}(2M + \psi(\sigma))(\sigma + \sqrt{\psi(\sigma)}\,)^\beta + 1\,,$$

$$\upsilon_M(\sigma) := \Big(\tilde{K}(2M + \psi(\sigma))(\sigma + \sqrt{\psi(\sigma)}\,)^\beta + \sqrt{\psi(\sigma)}\Big)(1 + M^2)\,.$$

The following **comparison estimate** is (8.5) in [**16**].

Lemma 6. *Let $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^n))$ be a weak solution of the parabolic system* (10) *under the assumptions* (11)–(14) *and let $z_0 \in Q_T$ such that (27) is satisfied. Furthermore, for $Q_{2\rho}(z_0) \Subset Q_T$ let $v \in L^2(t_0 - 4\rho^2, t_0; W^{1,2}(B_{2\rho}(x_0), \mathbb{R}^n))$ be the unique solution of the initial boundary value problem* (29), (30). *Then there holds*

$$\fint_{Q_{2\rho}(z_0)} |Du - Dv|^2\, dz + \sup_{t_0 - 4\rho^2 \le t \le t_0} \rho^{-2} \fint_{B_{2\rho}(z_0)} |u(x,t) - v(x,t)|^2\, dx \tag{31}$$

$$\le C\Big[\mu_M(2\rho) \fint_{Q_{2\rho}(z_0)} |Du - (Du)_{z_0,2\rho}|^2 dz + \upsilon_M(2\rho)\Big] =: E(2\rho)$$

where $C = C(\lambda, L)$. □

We observe that in points $z_0 \in Q_T$ satisfying (26) and (27) we have that $\liminf_{\rho \downarrow 0} \psi(\rho) = 0$, $\liminf_{\rho \downarrow 0} \mu_M(\rho) = 1$ and $\liminf_{\rho \downarrow 0} \upsilon_M(\rho) = 0$. Taking also into account that $\liminf_{\rho \downarrow 0} \fint_{Q_\rho(z_0)} |Du - (Du)_{z_0,\rho}|^2 dz = 0$ it follows that:

$$\liminf_{\rho \downarrow 0} E(2\rho) = 0\,. \tag{32}$$

The proof of Lemma 6 starts with the identity

$$\int_{Q_{2\rho}(z_0)} \Big((u - v)\varphi_t - (a(z, u, Du) - a(z_0, (u)_{z_0,2\rho}, Dv))D\varphi\Big)\, dz = 0\,, \tag{33}$$

which follows immediately from (17) and (29). In (33) we choose $\varphi := \chi(t)(u - v)$ with $\chi \equiv 1$ for $(-\infty, s)$, $\chi \equiv 0$ on $(s + \varepsilon, \infty)$, and $\chi(t) = \dfrac{s + \varepsilon - t}{\varepsilon}$ for $s \le t \le s + \varepsilon$, where

$[s, s+\varepsilon] \in (t_0 - 4\rho^2, t_0)$ and then let $\varepsilon \downarrow 0$. Rewriting the resulting identity by adding and subtracting $\int_{B_{2\rho}(x_0)\times(t_0-4\rho^2,s)} a(z_0, (u)_{z_0,2\rho}, Du) D(u-v) dz$ und using the monotonicity of the vectorfield $p \mapsto a(z_0, (u)_{z_0,2\rho}, p)$ we easily infer that for a.e. $s \in (t_0 - 4\rho^2, t_0)$:

$$\frac{1}{2}\|u(\cdot,t) - v(\cdot,t)\|^2_{L^2(B_{2\rho}(x_0))} + \lambda \int_{B_{2\rho}(x_0)\times(t_0-4\rho^2,t)} |Du - Dv|^2\, dz$$

$$\leq \int_{B_{2\rho}(x_0)\times(t_0-4\rho^2,s)} (a(z_0, (u)_{z_0,2\rho}, Du) - a(z, u, Du)) D(u-v)\, dz\,.$$

Using the modulus of continuity θ of $(x, u) \mapsto a(x, u, p)$ from (15) and Young's inequality the right-hand side of the previous estimate can be estimated by

$$\frac{\lambda}{2} \int_{B_{2\rho}(x_0)\times(t_0-4\rho^2,t)} |Du - Dv|^2\, dz + CI$$

where

$$I \leq \int_{Q_{2\rho}(z_0)} \theta(|2(u)_{z_0,2\rho}| + |u - (u)_{z_0,2\rho}|, 4\rho + |u - (u)_{z_0,2\rho}|)(1 + |Du|^2)\, dz\,.$$

The first term is absorbed as usual on the left-hand side. The main idea to estimate the second integral I appearing on the right-hand side is to split the domain of integration $Q_{2\rho}(z_0)$ into two parts: $A_t := \{z \in Q_{2\rho}(z_0) : |u - (u)_{z_0,\rho}| \geq t\}$ and $Q_{2\rho}(z_0) \setminus A_t$, where $t > 0$. Then $|A_t| \leq |Q_{2\rho}| t^{-1} \psi(2\rho)$. On A_t we use $\theta(\dots) \leq L$ while on $Q_{2\rho}(z_0) \setminus A_t$ we use instead $\theta(\dots) \leq 4L\tilde{K}(2M + t)(\rho + t)^{\beta}$. The desired result then follows by suitably choosing $t = \sqrt{\psi(2\rho)}$. For the somewhat lengthy calculations we refer to [**16**], Section 8.

To obtain the Poincaré type inequality for u we need a Poincaré inequality for v. Recalling that v is a weak solution of the parabolic system (29) we see that $\tilde{v} := v - (Dv)_{z_0,2\rho}(x - x_0)$ solves $\tilde{v}_t - \operatorname{div} \tilde{a}(D\tilde{v}) = 0$ weakly on $Q_{2\rho}(z_0)$ where $\tilde{a}(p) := a(z_0, (u)_{z_0,2\rho}, (Dv)_{z_0,2\rho} + p)$ for every $p \in \mathbb{R}^{nN}$. Then the Poincaré inequality is a consequence of the relevant regularity theory for parabolic systems with coefficients independent of (z, u); see [**6**], Theorem 3.1. Indeed, we have

$$\rho^2 \fint_{Q_\rho(z_0)} |\partial_t v|^2\, dz \leq C \fint_{Q_{2\rho}(z_0)} |Dv - (Dv)_{z_0,2\rho}|^2\, dz\,,$$

and from the usual Poincaré inequality we therefore infer that

$$\rho^{-2} \fint_{Q_\rho(z_0)} |v - (v)_{z_0,\rho} - (Dv)_{z_0,\rho}(x - x_0)|^2 dz \leq C \fint_{Q_{2\rho}(z_0)} |Dv - (Dv)_{z_0,2\rho}|^2 dz,$$

where $C = C(n, \lambda, L)$.

By comparison, i.e. by adding and subtracting v, we finally obtain the Poincaré type inequality for u from the previous estimate and (31):

$$\rho^{-2} \fint_{Q_\rho(_0)} |u - (u)_{z_0,\rho} - (Du)_{z_0,\rho}(x - x_0)|^2 \, dz \leq CE(2\rho)\,. \tag{34}$$

Now the proof of **partial regularity theorem** easily follows from (34) and (32). Indeed, if $z_0 \in Q_T$ satisfies (26) and (27) then we have (28) finishing the proof of Theorem 2.

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Abstract

We discuss a new approach to the regularity theory for solutions of second order nonlinear parabolic systems. This method is direct, exhibiting the dependence of the regularity estimates on structural data of the coefficients of the parabolic system in explicit form; it requires only weak growth and smoothness assumptions on the coefficients, and it leads to new regularity results which give the best possible modulus of continuity for the spatial derivative of the solutions. The present note is a summary of parts of the results obtained in [**16**].

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