

FRANK PACARD (\*)

**Constant mean curvature hypersurfaces  
in Riemannian manifolds (\*\*)**

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## 1 - Introduction

Let  $(M^{m+1}, g)$  be a compact  $(m + 1)$ -dimensional Riemannian manifold. We are interested in the set  $\mathcal{S}(\Sigma, M, g)$  of embedded hypersurfaces  $\Sigma \hookrightarrow M$  which have mean curvature constant (but not fixed). Observe that the topology of the elements of  $\mathcal{S}(\Sigma, M, g)$  is fixed by the topology of  $\Sigma$ .

For a generic choice of metric  $g$  on  $M$  (i.e. for  $g$  belonging to an open and dense set of metrics in  $\mathcal{C}^3$  topology) the set  $\mathcal{S}(\Sigma, M, g)$  is the union of smooth 1-dimensional manifolds [25], [34]. Some of these 1-dimensional manifolds might be compact but others might be noncompact and it is of interest to understand the compactification of the later since they give information about the set  $\mathcal{S}(\Sigma, M, g)$  itself. We report some recent progress in this direction.

We will also compare the results obtained in this geometric framework with recent results which have been obtained in the study of singularly perturbed semi-linear elliptic partial differential equations or also in the study of singularly perturbed Hamiltonian systems.

## 2 - The mean curvature

We briefly recall the definition of the mean curvature of a hypersurface since this is the central object of this note. We consider a hypersurface  $S$  which is embedded in  $M$  and we denote by  $\mathbf{n} = \mathbf{n}_S$  the unit normal vector field about  $S$  which is compatible with the orientation. Given a (small) smooth function  $w$  defined on  $S$ , we define the hypersurface  $S_w$  as the normal graph of the function  $w$  over  $S$ . Namely, the hypersurface  $S_w$  is parameterized by

$$p \in S \longrightarrow \text{Exp}_p(w(p)\mathbf{n}(p))$$

where  $\text{Exp}$  denotes the exponential map in  $(M, g)$ . Conversely, any embedded hypersurface  $\tilde{S}$  which is close enough to  $S$  can be written as  $\tilde{S} = S_w$  for some (small) smooth function  $w$ . We are now in a position to define the mean curvature of  $S$  first using some variational approach and then using some differential geometry.

We consider the  $m$ -volume functional

$$(1) \quad \mathcal{A}(w) := \int_{S_w} d\text{vol}_{S_w}.$$

Then, the differential of  $\mathcal{A}$  computed at  $w = 0$  can be written as

$$(2) \quad D\mathcal{A}|_{w=0}(v) = - \int_S H v d\text{vol}_S$$

for some function  $H := H(S)$  which is precisely the mean curvature function of  $S$ . Observe that the sign of the mean curvature  $H$  depends on the choice of the orientation but the mean curvature vector  $\mathbf{H} := H\mathbf{n}$  does not.

For practical purposes (i.e. computational issues), it is convenient to give another definition of the mean curvature  $H(S)$  of the hypersurface  $S$ . We define the second fundamental form by

$$b_S(X, Y) := g(\nabla_X Y, \mathbf{n}), \quad \forall X, Y \in TS$$

where  $\nabla$  denotes the covariant derivative on  $(M, g)$ . Using the metric  $g$ , we can define the shape operator  $A_S$  as a symmetric endomorphism of  $TS$  by

$$g(A_S X, Y) = b_S(X, Y), \quad \forall X, Y \in TS.$$

The mean curvature of  $S$  is defined to be the sum of the eigenvalues  $\kappa_i$  of  $A_S$ , which are usually referred to as the principal curvatures of the hypersurface  $S$ . Hence

$$H(S) := \kappa_1 + \dots + \kappa_m.$$

Observe that we have defined the mean curvature to be the sum of the principal curvatures instead of their average.

### 3 - Embedded constant mean curvature hypersurfaces

#### 3.1 - Variational approach

Let us consider an embedded oriented hypersurface  $S$ . Given any small function  $w$  which we decompose into  $w = w^+ - w^-$  where  $w^\pm := \max(\pm w, 0)$ , we define

$$B_{w^\pm} := \{\text{Exp}_p(t\mathbf{n}(p)) \quad : \quad \pm t \in (0, w^\pm(p))\}$$

to be the domain between  $S$  and  $S_{w^\pm}$ . With this notation, we define the  $(m + 1)$ -volume functional

$$\mathcal{V}(w) := \int_{B_{w^+}} d\text{vol}_M - \int_{B_{w^-}} d\text{vol}_M$$

where volumes are counted positively when  $w > 0$  and negatively when  $w < 0$ . The first variation of  $\mathcal{V}$  is given by

$$(3) \quad D\mathcal{V}|_{w=0}(v) = \int_S v \, d\text{vol}_S$$

and its second variation is given by

$$(4) \quad D^2\mathcal{V}|_{w=0}(v, v) = - \int_S H v^2 \, d\text{vol}_S.$$

Of interest will also be the operator which appears in the second variation of the  $m$ -volume functional  $\mathcal{A}$  which has been defined above. We have

$$(5) \quad D^2\mathcal{A}|_{w=0}(v, v) = \int_S (|\nabla_g v|^2 - |A_S|^2 v^2 - \text{Ric}(\mathbf{n}, \mathbf{n}) v^2 + H^2 v^2) d\text{vol}_S$$

where  $|A_S|^2 := \kappa_1^2 + \dots + \kappa_m^2$  is the square of the norm of the shape operator  $A_S$  and  $\text{Ric}$  is the Ricci tensor on  $(M, g)$ .

In view of (2) and (3) we see that critical points of the functional  $\mathcal{A}$  with respect to some volume constraint  $\mathcal{V} = \text{cte}$  have constant mean curvature. Here the mean curvature appears as a multiple of the Lagrange multiplier associated to the constraint (and hence it is constant). From a slightly different point of view, it follows from (2) and (3) that hypersurfaces with constant mean curvature equal to  $\lambda$  are critical points of

$$\mathcal{E}(S) := \mathcal{A}(S) + \lambda \mathcal{V}(S).$$

Indeed, observe that the first variation of this functional reads

$$D\mathcal{E}|_{w=0}(v) = \int_S (\lambda - H) v d\text{vol}_S$$

and hence is equal to 0 if  $H = \lambda$ , while its second variation reads

$$D^2\mathcal{E}|_{w=0}(v, v) = \int_S (|\nabla_g v|^2 - |A_S|^2 v^2 - \text{Ric}(\mathbf{n}, \mathbf{n}) v^2 + (H^2 - \lambda H) v^2) d\text{vol}_S.$$

Hence, if  $H = \lambda$ , i.e.  $S$  has constant mean curvature equal to  $\lambda$ , then the second variation of  $\mathcal{E}$  reduces to

$$D^2\mathcal{E}|_{w=0}(v, v) = \int_S (|\nabla_g v|^2 - |A_S|^2 v^2 - \text{Ric}(\mathbf{n}, \mathbf{n}) v^2) d\text{vol}_S.$$

This quadratic form can also be written as

$$D^2\mathcal{E}|_{w=0}(v, v) = - \int_S v \mathfrak{J}_S v d\text{vol}_S$$

where by definition  $\mathfrak{J}_S$  is the Jacobi operator given by

$$\mathfrak{J}_S := \Delta_S + |A_S|^2 + \text{Ric}(\mathbf{n}, \mathbf{n}).$$

Here  $\Delta_S$  is the Laplace Beltrami operator on  $S$ .

Minimizers of the functional  $\mathcal{A}$  with respect to some volume constraint  $\mathcal{V} = cte$  play a very important role since they are solutions of the well known *isoperimetric problem* concerning which an important literature is available. We shall refer to [27] and to the references therein for precise results. Finally, critical points of the functional  $\mathcal{A}$  have (constant) mean curvature equal to 0 and are usually referred to as minimal hypersurfaces. Again, there is a huge literature on minimal hypersurfaces in Riemannian manifolds and we shall refer to [4] for references and recent developments.

### 3.2 - Embedded constant mean curvature hypersurfaces in $\mathbb{R}^{m+1}$

The sphere of radius  $\rho$ ,  $S_\rho^m \subset \mathbb{R}^{m+1}$ , has constant mean curvature  $H = \frac{m}{\rho}$ . The question whether, in Euclidean space, there are other embedded compact examples has been negatively answered by A.D. Alexandrov [1] in the celebrated:

**Theorem 3.1 (A.D. Alexandrov).** *The spheres are the only compact embedded constant mean curvature hypersurface in  $\mathbb{R}^{m+1}$ .*

It turns out that relaxing the embeddedness condition changes the picture completely and, since the early 1980's there has been a lot of results concerning the understanding of compact constant mean curvature hypersurfaces in Euclidean space [12] and [11], triggered by the pioneer work of H. Wente [33].

Relaxing the compactness assumption also changes the picture completely. Obviously, the cylinder  $S_\rho^{m-1} \times \mathbb{R}^1$  has mean curvature  $H = \frac{m-1}{\rho}$ . But there is also a one parameter family of embedded hypersurfaces of revolution which have constant mean curvature (in dimension  $m = 2$ , these surfaces had been found by C.E. Delaunay in the middle of the 19-th century [5]). In any dimension, these hypersurfaces can be parameterized by

$$(s, x) \in \mathbb{R} \times S^m \longrightarrow \rho(\tau e^{\sigma(s)} x, \kappa(s)) \in \mathbb{R}^{m+1},$$

where  $\rho > 0$  and  $\tau \in \left(0, \frac{1}{m}(m-1)^{\frac{m-1}{m}}\right)$ . The function  $s \longrightarrow \sigma(s)$  is a smooth non-constant solution of

$$(\partial_s \sigma)^2 + \tau^2 (e^\sigma + e^{(1-m)\sigma})^2 = 1$$

and the function  $s \longrightarrow \kappa(s)$  is defined by

$$\partial_s \kappa = \tau^2 (e^{2\sigma} + e^{(2-m)\sigma}).$$

It is easy to check that these are constant mean curvature hypersurfaces [10]. They have been the starting point of the development of the theory of complete non-compact constant mean curvature hypersurfaces [16], [13], [17].

#### 4 - Local foliations by CMC hypersurfaces

In the early 1990, motivated by possible applications in general relativity, R. Ye proved the existence of constant mean curvature spheres in Riemannian manifolds. More precisely:

**Theorem 4.1 (R. Ye).** *Assume that  $p \in M$  is a nondegenerate critical point of the scalar curvature  $R$  on  $(M, g)$ . Then, a neighborhood of  $p$  is foliated by constant mean curvature topological spheres  $\Sigma(\rho)$ , for  $\rho \in (0, \rho_0)$ .*

In fact, if  $S_\rho(q)$  denotes the geodesic sphere of radius  $\rho$  centered at  $q$ , the leaf  $\Sigma(\rho)$  of this foliation is a normal graph over  $S_\rho(q)$  for some function  $w = \mathcal{O}(\rho^3)$  and some point  $q \in M$  such that  $\text{dist}_g(p, q) = \mathcal{O}(\rho^2)$ . We shall make this statement more precise later on.

This result gives a local description of some noncompact components of the space  $\mathcal{S}(S^m, M, g)$ . The condition on  $p$  to be a critical point of the scalar curvature is “almost” necessary for the existence of such foliation and R. Ye has proved, under some extra technical assumption, that the existence of such a foliations is necessarily associated to a critical point of the scalar curvature on  $M$ .

The  $m$ -volume of the leaf  $\Sigma(\rho)$  can be expanded as

$$\mathcal{A}(\Sigma(\rho)) = \rho^m |S^m| \left( 1 - \frac{1}{2(m+1)} R_p \rho^2 + \mathcal{O}(\rho^4) \right)$$

while the volume of the mean convex part of  $M - \Sigma(\rho)$ , which is a domain  $B_{\Sigma(\rho)}$  enclosed by the leaf  $\Sigma(\rho)$ , can be expanded as

$$\mathcal{V}(B_{\Sigma(\rho)}) = \frac{\rho^{m+1}}{m+1} |S^m| \left( 1 - \frac{m+2}{2m(m+3)} R_p \rho^2 + \mathcal{O}(\rho^4) \right)$$

where  $R_p$  is the scalar curvature at the point  $p$ . It is known that solutions of the isoperimetric problem for small volumes are close to small geodesic spheres [27] and it is very likely that, for generic metrics  $g$  on  $M$ , these solutions belong to one of the foliations obtained by R. Ye. This has not yet been proven since some additional work is required, but this would allow one to expand, in terms of the small parameter  $t > 0$ , the isoperimetric profile  $\mathcal{I}(t)$  defined by

$$\mathcal{I}(t) := \inf_{S: \mathcal{V}(B_S)=t} \mathcal{A}(S)$$

where  $B_S$  is the domain enclosed by  $S$  and  $\mathcal{V}(\cdot)$  is the  $(m+1)$ -dimensional volume functional.

It would be interesting to extend R. Ye’s result in the case where the metric has constant scalar curvature (in which case all points are degenerate critical points of

the scalar curvature) or even when the metric is Einstein. The solutions of the isoperimetric problem for small volumes are known to be close to geodesic spheres and a natural question is understand where these spheres can be on the manifold.

### 5 - Proof of Theorem 4.1

We give a sketch of the proof of R. Ye's result. This proof involves the same ingredients as the original proof but has the advantage not to be specific to the mean curvature operator and hence it can be extended to prove the existence of hypersurfaces with constant  $k$ -th symmetric function of the eigenvalues of the shape operator [19]. Assume that we are given  $q \in M$  and let  $E_1, \dots, E_{m+1}$  denote an orthonormal basis of  $T_q M$ . The geodesic sphere of radius  $\rho$  centered at  $q \in M$  is denoted by  $S_\rho(q)$  and can be parameterized by

$$x \in S^m \longrightarrow \text{Exp}_q^M(\rho \Theta(x))$$

where

$$x = (x^1, \dots, x^{m+1}) \in S^m \quad \text{and} \quad \Theta(x) := \sum_j x^j E_j$$

parameterizes the unit sphere in  $T_q M$ .

Given a small function  $w$  defined on  $S^m$ , we consider the perturbed geodesic sphere  $S_\rho(q, w)$  which is parameterized by

$$x \in S^m \longrightarrow \text{Exp}_q^M(\rho(1 - w(x)) \Theta(x)).$$

The mean curvature of the hypersurface  $S_\rho(q, w)$  cannot be computed explicitly, however we can easily obtain its expansion in powers of the constant  $\rho$  and the function  $w$  together with its partial derivatives. This expansion reads

$$\begin{aligned} H(S_\rho(q, w)) &= \frac{m}{\rho} - \frac{1}{3} \text{Ric}_q(\Theta, \Theta) \rho - \frac{1}{4} \nabla_\Theta \text{Ric}_q(\Theta, \Theta) \rho^2 + \mathcal{O}_q(\rho^3) \\ &\quad + \frac{1}{\rho} (\Delta_{S^m} + m) w + \rho L_q(w) + \frac{1}{\rho} Q_q(w) \end{aligned}$$

where  $\text{Ric}$  denotes the Ricci curvature tensor on  $(M, g)$ . Here  $L_q$  is a linear second order differential operator whose coefficients are bounded independently of  $\rho$  and  $Q_q$  is a second order nonlinear differential operator which gathers all the nonlinear terms and has Taylor expansion whose coefficients are bounded independently of  $\rho$  and which does not involve any constant nor any linear term. We refer to [19] for precise statements.

Since the mean curvature of  $S_\rho(q)$  is close to the constant  $\frac{m}{\rho}$  it is tempting to try to find a function  $w$  such that

$$H(S_\rho(q, w)) = \frac{m}{\rho}.$$

This problem is clearly equivalent to finding a solution to the following nonlinear elliptic problem

$$(6) \quad (\Delta_{S^m} + m)w = \frac{1}{3} \operatorname{Ric}(\Theta, \Theta)\rho^2 + \frac{1}{4} \nabla_\Theta \operatorname{Ric}(\Theta, \Theta)\rho^3 - \mathcal{O}(\rho^4) - \rho^2 L(w) - Q(w)$$

where we have dropped the  $q$  indices for the sake of simplicity in the notations. In order to solve this equation, we would like to invert the elliptic operator on the left hand side and then apply some fixed point theorem for contraction mappings. Unfortunately, the operator

$$\Delta_{S^m} + m$$

has a nontrivial kernel and in fact

$$\operatorname{Ker}(\Delta_{S^m} + m) = \operatorname{Span}\{x^1, \dots, x^{m+1}\}.$$

From the analytical point of view, this is nothing but the fact that  $m$  is an eigenvalue of  $-\Delta_{S^m}$  and from the geometric point of view this follows from the fact that the operator  $\Delta_{S^m} + m$  is the Jacobi operator about the unit sphere which has some nontrivial Jacobi fields associated to rigid motions. We denote by  $\Pi$  the  $L^2(S^m)$ -projection onto  $\operatorname{Ker}(\Delta_{S^m} + m)$  and also denote by  $\Pi^\perp$  the  $L^2(S^m)$ -projection onto  $\operatorname{Ker}(\Delta_{S^m} + m)^\perp$ . Observe that  $\Pi$  and  $\Pi^\perp$  commute with  $\Delta_{S^m} + m$ .

We first consider the projection of the equation over  $\operatorname{Ker}(\Delta_{S^m} + m)^\perp$ . Assume that  $w \in \operatorname{Ker}(\Delta_{S^m} + m)^\perp$  in the  $L^2(S^m)$  sense then the projection of (6) over  $\operatorname{Ker}(\Delta_{S^m} + m)^\perp$  reads

$$(7) \quad (\Delta_{S^m} + m)w = \Pi^\perp \left( \frac{1}{3} \operatorname{Ric}(\Theta, \Theta)\rho^2 + \frac{1}{4} \nabla_\Theta \operatorname{Ric}(\Theta, \Theta)\rho^3 - \mathcal{O}(\rho^4) \right) \\ - \Pi^\perp (\rho^2 L(w) + Q(w)).$$

Now the operator  $\Delta_{S^m} + m$  restricted to functions which are  $L^2(S^m)$ -orthogonal to  $\operatorname{Ker}(\Delta_{S^m} + m)$  is invertible and a fixed point argument (for contraction mappings) implies that there exists  $w := w(\rho, q) \in C^{2,\alpha}(S^m)$  (depending on  $\rho$  and  $q$ ) solution of (7). Moreover

$$w(\rho, q) = \mathcal{O}_{C^{2,\alpha}(S^m)}(\rho^2).$$



Once this is understood, we consider the  $L^2$ -projection over  $\text{Ker}(\Delta_{S^m} + m)$  of the equation (6) when  $w = w(\rho, q)$ . Simple observations show that:

- (i) Since  $w \in \text{Ker}(\Delta_{S^m} + m)^\perp$ , then  $\Pi((\Delta_{S^m} + m)w) = 0$ .
- (ii) Since  $\text{Ric}(\Theta, \Theta)$  is homogeneous of degree 2 in  $x^j$  and the elements of  $\text{Ker}(\Delta_{S^m} + m)$  are homogeneous of degree 1 in  $x^j$ , then  $\Pi\left(\frac{1}{3} \text{Ric}(\Theta, \Theta)\rho^2\right) = 0$ .
- (iii) We have  $\Pi\left(\frac{1}{4} \nabla_\Theta \text{Ric}(\Theta, \Theta)\rho^3\right) = c_m \rho^3 \sum (\nabla_{E_i} R) x^i$  where  $c_m \neq 0$  only depends on  $m$  and  $R$  is the scalar curvature of  $(M^i, g)$ . The proof of this fact uses Bianchi's second identity. This is precisely the point where the gradient of the scalar curvature appears in the analysis.
- (iv) Next  $\Pi(\mathcal{O}(\rho^4)) = \mathcal{O}(\rho^5)$ . This uses the fact that  $\mathcal{O}(\rho^4)$  can be decomposed into the sum of a term which is homogeneous of degree 4 in  $x^j$  and hence whose projection over  $\text{Ker}(\Delta_{S^m} + m)$  is 0 and a term  $\mathcal{O}(\rho^5)$ .
- (v) Finally,  $\Pi(\rho^2 L(w(\rho, q)) + Q(w(\rho, q))) = \mathcal{O}(\rho^5)$ . The proof of this fact uses the special structure of the operators  $L$  and  $Q$  and the special structure of  $w(\rho, q)$ .

Collecting these observations, we conclude that the  $L^2$ -projection over  $\text{Ker}(\Delta_{S^m} + m)$  of the equation (6) where  $w := w(\rho, q)$ , reduces to solving the non-linear equation on  $q \in M$

$$(8) \quad \sum_j \nabla_{E_j} R_q x^j = \mathcal{O}_q(\rho^2)$$

where the right hand side depends nonlinearly on  $q$  and belongs to  $\text{Ker}(\Delta_{S^m} + m)$ . At this point, it is straightforward to see that, a necessary condition for the equation to be solvable for any  $\rho$  tending to 0 is that the point  $p$  towards which  $q$  converges is a critical point of the scalar curvature function. If  $p$  is a nondegenerate critical point of  $R$ , then this equation can be solved using either a fixed point argument of a degree argument. In addition the point  $q$  solution of (8) satisfies  $\text{dist}(p, q) = \mathcal{O}(\rho^2)$ . This completes the existence of constant mean curvature leaves in R. Ye's result. The fact that these leaves constitute a foliation require slightly more work.

Nevertheless, observe that the obvious estimate  $\Pi(\rho^2 L(w(\rho, q)) + Q(w(\rho, q))) = \mathcal{O}(\rho^4)$  is sufficient to prove the existence of the constant mean curvature leaves, but is not enough to guarantee that the leaves constitute a foliation.

## 6 - Constant mean curvature clusters

In an ongoing work with A. Malchiodi, we construct further branches of  $S(S^m, \Sigma, g)$ . To keep the level of technicalities as simple as possible, let us just mention the:

**Theorem 6.1** (A. Malchiodi, F. Pacard). *Assume that  $p$  is a nondegenerate critical point of the scalar curvature  $R$  of  $(M, g)$  and further assume that  $\nabla^2 R_p$  has a simple positive eigenvalue. Then there exists a one parameter family of constant mean curvature topological spheres  $\Sigma(\rho)$ ,  $\rho \in (0, \rho)$  which are embedded and are close to the connected sum of two geodesic spheres  $S_\rho(q_1)$  and  $S_\rho(q_2)$  with*

$$\text{dist}(p, q_j) = \rho + o(\rho) \quad \text{and} \quad \text{dist}(q_1, q_2) = 2\rho + o(\rho).$$

This result shows that the result of R. Ye does not exhaust the description of the ends of the branches of  $\mathcal{S}(S^m, M, g)$ . This last result generalizes to arbitrary number of spheres under some appropriate conditions.

In view of these results, we can ask the following question: Is it true that, as  $H(S)$  tends to  $+\infty$ ,

$$(9) \quad H(S)^m \mathcal{H}^{m-1} S \rightarrow j m^m |S^m| \delta_p,$$

for some  $j \in \mathbb{N} - \{0\}$  and  $p \in M$ , where hypersurfaces  $S$  which belong to some fixed noncompact branch of  $\mathcal{S}(S^m, M, g)$  are considered?

## 7 - Condensation over submanifolds

We are interested in the existence of families of constant mean curvature hypersurfaces which condensate over a submanifold. Assume that  $K$  is a  $k$ -dimensional (embedded) submanifold in  $M$  with  $1 \leq k \leq m - 1$ .

We define the geodesic tube of radius  $\rho > 0$  around  $K$  by

$$S_\rho(K) := \{p \in M \quad : \quad \text{dist}(p, K) = \rho\}.$$

The mean curvature of  $S_\rho(K)$  is given by

$$(10) \quad H(S_\rho(K)) = \frac{m-k}{\rho} + \mathcal{O}(1),$$

and is close to be constant. Based on the previous experience, it sounds reasonable to perturb the geodesic tube into a constant mean curvature hypersurface, at least when  $\rho$  is small enough. In fact, we have the:

**Theorem 7.1** (F. Mahmoudi, R. Mazzeo, F. Pacard). *Assume that  $K$  is a nondegenerate minimal submanifold. Then, there exists  $I \subset (0, 1)$ , a countable union of closed nonempty intervals, such that:*

- $\forall \rho \in I, S_\rho(K)$  can be perturbed into  $\Sigma(\rho)$  a constant mean curvature hypersurface with

$$H(\Sigma(\rho)) = \frac{m-k}{\rho}.$$

- For all  $t \geq 2$ , there exists  $c_t > 0$  such that

$$|I \cap (0, r) - r| \leq c_t r^t.$$

Some remarks are due. This result proves the existence of constant mean curvature hypersurfaces whose topology is given by the topology of  $SNK$ , the spherical normal bundle of  $K$  in  $M$ . In general, the result does not hold for all small values of  $\rho$  and in fact  $I$  is the union of countably many disjoint closed nonempty intervals. This is related to some underlying bifurcation phenomena which prevents us to carry over the construction for any small values of  $\rho$ , as was the case in R. Ye's result. The second property shows that, even though  $I$  might not contain any interval of the form  $(0, r)$  it is denser and denser in this interval as  $r$  tends to 0.

In addition, for a generic choice of the metric  $g$  on  $M$ , this result proves the existence of infinitely many distinct noncompact branches in  $\mathcal{S}(SNK, M, g)$ . This follows from the fact that the index of  $\Sigma(\rho)$  tends to  $+\infty$  as  $\rho \in I$  tends to 0.

Again, the hypersurface  $\Sigma(\rho)$  is a normal graph over  $S_\rho(\tilde{K}_\rho)$  for some function  $w = \mathcal{O}(\rho^3)$ , where  $\tilde{K}_\rho$  is itself a normal graph over  $K$  for some normal variation  $\Phi = \mathcal{O}(\rho^2)$ .

Observe that, in this result, the submanifold  $K$  cannot be any submanifold but has to be a minimal submanifold and indeed, if one were to compare this result with R. Ye's result then the assumption on  $p$  being a critical point of the scalar curvature in R. Ye's result is now replaced by the fact that  $K$  has to be a minimal submanifold.

We end up this discussion by giving explicit examples of such hypersurfaces when  $M^{m+1} = S^{m+1}$  with the standard metric and  $K = \{0\} \times S_1^k$ . Given  $r \in (0, 1)$ , we consider the hypersurface

$$\Sigma(r) := S_r^{m-k} \times S_{\sqrt{1-r^2}}^k$$

which has constant mean curvature given by

$$H(\Sigma(r)) = (m-k) \frac{\sqrt{1-r^2}}{r} - k \frac{r}{\sqrt{1-r^2}}.$$

This gives explicit the hypersurfaces whose existence is proven in Theorem 7.1. Observe that these hypersurfaces exist for all  $r \in (0, 1)$  but on the other end the standard metric on  $S^{m+1}$  is not generic! Other explicit hypersurfaces can be defined analogously in appropriate quotients of  $\mathbb{R}^{m+1}$  or  $\mathbb{H}^{m+1}$ .

### 8 - Proof of Theorem 7.1

We consider the perturbed geodesic tube  $S_\rho(K, w, \Phi)$  parameterized by

$$(11) \quad (p, x) \in SNK \longrightarrow \text{Exp}_p(\rho(1 - w(p, x))\Theta(x) - \Phi(p))$$

where  $w$  is a function defined on the spherical normal bundle  $SNK$  and  $\Phi$  is a section of the normal bundle  $NK$ . The role of  $\Phi$  is to parameterize all  $k$ -dimensional submanifolds close to  $K$  just as in R. Ye's proof the center of the geodesic sphere had to be chosen close to the critical point of the scalar curvature. The function  $w$  plays the role of the corresponding function in R. Ye's proof.

As in §5, we expand the mean curvature of the hypersurface parameterized by (11) in terms of  $\rho$ ,  $w$ ,  $\Phi$  and their partial derivatives. We obtain

$$\begin{aligned} H(S_\rho(K, w, \Phi)) &= \frac{m-k}{\rho} - B(\Theta, \Theta)\rho + \mathcal{O}(\rho^2) \\ &+ \left( \rho \Delta_K + \frac{1}{\rho}(\Delta_{S^{m-k}} + m - k) \right) w + g(\mathfrak{S}_K \Phi, \Theta) \\ &+ \rho L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi) \end{aligned}$$

where  $B$  is a quadratic form defined on  $NK$  and  $\mathfrak{S}_K$  is the Jacobi operator about  $K$ . We refer to [20] for the details. We shall not give the exact expression of the operator  $B$  but only mention that, in a local trivialization,  $B(\Theta, \Theta)$  is homogeneous of degree 2 in the coordinates  $x^j$ . The Jacobi operator  $\mathfrak{S}_K$  is nothing but the operator which appears in the second variation of the  $k$ -dimensional volume functional, or equivalently in the differential of the mean curvature operator about  $K$ . It is given by

$$\mathfrak{S}_K = \Delta_K^N + \mathfrak{B}_K + \mathfrak{R}^N$$

where  $\Delta_K^N$  is the rough Laplacian on  $NK$ ,  $\mathfrak{B}_K$  is a potential associated to the second fundamental form about  $K$  and  $\mathfrak{R}^N$  is some contraction of the curvature tensor associated to the connection on  $NK$ . We refer to [18] for further details. The important fact for us is that this operator, acting on sections of the normal bundle, is elliptic and is invertible, since we have assumed that the minimal submanifold  $K$  is non-degenerate.

Observe that, when  $w = 0$  and  $\Phi = 0$  we have

$$H(S_\rho(K, w, \Phi)) = \frac{m-k}{\rho} + \mathcal{O}(\rho)$$

which is better than what was claimed in (10). This improvement is due to the fact that we have assumed that  $K$  is a minimal submanifold.

We would like to perturb  $S_\rho(K)$  in such a way that the mean curvature of the perturbed hypersurface is equal to  $H = \frac{m-k}{\rho}$ . The problem amounts to find a function  $w$  and a vector field  $\Phi$  satisfying

$$(12) \quad \begin{aligned} & \left( \rho \Delta_K + \frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) \right) w + g(\mathfrak{S}_K \Phi, \Theta) \\ & = B(\Theta, \Theta) \rho - \mathcal{O}(\rho^2) - \rho L(w, \Phi) - \frac{1}{\rho} Q(w, \Phi). \end{aligned}$$

We decompose any function  $v$  defined on  $SNK$  as

$$v = \rho w + g(\Phi, \Theta)$$

where, for all  $p \in K$

$$w(p, \cdot) \perp \text{Ker}(\Delta_{S^{m-k}} + m - k)$$

and  $\Phi$  is a section of  $NK$ . i.e. for every  $p \in K$  we consider the eigenfunction decomposition of  $w(p, \cdot)$  in terms of eigenfunctions of  $\Delta_{S^{m-k}}$ . We denote by  $\Pi$  the  $L^2(S^{m-k})$ -projection over  $\text{Ker}(\Delta_{S^{m-k}} + m - k)$  and by  $\Pi^\perp$  the  $L^2(S^{m-k})$ -projection over  $(\Delta_{S^{m-k}} + m - k)^\perp$  so that

$$\Pi v(p, \cdot) = g(\Phi(p), \Theta) \quad \text{and} \quad \Pi^\perp v(p, \cdot) = \rho w(p, \cdot).$$

Finally, we denote by  $\mathbb{L}_\rho$  the operator

$$\mathbb{L}_\rho(\rho w + g(\Phi, \Theta)) := \left( \rho \Delta_K + \frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) \right) w + g(\mathfrak{S}_K \Phi, \Theta)$$

which appears on the left hand side of (12).

### 8.1 - Analysis of the spectrum of $\mathbb{L}_\rho$

The spectrum of  $\mathbb{L}_\rho$  is the union of:

- The set of

$$A_{ij} := \lambda_i + \frac{1}{\rho^2} (\mu_j - m + k)$$

where  $\lambda_i$  are the eigenvalues of  $\Delta_K$  and  $\mu_j$  are the eigenvalues of  $\Delta_{S^{m-k}}$ .

- The set of eigenvalues of the Jacobi operator  $\mathfrak{S}_K$ , none of which is equal to 0 since  $K$  is assumed to be nondegenerate.

Obviously, an important new difficulty arises since

$$(13) \quad A_{i0} := \lambda_i - \frac{1}{\rho^2} (m - k) = 0 \quad \text{when} \quad \rho = \sqrt{\frac{m-k}{\lambda_i}}$$

and hence the operator  $\mathbb{L}_\rho$  is not invertible for this value of  $\rho$ .

Nevertheless, when

$$(14) \quad \rho \notin \left\{ \sqrt{\frac{m-k}{\lambda_j}} \quad : \quad j \geq 1 \right\},$$

one can evaluate the distance between the spectrum of  $\mathbb{L}_\rho$  and 0. This yields some estimate of the norm of  $(\mathbb{L}_\rho)^{-1}$  (say as a operator from  $L^2(SNK)$  into  $W^{2,2}(SNK)$ ). Unfortunately, formal computations show that, when  $k \geq 2$ , the discrepancy  $\rho L$  is not small enough to consider  $\mathbb{L}_\rho + \rho L$  as a small perturbation of  $\mathbb{L}_\rho$ . Also, still when  $k \geq 2$ , the error term  $B(\Theta, \Theta)\rho + \mathcal{O}(\rho^2)$  is too large and these two facts annihilates any hope to apply some fixed point argument to solve (12) when  $k \geq 2$ . When  $k = 1$ , i.e. when  $K$  is a geodesic, the distance between two consecutive eigenvalues of  $\Delta_K (= \partial_s^2)$  is “large enough” and yields a reasonable estimate of the norm of  $(\mathbb{L}_\rho)^{-1}$ . Therefore, in this case one can obtain a solution of the nonlinear equation by application of a fixed point theorem for contraction mappings. This is precisely what has been done in [25].

In order to overcome this difficulty when  $k \geq 2$ , we have adopted a strategy which was developed originally by A. Malchiodi and M. Montenegro in a different context [24].

### 8.2 - Finite improvement of the approximate solution

The first important idea is to improve the approximate solution we are working with. To do so, it is natural to apply some iteration scheme and define by induction  $(w_i, \Phi_i)$ ,  $i \geq 0$ , as the solution of

$$\begin{aligned} \rho \Delta_K w_{i+1} + \frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) w_{i+1} + g(\mathfrak{S}_K \Phi_{i+1}, \Theta) \\ = B(\Theta, \Theta)\rho - \mathcal{O}(\rho^2) - \rho L(w_i, \Phi_i) - \frac{1}{\rho} Q(w_i, \Phi_i) \end{aligned}$$

and, for example  $w_0 = 0$ ,  $\Phi_0 = 0$ . Needless to say, one has to assume that (14) is fulfilled. It turns out that this is not the right iteration scheme to consider since, as mentioned above, the operator on the left hand side might not be invertible for all values of  $\rho$  and, even in the cases where it is invertible, the norm of its inverse is large with respect to  $\rho$ . Instead, the idea of A. Malchiodi and M. Montenegro is to use the iteration scheme defined by

$$(15) \quad \begin{aligned} \frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) w_{i+1} + g(\mathfrak{S}_K \Phi_{i+1}, \Theta) \\ = B(\Theta, \Theta)\rho - \mathcal{O}(\rho^2) - \rho L(w_i, \Phi_i) - \frac{1}{\rho} Q(w_i, \Phi_i) - \rho \Delta_K w_i. \end{aligned}$$

The advantage of this iteration scheme versus the previous one is that, this time, the operator on the left hand side has a nice bounded right inverse in the following non conventional sense: Given any function  $\rho z + g(\Psi, \Theta)$  defined on  $SNK$  where  $z(p, \cdot)$  is  $L^2(S^{m-k})$ -orthogonal to  $\text{Ker}(\Delta_{S^{m-k}} + m - k)$  and  $\Psi$  is a section of  $NK$ . We first solve the equation

$$\mathfrak{S}_K \Phi = \Psi.$$

This is where the assumption on the nondegeneracy of  $K$  is used since it implies that the Jacobi operator  $\mathfrak{S}_K$  is invertible. Next, we solve

$$\frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) w(p, \cdot) = z(p, \cdot)$$

for each  $p \in K$ . This amounts to consider  $p \in K$  as a parameter. Finally, direct computation shows that we have

$$\frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) w + g(\mathfrak{S}_K \Phi, \Theta) = \rho z + g(\Psi, \Theta).$$

This recipe yields the existence of a well defined operator

$$\rho z + g(\Psi, \Theta) \in L^2(SNK) \longrightarrow \rho w + g(\Phi, \Theta) \in L^2(SNK)$$

whose norm is bounded independently of  $\rho$  and which is a “right inverse” for the operator which appears on the left hand side of (15). Naturally, the main drawback is that we do not have any gain of regularity along  $K$ ! since the operator on the left hand side of (15) is not elliptic.

One might wonder if one can use this “right inverse” to solve (14) using a fixed point argument. Unfortunately, on the right hand side of (14), the presence of  $\Delta_K$  again annihilates any hope for such a strategy since 2 degrees of regularity are lost at each iteration.

Nevertheless, this “right inverse” is good enough for the iteration scheme to be applied a finite number of times since  $B(\Theta, \Theta) \rho - \mathcal{O}(\rho^2)$ , the error term in (14), is a smooth function. Using this strategy, we obtain easily

$$w_i = \mathcal{O}(\rho^2) \quad \text{and} \quad \Phi_i = \mathcal{O}(\rho^2)$$

and one checks that the mean curvature of  $S_\rho(K, w_i, \Phi_i)$  is given by

$$H(S_\rho(K, w_i, \Phi_i)) = \frac{m - k}{\rho} + \mathcal{O}(\rho^{2+i})$$

for all  $i \geq 1$ .

The estimate for  $\Phi_i$  follow from the fact that  $\Pi(B(\Theta, \Theta)) = 0$ .

### 8.3 - Estimate of the spectrum of some operator

Now the problem reduces to find  $w = w_i + \tilde{w}$  and  $\Phi = \Phi_i + \tilde{\Phi}$  such that  $H(S_\rho(K, w, \Phi)) = \frac{m-k}{\rho}$ . This amounts to solve an equation of the form

$$\left( \rho \Delta_K + \frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) \right) \tilde{w} + g(\mathfrak{S}_K \tilde{\Phi}, \Theta) + \rho L_i(\tilde{w}, \tilde{\Phi}) = \mathcal{O}(\rho^{2+i}) - \frac{1}{\rho} Q_i(\tilde{w}, \tilde{\Phi}).$$

We will be able to solve this nonlinear problem using a fixed point theorem provided we can show that, for some values of  $\rho$ , the inverse of the linear operator on the left hand side has a norm bounded by some fixed power (independently of  $i$ ) of  $1/\rho$ . Therefore, it remains to estimate the distance between 0 and the spectrum of the operator

$$\mathbb{L}_{\rho,i} \tilde{v} := \left( \rho \Delta_K + \frac{1}{\rho} (\Delta_{S^{m-k}} + m - k) \right) \tilde{w} + g(\mathfrak{S}_K \tilde{\Phi}, \Theta) + \rho L_i(\tilde{w}, \tilde{\Phi}).$$

To this aim, one estimates the index of  $\mathbb{L}_{\rho,i}$  (i.e. the number of negative eigenvalues of  $\mathbb{L}_{\rho,i}$ ). We get

$$(16) \quad \text{Index } \mathbb{L}_{\rho,i} \sim c \rho^{-k}.$$

The proof of this estimate uses Weyl's asymptotic formula to estimate the number of eigenvalues of  $\Delta_K$  which are less than a large constant

$$\#\{j \in \mathbb{N} : \lambda_j \leq \lambda\} \sim \lambda^{-\frac{k}{2}}.$$

Next, we estimate the derivative of the small eigenvalues  $\sigma$  of  $\mathbb{L}_{\rho,i}$  (considered as multivalued functions) by

$$(17) \quad \rho \partial_\rho \sigma \geq 2(m-k) - c\rho,$$

as  $\rho$  tends to 0. Let us point out that this estimate is far from being optimal since formal computations (for example starting from (13)) would predict that  $\rho^3 \partial_\rho \sigma \sim 2(m-k)$ . The proof of (17) uses a localization result for the eigenfunctions associated to small eigenvalues: If  $\mathbb{L}_{\rho,i} \tilde{v} = -\sigma \tilde{v}$  and  $\sigma$  is small, then  $\tilde{v}$  is essentially a function defined on  $K$ .

The estimate (17) has important consequences. For example, it implies that, for small values of  $\rho$ , the small eigenvalues of  $\mathbb{L}_{\rho,i}$  are monotone increasing functions of  $\rho$  and therefore, the index of  $\mathbb{L}_{\rho,i}$  is a monotone (decreasing) function of  $\rho$ . Now that we know that the index is a monotone function of  $\rho$ , we can use (16) to show that, given  $t \geq 2$ , the intervals  $[r_1, r_2] \subset (\rho, 2\rho)$  such that

$$r_2 - r_1 \geq \rho^{k+t}$$



and for which  $\mathbb{L}_{r,i}$  has no kernel for all  $r \in [r_1, r_2]$  cover at least a subset of size  $\rho - c\rho^t$  of  $(\rho, 2\rho)$ .

For any such interval  $[r_1, r_2]$ , using (17), we obtain a lower bound for the distance between 0 and the spectrum of  $\mathbb{L}_{\rho,i}$  and hence a uniform estimate for the norm of

$$(\mathbb{L}_{\rho,i})^{-1} : L^2(SNK) \longrightarrow L^2(SNK)$$

by a constant times  $\rho^{-(k+t)}$  when  $r \in (r_1 + \rho^{k+t+1}, r_2 - \rho^{k+t+1})$ . This is precisely the bound on the norm of the inverse by a power (independent of  $i$ ) of  $1/\rho$  we were looking for. Once such a uniform bound has been obtained in the  $L^2$  setting, elliptic regularity theory provides a uniform bound (with a worst exponent) in the Hölder setting. The proof of the result is therefore complete.

#### 8.4 - Open problems

It is an interesting open problem to prove that the above results have a suitable converse, in other words if it is possible to characterize the possible condensation sets of families of constant mean curvature hypersurfaces as their mean curvature tends to  $\infty$ . In this direction, let us mention the recent:

**Theorem 8.1 (H. Rosenberg).** *There exists a constant  $H_0 > 0$  only depending on the geometry of  $(M, g)$  such that, if  $S$  is an embedded constant mean curvature hypersurface with mean curvature greater than  $H_0$  then  $S$  is homologically trivial.*

*Moreover, the distance from any point  $p$  in the mean convex part of  $M - S$  and  $S$  is bounded by  $c/H$ , where  $c > 0$  only depends on the geometry of  $(M, g)$ .*

It follows from §3.2 that, in  $\mathbb{R}^{m+1}$ , Delaunay hypersurfaces do exist. It turns out that Delaunay type hypersurfaces also exist in  $\mathbb{H}^{m+1}$  and they can be obtained by reducing the problem to some nonlinear ordinary differential equation. Moreover Delaunay type hypersurfaces also exist in  $S^{m+1}$  and these can be understood as bifurcated branches from the one parameter family of hypersurfaces described at the end of §7.

It is then a natural question to investigate the existence of these Delaunay type constant mean curvature hypersurfaces in any compact Riemannian manifold. Despite some partial result in this direction [19] the problem remains completely open. The existence or non existence of these Delaunay type hypersurfaces for generic choice of the metric  $g$  is intimately related to a quantization result in the following sense: The hypersurfaces constructed in Theorem 7.1 satisfy

$$(18) \quad H(S)^{m-k} \mathcal{H}^m \lrcorner S \quad \rightarrow \quad m^{m-k} |S^{m-k}| \mathcal{H}^k \lrcorner K,$$

as  $H(S)$  tends to  $+\infty$ . In other words, properly rescaled, the  $m$ -dimensional volume density restricted to  $S$  converges uniformly to the  $k$ -dimensional volume density restricted to  $K$  as the mean curvature tends to  $+\infty$ . Is such a quantization result also true for generic choice of the metric  $g$  and hypersurfaces  $S$  belonging to some fixed noncompact branch of  $\mathcal{S}(SNK, M, g)$ ?

In another direction, is it possible to construct for generic metric, families of constant mean curvature hypersurfaces which condensate along lower dimensional sets which are still minimal in an appropriate sense, but which have singularities (for example, a Steiner tree with geodesic edges)? When  $M^{m+1} = S^{m+1}$  with the canonical metric such constructions seem possible, however again the canonical metric on  $S^{m+1}$  is a non generic metric!

## 9 - Concentration phenomena in singularly perturbed nonlinear problems

The previous results can be understood as typical results which can be obtained in many singularly perturbed nonlinear problems. We give here a few instances of this phenomena.

### 9.1 - Example 1

We consider the semilinear elliptic equation

$$(19) \quad \varepsilon^2 \Delta u - u + u^p = 0$$

where the function  $u$  is defined in  $\Omega \subset \mathbb{R}^n$  and satisfies  $\partial_\nu u = 0$  on  $\partial\Omega$  and  $u > 0$  in  $\Omega$ . The problem is to describe some solutions of this problem as the parameter  $\varepsilon > 0$  tends to 0. First of all, there exist solutions of (19) which concentrate at nondegenerate critical points of the mean curvature of  $\partial\Omega$ , as  $\varepsilon$  tends to 0 [9]. These solutions are precisely the counterpart, in this setting, of the solutions obtained in Theorem 4.1.

Also, solutions with multiple concentration at nondegenerate critical points of the mean curvature of  $\partial\Omega$ , as  $\varepsilon$  tends to 0 have been proven to exist [8]. These solutions are the counterpart, in this setting, of the solutions obtained in Theorem 6.1.

Finally, A. Malchiodi and M. Montenegro have recently proved the existence of solutions of (19) concentrating along nondegenerate geodesics of  $\partial\Omega$  or along all  $\partial\Omega$ , for a sequence of  $\varepsilon$  which tends to 0 [23], [24] and [21].

### 9.2 - Example 2

We consider the nonlinear Schrödinger equations

$$(20) \quad -i \varepsilon \partial_t \psi = \varepsilon^2 \Delta \psi - V \psi + |\psi|^{p-1} \psi$$

for which we are looking for standing waves

$$\psi(t, x) = e^{i\lambda t/\varepsilon} u(x).$$

It is easy to check that the function  $u$  has to be a solution of a singularly perturbed semilinear elliptic equation.

A. Floer and A. Weinstein [7] have proved the existence of solutions of (20) concentrating at nondegenerate critical points of  $V$  as  $\varepsilon$  tends to 0. This results parallel the result of Theorem 4.1.

Recently, A. Ambrosetti, A. Malchiodi and W.M. Ni on the one hand and M. Montenegro, M. del Pino, M. Kowalczyk and J. Wei on the other hand have proved that there exist solutions of (20) concentrating along curves which are nondegenerate critical points of the weighted length functional

$$K \longrightarrow \int_K (\lambda + V)^{\frac{p+1}{p-1} - \frac{1}{2}} d\text{vol}_K$$

for a sequence of  $\varepsilon$  which tends to 0. Again, these results parallel the result of Theorem 7.1.

### 9.3 - Example 3

We give a last example in the framework of Hamiltonian systems. We consider the Hamiltonian system associated to the action

$$A(y) := \int -\frac{1}{2} |\partial_s y|^2 + w(y)$$

where the curve  $y$  is restricted to some submanifold  $M \subset \mathbb{R}^n$ . For all  $\varepsilon > 0$ , we consider the singularly perturbed action

$$A_\varepsilon(x) := \int -\frac{1}{2} |\partial_s x|^2 + w(x) + \frac{1}{\varepsilon^2} G(x)$$

where this time  $x$  is a curve in  $\mathbb{R}^n$  and  $G(x) := \text{dist}(x, M)^2$ .

J. Shatah and C. Zeng [29] have proved that nondegenerate periodic solutions of the Hamiltonian system associated to  $A$  can be perturbed into periodic solutions of the Hamiltonian system associated to  $A_\varepsilon$  for a suitably chosen sequence of  $\varepsilon$  which tends to 0. Again, this result parallels the result of Theorem 7.1. A similar resonance phenomena arises in the work of A. Malchiodi [22] where the equation

$$(21) \quad \partial_s^2 x + \frac{1}{\varepsilon^2} \nabla V(x) = 0$$

is considered for a curve  $x$  in  $\mathbb{R}^n$ . Here the potential  $V$  is assumed to have a sub-

manifold  $M \subset \mathbb{R}^n$  as set of critical points and  $(X, Y) \longrightarrow D^2V(X, Y)$  is positive definite on the normal bundle of  $M$ . Then, periodic solutions of (21) are shown to exist for a sequence of  $\varepsilon$  tending to 0 and these converge to a closed geodesic of  $M$  as  $\varepsilon$  tends to 0.

### 10 - Further comments

If the concentration phenomena at isolated points has been extensively studied over the last 30 years, the concentration phenomena along submanifolds has just been the object of more recent attention.

Beside the above examples, there are important works for which a concentration phenomena along submanifolds has been pointed out. This is for example the case in the study of Sibert-Witten equations [30], the study of Ginzburg-Landau equation in higher dimensions [3], the study of Cahn-Hilliard or Allen-Cahn equations [26], [15], ... However, in all these last examples the resonance phenomena (which is present in the previous list of examples) does not arise and the only problem one has to deal with is usually the perturbation of the concentration set as the parameter tends to 0.

We end up this note by the following observation: In Example 3, if one assumes that  $(X, Y) \longrightarrow D^2V(X, Y)$  is negative definite on the normal bundle of  $M$ , then, periodic solutions of (21) do exist for all small values of the parameter  $\varepsilon$  and these converge to a closed geodesic of  $M$  as  $\varepsilon$  tends to 0. Hence the resonance phenomena does not appear any more. Similarly, the equation which arises in the Allen-Cahn theory [15] reads

$$(22) \quad \varepsilon^2 \Delta + u - u^3 = 0.$$

The nonlinearity has the opposite sign as the one which appears in (19) and this change of sign is responsible of the loss of the resonance phenomena in the study of (22).

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### Abstract

*In this short note, we report some recent progress on the existence of constant mean curvature hypersurfaces in Riemannian manifolds. We also compare these results with similar results which have been obtained for singularly perturbed nonlinear problems.*

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