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Hydrodynamic Limits for the Boltzmann Equation (**)

Contents

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1 - The Boltzmann Equation

The Boltzmann equation governs the evolution of the distribution of molecules in rarefied gases. Originally, the equation was written by Boltzmann for monatomic gases. Various generalizations have been proposed more recently (for polyatomic gases, with exchange of internal energy at the molecular level or chemical reactions). However, for the sake of simplicity, the present survey will only address the case of monatomic gases.

This chapter follows mostly the presentation in [**12**], [**55**] and [**8**].

In kinetic theory, the state of a (rarefied) gas is adequately described by the distribution of molecules in phase-space (also called the distribution function or the number density), $F \equiv F(t, x, v)$ which is the density of particles located at the position $x \in \boldsymbol{R}^3$ with velocity $v \in \boldsymbol{R}^3$ at time $t \geq 0$.

In the absence of external forces (such as gravity, Coriolis force, electromagnetic force in the case of ionized gases), the number density $F \equiv F(t, x, v)$ satisfies

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F) \tag{1}$$

where $\mathcal{B}(F, F)$ is the *Boltzmann collision integral*.

The following simple remarks have important consequences on the structure of the Boltzmann equation:

- because the Boltzmann equation is meant to describe a rarefied gas, molecular collisions other than binary are neglected;
- at the kinetic level of description, the molecular radius is neglected everywhere except in the expression giving the mean free path, so that
- in Boltzmann's theory, collisions are a purely local and instantaneous process.

In view of these remarks, one anticipates that

- the collision integral is quadratic in the number density F, and
- the collision integral acts only on the v variable in $F(t, x, v)$.

1.1 - *The Boltzmann collision integral*

For a gas of hard spheres with radius r, the action of the Boltzmann collision integral on a function $f \equiv f(v)$ is

$$\mathcal{B}(f, f)(v) = 2r^2 \iint_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} \big(f(v')f(v'_*) - f(v)f(v_*)\big)|(v - v_*) \cdot \omega| d\omega \, dv_* \,, \tag{2}$$

where the velocities v' and v'_* are defined in terms of v, v_* and ω by the formulas

$$\begin{aligned} v' &\equiv v'(v, v_*, \omega) = v - (v - v_*) \cdot \omega\omega \,, \\ v'_* &\equiv v'(v, v_*, \omega) = v_* + (v - v_*) \cdot \omega\omega \,. \end{aligned} \tag{3}$$

That the collision integral acts only on the v variable in F means that the right hand side of the Boltzmann equation (1) is

$$\mathcal{B}(F, F)(t, x, v) :\equiv \mathcal{B}(F(t, x, \cdot), F(t, x, \cdot))(v)$$

with the definition (2) above for the collision integral acting on a function of v alone.

The following notation may seem unfelicitous; it is however customary in the literature devoted to the Boltzmann equation and must not be ignored.

Notation: one designates $F(t,x,v_*)$, $F(t,x,v')$ and $F(t,x,v'_*)$ respectively by F_*, F' and F'_*.

With this notation, the collision integral in the right hand side of (1) is written as

$$\mathcal{B}(F,F) = 2r^2 \iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*)|(v-v_*)\cdot\omega|d\omega\, dv_*\,.$$

Later on, we also designate by $\mathcal{B}$ the symmetric bilinear operator associated to the quadratic expression above:

$$\mathcal{B}(F,G) = \frac{1}{2}(\mathcal{B}(F+G,F+G) - \mathcal{B}(F,F) - \mathcal{B}(G,G))\,.$$

Let us discuss the geometrical and mechanical meaning of the relations (3). Observe first that these relations can be equivalently formulated as

$$\begin{aligned} v' + v'_* &= v + v_* \\ v' - v'_* &= (v-v_*) - 2(v-v_*)\cdot\omega\omega = \mathcal{R}_\omega(v-v_*) \end{aligned} \tag{4}$$

where $\mathcal{R}_\omega$ designates the specular reflection on the plane orthogonal to the vector ω. In particular, one has

$$|v' - v'_*| = |v - v_*|\,. \tag{5}$$

Therefore the 4 points v, v_*, v' and v'_* lie on a same circle, and ω is one of the (external) bissectors of the angle $(\widehat{v-v_*, v'-v'_*})$ — see figure 1.

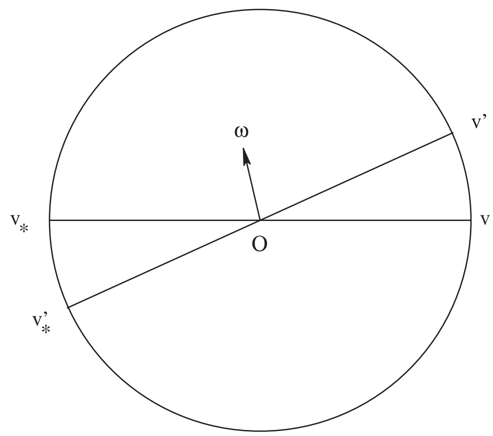

Fig. 1. The pre- and post-collision velocities in the reference frame of the center of mass of the particle pair.

From the mechanical viewpoint, the origin $\frac{1}{2}(v+v_*)$ is the velocity of the center of mass for any pair of molecules with velocities v and v_*; in (4), the first equality is the conservation of momentum for any pair of colliding molecules with velocities v, v_* after collision, and v', v'_* before collision. The equality of relative speeds before and after collision is equivalent to the conservation of kinetic energy by the collision process — i.e. the collisions considered are purely elastic. In other words, $v'(v, v_*, \omega)$ and $v'_*(v, v_*, \omega)$ represent all possible solutions in the unknowns v' and v'_* of the system of equations

$$
\begin{aligned}
v' + v'_* &= v + v_* \,, \\
|v'|^2 + |v'_*|^2 &= |v|^2 + |v_*|^2 \,.
\end{aligned}
\tag{6}
$$

Momentum and kinetic energy, together with the number of gas molecules, are the only natural conserved quantities at the microscopic level.

In this survey, when considering a gas of hard spheres, we shall assume without loss of generality that the molecular radius is $1/\sqrt{2}$.

The most important properties of the Boltmann equation, described in the next two sections, are straightforward consequences of the structure of the collision integral, and more specifically of the conservation laws at the microscopic level established above.

1.2 - *Local conservation laws*

First, one expects that the conservation laws (6) should have analogues at the macroscopic (fluid) level. These analogues are formulated according to the general recipe for defining macroscopic observables starting with microscopic quantities.

Proposition 2.1. *Assume that* $f \equiv f(v) \in L^1_{loc}(\boldsymbol{R}^3)$ *is rapidly decaying at infinity, i.e.*

$$f(v) = O(|v|^{-n}) \text{ as } |v| \to +\infty \quad \text{for all } n \geq 0 \,,$$

while $\phi \in C(\boldsymbol{R}^3)$ *has at most polynomial growth at infinity, i.e.*

$$\phi(v) = O(|v|^m) \text{ as } |v| \to +\infty \quad \text{for some } m \geq 0 \,.$$

Then one has

$$\int_{\boldsymbol{R}^3} \mathcal{B}(f,f)\phi dv = \frac{1}{4} \iiint_{\boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{S}^2} (f'f'_* - ff_*)(\phi + \phi_* - \phi' - \phi'_*)|(v - v_*) \cdot \omega| d\omega\, dv dv_* \,.$$

Since the proof of this proposition involves some of the most fundamental tricks in the theory of the Boltzmann collision operator, we give it in detail.

Proof. The assumptions on the decay of f and the growth of ϕ at infinity guarantee that all the integrals considered in the course of this proof are absolutely convergent.

Start with the obvious equality

$$\int_{\boldsymbol{R}^3} \mathcal{B}(f,f)\phi dv = \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f'f'_* - ff_*)\phi|(v-v_*)\cdot\omega|d\omega\, dvdv_* .$$

In the right-hand side of this equality, for each fixed $\omega \in \boldsymbol{S}^2$, apply the change of variables $(v,v_*)\mapsto(v_*,v)$. The formulas (3) show that, under this change of variables $(v',v'_*)\mapsto(v'_*,v')$. Hence

$$\begin{aligned}\int_{\boldsymbol{R}^3} \mathcal{B}(f,f)\phi dv &= \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f'f'_* - ff_*)\phi|(v-v_*)\cdot\omega|d\omega\, dvdv_* \\ &= \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f'f'_* - ff_*)\phi_*|(v-v_*)\cdot\omega|d\omega\, dvdv_* \\ &= \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f'f'_* - ff_*)\frac{\phi+\phi_*}{2}|(v-v_*)\cdot\omega|d\omega\, dvdv_* .\end{aligned}$$

Next, apply the change of variables $(v,v_*)\mapsto(v',v'_*)$ for each fixed $\omega\in\boldsymbol{S}^2$ in the last integral above. In the reference frame of the center of mass, this change of variables essentially reduces to the specular reflection $\mathcal{R}_\omega$ that exchanges the relative velocities:

$$\mathcal{R}_\omega : v - v_* \mapsto v' - v'_* .$$

Because $\mathcal{R}_\omega$ is an involution (meaning that $\mathcal{R}_\omega^2 = Id$), the change of variables above also is an involution and maps (v',v'_*) onto (v,v_*). Moreover, the second relation in (6) implies that this change of variables is an isometry of $\boldsymbol{R}^3\times\boldsymbol{R}^3$, and therefore leaves the Lebesgue measure $dvdv_*$ invariant. Since $(v'-v'_*)\cdot\omega = -(v-v_*)\cdot\omega$, applying this change of variables in the right hand side of the above equality implies that

$$\begin{aligned}&\int_{\boldsymbol{R}^3} \mathcal{B}(f,f)\phi dv \\ &= \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f'f'_* - ff_*)\frac{\phi+\phi_*}{2}|(v-v_*)\cdot\omega|d\omega\, dvdv_* \\ &= \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (ff_* - f'f'_*)\frac{\phi'+\phi'_*}{2}|(v-v_*)\cdot\omega|d\omega\, dvdv_* \\ &= \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f'f'_* - ff_*)\frac{\phi+\phi_*-\phi'-\phi'_*}{4}|(v-v_*)\cdot\omega|d\omega\, dvdv_*\end{aligned}$$

as announced. □

In view of the proposition above, the following class of functions is of particular importance.

Definition 2.2. *A collision invariant is a measurable a.e. finite function* $\phi \equiv \phi(v)$ *such that, for each* $(v, v_*) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3$ *and each* $\omega \in \boldsymbol{S}^2$, *one has*

$$\phi(v) + \phi(v_*) = \phi(v') + \phi(v'_*) .$$

Constants are obviously collision invariants. In view of (6), other interesting examples of collision invariants are $\phi(v) = v_k$ for $k = 1, 2, 3$ — i.e. the 3 components of v — or $\phi(v) \equiv |v|^2$.

An important result in the theory of the Boltzmann equation asserts that the examples above provide all the collision invariants, up to linear combinations.

Proposition 2.3. *Any collision invariant is a function of the form*

$$\phi(v) = a + b_1 v_1 + b_2 v_2 + b_3 v_3 + c|v|^2 ,$$

where a, b_1, b_2, b_3 *and* c *are arbitrary elements of* $\boldsymbol{R}$.

The proof of this proposition is far from obvious; see for instance [**12**] on pp. 36-42.

In any case, whenever ϕ is a collision invariant and f is a measurable, rapidly decaying function, it follows from Proposition 2.1 that

$$\int_{\boldsymbol{R}^3} \mathcal{B}(f, f)\phi dv = 0 .$$

This entails in particular the following

Corollary 2.4. *Let* $F \equiv F(t, x, v)$ *be a solution of the Boltzmann equation* (1) *that is locally integrable and rapidly decaying in* v *for each* (t, x). *Then*

$$\int_{\boldsymbol{R}^3} \mathcal{B}(F, F) dv = \int_{\boldsymbol{R}^3} \mathcal{B}(F, F) v_k dv = \int_{\boldsymbol{R}^3} \mathcal{B}(F, F)|v|^2 dv = 0 \tag{7}$$

for $k = 1, 2, 3$, *and the following local conservation laws hold:*

$$\begin{aligned} \partial_t \int_{\boldsymbol{R}^3} F dv + \operatorname{div}_x \int_{\boldsymbol{R}^3} v F dv = 0 , \\ \partial_t \int_{\boldsymbol{R}^3} v F dv + \operatorname{div}_x \int_{\boldsymbol{R}^3} v \otimes v F dv = 0 , \\ \partial_t \int_{\boldsymbol{R}^3} \frac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\boldsymbol{R}^3} v \frac{1}{2}|v|^2 F dv = 0 , \end{aligned} \tag{8}$$

respectively the local conservation of mass (or continuity equation), momentum and energy.

Define the following fields:

$$\rho = \int_{\mathbf{R}^3} F dv \,, \quad u = \frac{1}{\rho} \int_{\mathbf{R}^3} v F dv \,, \quad P = \int_{\mathbf{R}^3} (v-u)^{\otimes 2} F dv \,,$$
$$C = \int_{\mathbf{R}^3} (v-u)|v-u|^2 F dv \,.$$

Notice that, by definition of u, one has

$$\begin{aligned} \int_{\mathbf{R}^3} v^{\otimes 2} F dv &= \rho u^{\otimes 2} + P \,, \\ \int_{\mathbf{R}^3} |v|^2 F dv &= \rho |u|^2 + \mathrm{trace}(P) \,, \\ \int_{\mathbf{R}^3} v|v|^2 F dv &= (\rho |u|^2 + \mathrm{trace}(P)) u + 2P\cdot u + C \,. \end{aligned}$$

Therefore, the system of conservation laws above can be put in the form

$$\begin{aligned} \partial_t \rho + \mathrm{div}_x(\rho u) &= 0 \,, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u + P) &= 0 \,, \\ \partial_t \frac{1}{2}(\rho |u|^2 + \mathrm{trace}(P)) + \mathrm{div}_x \frac{1}{2}((\rho|u|^2 + \mathrm{trace}(P))u + 2P\cdot u + C) &= 0 \,. \end{aligned} \tag{9}$$

If we knew that $P = pI$ and $C = 0$, this system would coincide exactly with the Euler equations for compressible fluids, with perfect gas pressure law.

However, one should bear in mind that (9) is satisfied by *any* solution of the Boltzmann equation, and therefore by any perfect gas in a kinetic regime. Thus one cannot expect that such a gas in a kinetic regime be in local thermodynamic equilibrium. In other words, one cannot hope that, for a generic solution of the Boltzmann equation, the tensor field P be of the form pI, for instance, or that $C = 0$. As we shall see, deriving the compressible Euler system from the Boltzmann equation requires additional arguments.

1.3 - *Boltzmann's H Theorem*

Undoubtedly, the most important feature of the Boltzmann equation, along with the conservation laws stated in Corollary 2.4 is Boltzmann's H Theorem. As in the case of the conservation laws, we begin with a statement that bears exclusively on the collision integral.

Theorem 3.1. (Boltzmann's H Theorem). *Let $f \equiv f(v) > 0$ be a locally integrable function that is rapidly decaying and such that $\ln f$ has at most polynomial growth as $|v| \to +\infty$. Then*

$$\int_{\mathbf{R}^3} \mathcal{B}(f,f) \ln f dv = -\frac{1}{4} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (f'f'_* - ff_*) \ln\left(\frac{f'f'_*}{ff_*}\right) |(v-v_*)\cdot\omega| d\omega\, dv dv_* \leq 0\,.$$

Moreover, the following conditions are equivalent:

(a) $\mathcal{B}(f,f) = 0$ a.e.,

(b) $\int_{\mathbf{R}^3} \mathcal{B}(f,f) \ln f dv = 0$,

(c) *f is a Maxwellian density, i.e.*

$$f(v) = \mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}}$$

for some $\rho, \theta > 0$ and $u \in \mathbf{R}^3$.

Proof. Applying Proposition 2.1 with $\phi = \ln f$ leads to the first equality above; since the logarithm is an increasing function, the right hand side of this first equality is nonnegative.

As for the equality case, observe that (a) obviously implies (b); that (c) implies (a) follows by inspection. The only non trivial point is that (b) implies (c). If one takes Proposition 2.3 for granted and assumes that f is continuous, it is immediate. Indeed, $\ln f$ is then a collision invariant, which is clearly equivalent to the fact that f is a Maxwellian. □

Since we do not know in general whether f is continuous, the implication $(b) \Rightarrow (c)$ is a consequence of the following

Lemma 3.2. (Perthame [**49**]). *Let $f > 0$ a.e. be a measurable function such that*

$$\int_{\mathbf{R}^3} (1+|v|^2) f(v) dv < +\infty\,.$$

If

$$f(v)f(v_*) = f(v')f(v'_*)$$

for a.e. $(v, v_*, \omega) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{S}^2$ with v' and v'_* given by (3), *then f is a Maxwellian.*

Perthame's proof uses the Fourier transform of the functional equation on f in a very clever way; see also [8] on pp. 47-48.

From the above statement on the collision integral, we deduce the following important consequence on solutions of the Boltzmann equation.

Corollary 3.3. *Let $F \equiv F(t, x, v) > 0$ be a solution of the Boltzmann equation that is rapidly decaying and such that* $\ln F$ *has at most polynomial growth as* $|v| \to +\infty$. *Then, one has*

$$\partial_t \int_{\boldsymbol{R}^3} F \ln F dv + \operatorname{div}_x \int_{\boldsymbol{R}^3} vF \ln F dv = -\frac{1}{4} \iiint_{\boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v - v_*) \cdot \omega| d\omega\, dv dv_* \leq 0\,. \tag{10}$$

Defining

$$S = -\frac{1}{\rho} \int_{\boldsymbol{R}^3} F \ln F dv\,, \quad N = -\int_{\boldsymbol{R}^3} (v - u) F \ln F dv$$

we see that the differential inequality (10) takes the form

$$\partial_t(\rho S) + \operatorname{div}_x(\rho S u + N) \geq 0\,. \tag{11}$$

Again, this differential inequality is formally reminiscent of the Lax-Friedrichs criterion that selects admissible solutions of hyperbolic systems of conservation laws, of which the Euler equations for compressible fluids are the most famous example. See [**13**], § 4.3 for a discussion of this criterion. In the case of the Euler equations for perfect gases, $N = 0$, so that the above differential inequality means that the specific entropy S is a nondecreasing quantity along the trajectory of each infinitesimal fluid element.

However, the inequality (11) is satisfied by any solution of the Boltzmann equation, therefore by any monatomic gas of hard spheres in kinetic regime.

A considerable difference with the theory of ideal fluids is that Boltzmann's H Theorem provides an expression for the entropy production rate in terms of the number density that is local in (t, x). In the theory of ideal fluids, one only knows that the entropy is produced across shock waves, but there is no expression of the entropy production there.

1.4 - *A priori estimates*

In the mathematical literature on the Boltzmann equation, Boltzmann's H Theorem is very often used as a tool for obtaining a priori estimates on the number density. In the discussion below, we describe two important examples of such bounds.

1.4.1 - The Euclidian space with Maxwellian equilibrium at infinity

Consider first the case of a gas which is in Maxwellian equilibrium at infinity. In other words, we consider the Cauchy problem

$$\begin{aligned} \partial_t F + v \cdot \nabla_x F &= \mathcal{B}(F,F)\,, \quad (t,x,v) \in \boldsymbol{R}_+^* \times \boldsymbol{R}^3 \times \boldsymbol{R}^3\,, \\ F(t,x,v) &\to \mathcal{M}_{(\rho,u,\theta)}\,, \quad |x| \to +\infty\,, \\ F\big|_{t=0} &= F^{in}\,. \end{aligned}$$

An important notion in this context is that of relative entropy.

Definition 4.1. *Let $F \geq 0$ a.e. and $G > 0$ be two measurable functions on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$; the relative entropy of F with respect to G is*

$$H(F|G) = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \left(F \ln\left(\frac{F}{G}\right) - F + G \right) dx dv\,.$$

Notice that the integrand in the definition of $H(F|G)$ is an a.e. nonnegative measurable function, so that the relative entropy $H(F|G)$ is well defined as an element of $[0,+\infty]$.

Going back to the Cauchy problem above, we shall assume that F converges to the Maxwellian state $\mathcal{M}_{(\rho,u,\theta)}$ rapidly enough so that the relative entropy

$$H(F(t)|\mathcal{M}_{(\rho,u,\theta)}) = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dx dv < +\infty$$

for each $t \geq 0$. We claim that

$$(12) \qquad \frac{1}{4}\int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v-v_*)\cdot\omega| dv dv_* d\omega dx ds$$

$$= H(F(0)|\mathcal{M}_{(\rho,u,\theta)}) - H(F(t)|\mathcal{M}_{(\rho,u,\theta)})\,,$$

for each $t \geq 0$.

Indeed,

$$\begin{aligned}\int_{\boldsymbol{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)dv &= \int_{\boldsymbol{R}^3} F\ln F dv\\ &\quad+\frac{1}{2\theta}\int_{\boldsymbol{R}^3}(|v|^2+|u|^2)Fdv-\frac{1}{\theta}\int_{\boldsymbol{R}^3}u\cdot vFdv\\ &\quad-\left(1+\ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right)\int_{\boldsymbol{R}^3}Fdv+\rho\,,\end{aligned}$$

while

$$\begin{aligned}\int_{\boldsymbol{R}^3}v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)dv &= \int_{\boldsymbol{R}^3} vF\ln F dv\\ &\quad+\frac{1}{2\theta}\int_{\boldsymbol{R}^3}v(|v|^2+|u|^2)Fdv-\frac{1}{\theta}\int_{\boldsymbol{R}^3}vu\cdot vFdv\\ &\quad-\left(1+\ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right)\int_{\boldsymbol{R}^3}vFdv\,.\end{aligned}$$

In other words,

$$\begin{aligned}\int_{\boldsymbol{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)dv &= \int_{\boldsymbol{R}^3} F\ln F dv\\ &\quad+\text{ locally conserved quantity}\end{aligned}$$

while

$$\begin{aligned}\int_{\boldsymbol{R}^3}v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)dv &= \int_{\boldsymbol{R}^3} vF\ln F dv\\ &\quad+\text{ flux of that locally conserved quantity}\end{aligned}$$

so that

$$\begin{aligned}&\partial_t\int_{\boldsymbol{R}^3}\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)dv\\ &\quad+\operatorname{div}_x\int_{\boldsymbol{R}^3}v\left(F\ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right)-F+\mathcal{M}_{(\rho,u,\theta)}\right)dv\\ &\quad=\partial_t\int_{\boldsymbol{R}^3}F\ln F dv+\operatorname{div}_x\int_{\boldsymbol{R}^3}vF\ln F dv\,.\end{aligned}$$

Hence

$$\partial_t \int_{\boldsymbol{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$+ \operatorname{div}_x \int_{\boldsymbol{R}^3} v \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$= -\frac{1}{4} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v-v_*)\cdot\omega| dv dv_* d\omega \,.$$

Integrating further on $[0,t]\times\boldsymbol{R}^3$, one arrives at (12).

To summarize, we deduce from (12) that

- the relative entropy bound

$$0 \le H(F(t)|\mathcal{M}_{(\rho,u,\theta)}) \le H(F^{in}|\mathcal{M}_{(\rho,u,\theta)}) \,, \qquad \text{for each } t \ge 0 \,;$$

- the entropy production estimate

$$\frac{1}{4}\int_0^{+\infty}\int_{\boldsymbol{R}^3}\iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v-v_*)\cdot\omega| dv dv_* d\omega dx ds$$
$$\le H(F^{in}|\mathcal{M}_{(\rho,u,\theta)}) \,.$$

1.4.2 - The Euclidian space with vacuum at infinity

Next, we consider the case of a cloud of gas expanding in the vacuum. This case is slightly more involved than the previous one. Consider the Cauchy problem

$$\begin{aligned} &\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F,F)\,, \quad (t,x,v)\in\boldsymbol{R}^*_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3\,,\\ &F(t,x,v)\to 0\,, \quad |x|,|v|\to+\infty\,,\\ &F\big|_{t=0} = F^{in}\,. \end{aligned}$$

We shall assume that F vanishes rapidly enough at infinity so that the relative entropy

$$H(F(t)|\mathcal{G}) < +\infty \qquad \text{for each } t\ge 0\,,$$

where $\mathcal{G}$ is the centered reduced Gaussian

$$\mathcal{G}(x,v) = \frac{1}{(2\pi)^3} e^{-\frac{|x|^2+|v|^2}{2}}\,.$$

In addition to the relative entropy $H(F(t)|\mathcal{G})$, another important quantity is Boltzmann's H function:

Definition 4.2. *Let $F \geq 0$ a.e. be an element of $L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3)$ such that*

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |F \ln F(x,v)| dxdv < +\infty \,.$$

One denotes by $H(F)$ the quantity

$$H(F) = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F \ln F(x,v) dxdv \,.$$

Whenever there is no risk of ambiguity, we use the notation $H(t)$ to designate $H(F(t,\cdot,\cdot))$, when F is a solution of the Boltzmann equation.

Assume that

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in}(|\ln F^{in}| + |x|^2 + |v|^2 + 1) dxdv < +\infty \,.$$

We claim that, for each ≥ 0,

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |x - tv|^2 F(t,x,v) dxdv = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |x|^2 F(0,x,v) dxdv \,. \tag{13}$$

Indeed

$$\begin{aligned}
\frac{d}{dt} \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} & |x - tv|^2 F(t,x,v) dxdv \\
&= \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \partial_t \big(|x - tv|^2 F(t,x,v)\big) dxdv \\
&= \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (\partial_t + v \cdot \nabla_x)\Big(|x - tv|^2 F(t,x,v)\Big) dxdv \\
&= \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |x - tv|^2 (\partial_t + v \cdot \nabla_x) F(t,x,v) dxdv \\
&= \int_{\boldsymbol{R}^3} \left(\int_{\boldsymbol{R}^3} |x - tv|^2 \mathcal{B}(F,F)(t,x,v) dv \right) dx = 0 \,.
\end{aligned}$$

Observe that

$$\begin{aligned}
H(F|\mathcal{G}) = & H(F) + \frac{1}{2} \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (|x|^2 + |v|^2) F dxdv \\
& + (3 \ln(2\pi) - 1) \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F dxdv + 1 \,.
\end{aligned}$$

Because of (13), one has

$$\begin{aligned}\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} &|x|^2 F(t)dxdv\\ &\le 2\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |x-tv|^2 F(t)dxdv + 2t^2 \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F(t)dxdv\\ &= 2\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |x|^2 F(0)dxdv + 2t^2 \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F(0)dxdv\end{aligned}$$

so that

$$\begin{aligned}&-\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |x|^2 F^{in}dxdv - (\frac{1}{2}+t^2)\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F^{in}dxdv\\ &-(3\ln(2\pi)-1)\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in}dxdv - 1 \le H(t) \le H(0)\,.\end{aligned}$$

Integrating on $[0,t]\times\boldsymbol{R}^3$ the local entropy equality (10), one arrives at the equality

$$\begin{aligned}H(F(0))&-H(F(t))\\ &=\frac{1}{4}\int_0^t\int_{\boldsymbol{R}^3}\iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*)\ln\left(\frac{F'F'_*}{FF_*}\right)|(v-v_*)\cdot\omega|dvdv_*d\omega dxds \qquad (14)\\ &\le C(1+t^2)\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3}(1+|x|^2+|v|^2+|\ln F^{in}|)F^{in}dvdx\,.\end{aligned}$$

In particular, whenever F is a classical solution of the Boltzmann equation that satisfies $H(F(t)|\mathcal{G}) < +\infty$ for each $t > 0$ and decays rapidly enough at infinity, $H(F(t))$ is a nonincreasing function of t; it was precisely this property that Boltzmann called «the H Theorem». Moreover, $H(F)$ is stationary only if F is a Maxwellian (see the case of equality in Boltzmann's H Theorem above).

Hence, from the physical viewpoint, it is natural to think of $H(F(t))$ as *minus the entropy* of the system of particles distributed under $F(t,\cdot,\cdot)$.

The case of the Euclidian space with Maxwellian equilibrium at infinity is the most natural setting where to consider the hydrodynamic limit; the case of a cloud of gas expanding in the vacuum is also very natural, albeit not for the hydrodynamic limit. However, several important results on the Boltzmann equation (for instance the derivation of the Boltzmann equation by Lanford [**36**], the original version of the DiPerna-Lions global existence result [**17**]) have been established in this case.

1.5 - *More general collision kernels*

The Boltzmann collision integral considered so far involved the collision kernel

$$b(v-v_*,\omega)=2r^2|(v-v_*)\cdot\omega|$$

that corresponds to pairwise elastic collisions between hard spheres of radius r. But gas molecules are more complicated objects than just hard spheres, and their pairwise interaction is a rather complex combination of the electrostatic potential created by the elementary constituents of the molecules (electrons and protons).

In general, the collision operator is

$$\mathcal{B}(f,f)=\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2}(f'_*f'-f_*f)b(v_*-v,\omega)dv_*d\omega \tag{15}$$

while the collision kernel b has the classical form

$$b(v_*-v,\omega)=|v_*-v|\Sigma\left(|v_*-v|,\left|\omega\cdot\frac{v_*-v}{|v_*-v|}\right|\right). \tag{16}$$

In the formula above, $\Sigma\geq 0$ is the specific differential cross-section, which has units of area (length^2) over mass.

The specific dependence of b on (v,v_*,ω) — more specifically the fact that b only depends on $|v-v_*|$ and $\left|\omega\cdot\frac{v_*-v}{|v_*-v|}\right|$ — implies that

$$b(v-v_*,\omega)=b(v_*-v,\omega)=b(v'_*-v',\omega) \tag{17}$$

for each $(v,v_*,\omega)\in\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2$. These relations imply that the collision integral (15) satisfies Proposition 2.1 and the resulting local conservation laws (Corollary 2.4), as well as Boltzmann's H Theorem (Theorem 3.1) and its consequences (mainly Corollary 3.3 and the a priori estimates in section 1.4).

Of course all these properties are subject to the obvious requirement that the Boltzmann collision integral should converge in some sense. This is however far from obvious whenever the molecular interaction is given by a long-range potential.

A typical example of such a situation is the case of an inverse power-law repulsive potential of the form

$$U(r)=\frac{c}{r^k},$$

where c and k are positive constants, and r is the intermolecular distance. Instead of giving a complete derivation of the collision kernel b — or equivalently of the cross-section Σ — in this case, we refer the interested reader to pp. 67-71 of [**11**], and summarize the results there.

In this case, one can show that b has the factored form

$$b(v_* - v, \omega) = |v_* - v|^\beta \hat{b}\left(\left|\omega \cdot \frac{v_* - v}{|v_* - v|}\right|\right), \quad \text{with} \quad \beta = 1 - \frac{4}{k}\,.$$

This will be locally integrable with respect to dv_* provided $\beta > -3$, which leads to the constraint

$$k > 1$$

meaning that the marginal case of the Coulomb potential c/r is excluded.

We will not give the function $\hat{b}$ here. We will however remark that $\hat{b}$ is well-behaved except for a singularity at $\omega \cdot \frac{v_* - v}{|v_* - v|} = 0$ of the form

$$\hat{b}(s) \sim s^{-\hat{\beta}} \quad \text{as } s \to 0\,, \quad \text{with} \quad \hat{\beta} = 1 + \frac{2}{k}\,.$$

This singularity arises due to the infinite range of the c/r^k potential. It reflects the fact that there are many collisions in which the colliding molecules do not pass very close to each other and are therefore deflected only slightly. This singularity has proved difficult to analyze. For example, the fact that this singularity is not integrable with respect to $d\omega$ means that the gain and loss part of the Boltzmann collision integral, defined respectively as

$$\begin{aligned} \mathcal{B}^+(f,f) &= \iint_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} f'_* f' b(v_* - v, \omega) dv_* d\omega \\ \mathcal{B}^-(f,f) &= \iint_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} f_* f b(v_* - v, \omega) dv_* d\omega \end{aligned} \tag{18}$$

do not make sense. So-called cut-off collision kernels have therefore been introduced. These replace the exact $\hat{b}$ above with a more benign one, by replacing the angular part of the cross-section, i.e. the function $\hat{b}$ with its truncation for s below some small value s_0, that can be defined as

$$\overline{b}(s) = \inf\,(\hat{b}(s_0), \hat{b}(s))\,.$$

H. Grad [**31**] argued that this truncation is legitimate on physical grounds for neutral gases, since grazing collisions (which are responsible for the singularity of $\hat{b}$ at $s = 0$) are statistically negligeable in that case. In the case of plasmas such a truncation is of course not valid and grazing collisions are important in some variety of physical regimes.

In any case, these considerations led Grad to propose the notion of «cut-off potential» — a slightly improper terminology, since in this procedure, it is the collision kernel that is truncated and not the potential.

More specifically, we shall say that the collision kernel b comes from a «hard cut-off potential» if, for each $(z,\omega)\in \boldsymbol{R}^3\times \boldsymbol{S}^2$

$$0\le b(z,\omega)\le C(1+|z|)^{\beta} \text{ and } \int_{\boldsymbol{S}^2} b(z,\omega)d\omega \ge \frac{1}{C_b}(1+|z|)^{\beta}$$

for some $\beta\in[0,1]$ and $C_b>0$. Instead, we shall say that it comes from a «soft cut-off potential» if b satisfies the above conditions with $\beta\in(-3,0)$.

In addition to the case of hard spheres mentionned above, a notable particular case is that of a «cut-off Maxwellian interaction» corresponding to

$$b(z,\omega)=\hat{b}\left(\left|\omega\cdot\frac{v_*-v}{|v_*-v|}\right|\right), \tag{19}$$

with $0<\hat{b}\in C([0,1])$. This particular case attracted Maxwell's attention since the linearized collision integral can then be reduced to diagonal form explicitly by using Sonine polynomials (a multidimensional variant of Hermite polynomials).

In the sequel, we shall mostly consider hard cut-off potentials, and sometimes only the particular case of hard spheres.

1.6 - *The linearized collision integral*

In this section, we return to the hard sphere case. Let $\mathcal{M}_{(\rho,u,\theta)}$ be a uniform Maxwellian. The linearization at $\mathcal{M}_{(\rho,u,\theta)}$ of the collision integral is given by the formula

$$\begin{aligned}\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}\phi &= -2\mathcal{M}^{-1}_{(\rho,u,\theta)}\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)},\mathcal{M}_{(\rho,u,\theta)}\phi)\\ &=\iint(\phi+\phi_*-\phi'-\phi'_*)|(v-v_*)\cdot\omega|\mathcal{M}_{(\rho,u,\theta)}(v_*)dv_*d\omega\end{aligned}$$

— here we have used that

$$\mathcal{M}_{(\rho,u,\theta)}(v)\mathcal{M}_{(\rho,u,\theta)}(v_*)=\mathcal{M}_{(\rho,u,\theta)}(v')\mathcal{M}_{(\rho,u,\theta)}(v'_*)\,.$$

Because of the translation and scaling invariance of the collision kernel, we can actually restrict our discussion to the case where $M=\mathcal{M}_{(1,0,1)}$ is the centered reduced Gaussian.

Indeed, if τ_w and m_λ denote respectively the translation and scaling isometries on $L^1(\boldsymbol{R}^3)$ defined by

$$\tau_w F(v)=F(v-w),\quad (m_\lambda F)(v)=\lambda^{-3}F(\lambda^{-1}w)$$

one has

$$\mathcal{B}(\tau_w F,\tau_w F)=\tau_w\mathcal{B}(F,F)\,,\quad \mathcal{B}(m_\lambda F,m_\lambda F)=\lambda m_\lambda\mathcal{B}(F,F)\,.$$

We then deduce that

$$\mathcal{L}_{M_{(\rho,u,\theta)}}(\phi) = (\rho\sqrt{\theta})\tau_u m_{\sqrt{\theta}}\mathcal{L}_{M_{(1,0,1)}}(m_{1/\sqrt{\theta}}\tau_{-u}\phi)\,. \tag{20}$$

1.6.1 - Hilbert's decomposition

In order to establish the Fredholm alternative for the linearized collision operator $\mathcal{L}_M$ (where M is the normalized centered Gaussian), the first step is to obtain a convenient decomposition, showing that $\mathcal{L}_M$ is just a compact perturbation of some multiplication operator:

Theorem 6.1. (Hilbert [**33**]). *In the case of a hard sphere gas, the linear collision operator $\mathcal{L}_M$ can be decomposed as*

$$\mathcal{L}_M\phi(v) = \upsilon(|v|)\phi(v) - \mathcal{K}\phi(v)$$

where $\mathcal{K}$ is a compact integral operator on $L^2(Mdv)$ and $\upsilon = \upsilon(|v|)$ is a scalar called the collision frequency that satisfies, for some $C > 1$,

$$\frac{1}{C}(1+|v|) \leq \upsilon(|v|) \leq C(1+|v|)\,.$$

Sketch of the proof of Theorem 6.1. First, an explicit computation gives

$$\upsilon(|v|) = \sqrt{2\pi}\left(\exp\left(-\frac{1}{2}|v|^2\right) + |v|\int_0^{|v|}\exp\left(-\frac{1}{2}|w|^2\right)dw\right),$$

from which the upper and lower bounds on the collision frequency υ are easily established.

As for the integral operator $\mathcal{K}$, one first computes its integral kernel k. This computation is not entirely straightforward, as it is based on a clever change of variables sometimes called «Carleman's collision parametrization» although it goes back to Hilbert [**33**].

More precisely, one further splits the operator $\mathcal{K}$ as $\mathcal{K} = -\mathcal{K}_1 + \mathcal{K}_2$, where

$$(\mathcal{K}_1\phi)(v) = \iint \phi_*|(v_* - v)\cdot\omega|M_*dv_*d\omega\,,$$

and

$$(\mathcal{K}_2\phi)(v) = \iint (\phi'_* + \phi')|(v_* - v)\cdot\omega|M_*dv_*d\omega\,.$$

The integral kernels $K_1 \equiv k_1(v,w)$ and $k_2 \equiv k_2(v,w)$ of $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined by

$$(\mathcal{K}_j\phi)(v) = \int k_j(v,w)\phi(w)dw\,, \quad j = 1,2\,.$$

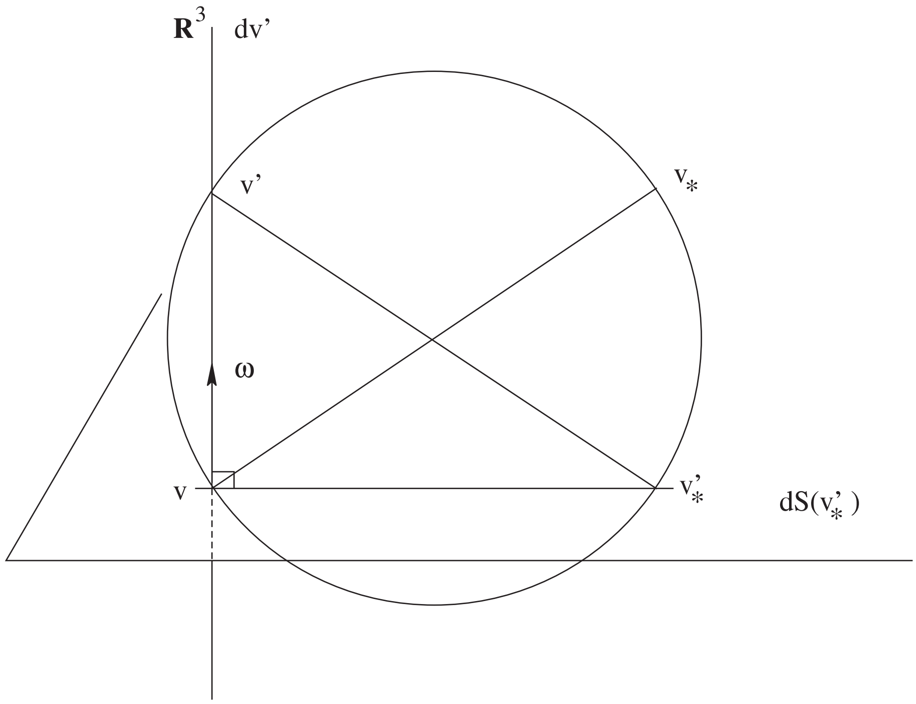

Fig. 2. Carleman's parametrization

One easily finds that

$$k_1(v,w) = \frac{1}{2\sqrt{2\pi}}|v-w|\exp\left(-\frac{1}{2}|w|^2\right).$$

The computation of k_2 is more complicated and Carleman's parametrization is used at this point. One changes variables in the integral defining $\mathcal{K}_2$ above, by using the transformation (see figure 2)

$$(v_*,\omega) \in \boldsymbol{R}^3 \times \boldsymbol{S}^2 \mapsto (v',v'_*) \in \mathcal{C},$$

where

$$\mathcal{C} = \{(v',v'_*) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 \,|\, (v'_* - v)\cdot(v'-v) = 0\}.$$

This transformation sends the measure $|(v-v_*)\cdot\omega| dv_* d\omega$ on the measure $dv' dS(v'_*)$, where dS is the surface element on the plane orthogonal to $(v'-v)$ passing through v. With this change of variables

$$k_2(v,w) = \frac{2}{\sqrt{2\pi}}\frac{1}{|v-w|}\exp\left(-\frac{1}{8}\frac{(|w|^2-|v|^2+|v-w|^2)^2}{|v-w|^2}\right).$$

That $\mathcal{K}_1$ and $\mathcal{K}_2$ are self-adjoint is easily seen on these formulas. That $\mathcal{K}_1$ is compact on $L^2(Mdv)$ is obvious; that $\mathcal{K}_2$ is also compact on $L^2(Mdv)$ follows from observing that $\mathcal{K}_2^4$ is in the Hilbert-Schmidt class on $L^2(Mdv)$. To see this, one computes from k_2 the integral kernel of $\mathcal{K}_2^4$, say $k_2^{(4)}$, and observe that

$$(v,w) \mapsto k_2^{(4)}(v,w)\frac{M(v)^{1/2}}{M(w)^{1/2}}$$

belongs to $L^2(\boldsymbol{R}^3 \times \boldsymbol{R}^3; dvdw)$. □

Further properties of the compact operator $\mathcal{K}$ were studied in detail by H. Grad in his fundamental paper [**31**]. His estimates, later improved by R. Caflisch [**9**], are recalled below.

Theorem 6.2. (Grad [**31**], Caflisch [**9**]). *With the previous notations, the compact integral operator $\mathcal{K}$ gains integrability with respect to the v-variable in the sense that*

- *it maps $L^2(Mdv)$ continuously into $L^\infty(M^{1/2}(1+|v|^{1/2})dv)$;*
- *for each $s>0$, it maps $L^\infty(M^{1/2}(1+|v|^s)dv)$ continuously into $L^\infty(M(1+|v|^{s+1})dv)$.*

1.6.2 - The relative coercivity estimate

With the above preliminary results, we establish the main property of the linearized collision operator $\mathcal{L}_M$, i.e. that it satisfies the Fredholm alternative in some weighted L^2 space.

Theorem 6.3. (Hilbert [**33**]). *For a hard sphere gas, the linear operator $\mathcal{L}_M$ is a nonnegative unbounded self-adjoint operator on $L^2(Mdv)$ with domain*

$$\mathcal{D}(\mathcal{L}_M) = \{\phi \in L^2(Mdv) \,|\, |v|\phi \in L^2(Mdv)\} = L^2(\boldsymbol{R}^3; (1+|v|^2)M(v)dv)$$

and nullspace

$$\ker(\mathcal{L}_M) = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

Moreover the following coercivity estimate holds: there exists $C>0$ such that, for each $\phi \in \mathcal{D}(\mathcal{L}_M) \cap (\ker(\mathcal{L}_M))^\perp$

$$\int \phi \mathcal{L}_M \phi(v) M(v) dv \geq C \int \phi(v)^2 \nu(|v|) M(v) dv\,.$$

Sketch of the proof of Theorem 6.3. The first step consists in characterizing the nullspace of $\mathcal{L}_M$. It must contain the collision invariants since the integrand in

$$\mathcal{L}_M\phi = \iint (\phi + \phi_* - \phi' - \phi'_*)|(v - v_*)\cdot\omega| M_* dv_* d\omega$$

vanishes identically if $\phi(v) = 1, v_1, v_2, v_3$ or $|v|^2$. Conversely, the same symmetries of the collision integral as in section 2 imply that

$$\int \psi\mathcal{L}_M\phi M dv = \frac{1}{4}\iiint (\psi+\psi_* - \psi' - \psi'_*)(\phi+\phi_* - \phi' - \phi'_*)|(v - v_*)\cdot\omega| MM_* dv_* dv d\omega\,.$$

Letting $\phi = \psi$ implies that $\mathcal{L}_M$ is a nonnegative self-adjoint operator on the weighted L^2 space

$$\{\phi \in L^2(Mdv)\,|\,|v|\phi \in L^2(Mdv)\} = L^2(\boldsymbol{R}^3; (1 + |v|^2)M(v)dv)\,.$$

In particular, if ϕ belongs to the nullspace of $\mathcal{L}_M$,

$$\frac{1}{4}\iiint (\phi + \phi_* - \phi' - \phi'_*)^2 |(v - v_*)\cdot\omega| MM_* dv dv_* d\omega = 0\,,$$

so that, for almost all $(v_*, \omega) \in \boldsymbol{R}^3 \times \boldsymbol{S}^2$

$$\phi + \phi_* = \phi' + \phi'_*.$$

In other words, ϕ is a collision invariant, which, as explained in Proposition 2.3, entails that ϕ is a linear combination of $1, v_1, v_2, v_3$ and $|v|^2$.

Next we prove the coercivity estimate. First the multiplication operator

$$\phi \mapsto \nu(|v|)\phi$$

is self-adjoint on $L^2(\boldsymbol{R}^3; Mdv)$ and has continuous spectrum which consists of the numerical range of ν, i.e. $[\inf_{v\in\boldsymbol{R}^3} \nu(|v|), +\infty)$. By Weyl's theorem, as $\mathcal{K}$ is self-adjoint and compact on $L^2(\boldsymbol{R}^3; Mdv)$, the spectrum of $\mathcal{L}_M$ consists of $[\inf_{v\in\boldsymbol{R}^3} \nu(|v|), +\infty)$ and of a sequence of eigenvalues in the interval $[0, \inf_{v\in\boldsymbol{R}^3} \nu(|v|)]$ with $\inf_{v\in\boldsymbol{R}^3} \nu(|v|)$ as its only possible accumulation point.

In particular, there exists a smallest positive element of the spectrum of $\mathcal{L}_M$, say λ_1, and one has

$$\int \phi\mathcal{L}_M\phi(v)M(v)dv \geq \lambda_1 \int \phi(v)^2 M(v)dv$$

for each $\phi \in \mathcal{D}(\mathcal{L}_M) \cap (\ker(\mathcal{L}_M))^\perp$.

The identity

$$\int \phi\mathcal{L}_M\phi M(v)dv = \int \phi^2\nu M(v)dv - \int \phi\mathcal{K}\phi M(v)dv$$

together with the continuity of $\mathcal{K}$ and the coercivity estimate above imply the stronger, weighted estimate announced in the statement of Theorem 6.3. □

1.6.3 - Invariance properties

Elastic collisions involving pairs of point particles are a purely isotropic process. It is therefore natural that the Boltzmann collision operator — or the linearization thereof about a centered Maxwellian distribution — should reflect the rotation invariance of this collision process.

Lemma 6.4. *For a hard sphere gas, the linear operator $\mathcal{L}_M$ commutes with rotations, which means that*

$$\mathcal{L}_M \phi_R = (\mathcal{L}_M \phi)_R$$

where

$$\phi_R(v) = \phi(R^T v) \text{ if } \phi \text{ is a scalar function},$$

$$\phi_R(v) = R\phi(R^T v) \text{ if } \phi \text{ is a vector field},$$

$$\phi_R(v) = R\phi(R^T v)R^T \text{ if } \phi \text{ is a } 2-\text{contravariant tensor field.}$$

Proof. The explicit computations in the proof of Theorem 6.1 show that the kernels k_1 and k_2 depend only on $|v-w|$, $|v|$ and $|w|$, while the collision frequency υ depends only on $|v|$. In particular, for each $R \in O_3(\boldsymbol{R})$, one has, for each $v, w \in \boldsymbol{R}^3$

$$k_1(Rv, Rw) = k_1(v, w)\,, \quad k_2(Rv, Rw) = k_2(v, w)\,, \quad \upsilon(Rv) = \upsilon(v)\,.$$

These relations entail the announced invariance properties of $\mathcal{L}_M$ under the orthogonal group $O_3(\boldsymbol{R})$. □

In the derivation of viscous hydrodynamic models from the Boltzmann equation, the following quantities play an important role, especially in the computation of transport coefficients such as the viscosity and heat conductivity in terms of the collision integral:

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 I\,, \quad B(v) = \frac{1}{2}v(|v|^2 - 5)\,. \tag{21}$$

Easy computations show that

$$\int A(v) \begin{pmatrix} 1 \\ v_j \\ |v|^2 \end{pmatrix} M dv = \int B(v) \begin{pmatrix} 1 \\ v_j \\ |v|^2 \end{pmatrix} M dv = 0\,, \quad j = 1, 2, 3\,. \tag{22}$$

By the Fredholm alternative applied to the linearized collision integral, there is a unique tensor field $\tilde{A}$ and a unique vector field $\tilde{B}$ such that

$$\mathcal{L}_M \tilde{A} = A \,, \quad \mathcal{L}_M \tilde{B} = B \,, \tag{23}$$

and

$$\int \tilde{A}(v) \begin{pmatrix} 1 \\ v_j \\ |v|^2 \end{pmatrix} M dv = \int \tilde{B}(v) \begin{pmatrix} 1 \\ v_j \\ |v|^2 \end{pmatrix} M dv = 0 \,, \quad j = 1, 2, 3 \,.$$

As an important consequence of the rotation invariance of $\mathcal{L}_M$, we obtain some additional information on the structure of $\tilde{A}$ and $\tilde{B}$. This additional structure explains why the viscosity and heat conductivity are nonnegative scalar fields.

Proposition 6.5. (Desvillettes & Golse [**16**], Golse & Saint-Raymond [**30**]). *There exist two scalar functions a and β such that*

$$\mathcal{L}_M^{-1} A(v) = a(|v|) A(v) \,, \quad \mathcal{L}_M^{-1} B(v) = \beta(|v|) B(v)$$

(where the operator $\mathcal{L}_M^{-1}$ is the pseudo-inverse of $\mathcal{L}_M$ defined on $\ker(\mathcal{L}_M)^\perp$ *by the Fredholm alternative).*

For a hard sphere gas, the functions a and β satisfy furthermore the growth estimate

$$|a(|v|)| + |\beta(|v|)| \leq C(1 + |v|) \,.$$

Sketch of the proof. The existence of the functions a and β established by Desvillettes and Golse [**16**] comes from the previous invariance properties (which are satisfied for all hard cut-off potentials coupled with geometrical arguments).

By definition of B, for each $R \in O_3(\boldsymbol{R}^3)$,

$$B_R(v) = B(Rv) \,.$$

On the other hand, according to Lemma 6.4,

$$\mathcal{L}_M(\tilde{B}_R) = (\mathcal{L}_M \tilde{B})_R = B_R \,,$$

and

$$B(Rv) = (\mathcal{L}_M \tilde{B})(Rv) = \mathcal{L}_M(\tilde{B} \circ R)(v) \,.$$

As $(\tilde{B})_R$ and $\tilde{B} \circ R$ are both orthogonal to the nullspace of $\mathcal{L}_M$, we deduce that

$$(\tilde{B})_R = \tilde{B} \circ R \,.$$

An elementary geometrical argument shows that the only such vector field is of the form

$$\tilde{B}(v) = b(|v|)v$$

which is equivalent to

$$\tilde{B}(v) = \beta(|v|)B(v)\,.$$

We proceed in the same manner to establish the analogous formula for $\tilde{A}$.

In order to obtain estimates on a and β, we first use the rotation invariance to reduce the relations

$$\mathcal{L}_M(a(|v|)A) = A\,, \qquad \mathcal{L}_M(\beta(|v|)B) = B$$

to scalar integral equations bearing on the functions a and β.

Let v be any fixed vector of $\boldsymbol{R}^3 \setminus \{0\}$. Denote by R_θ the rotation of axis v and angle θ:

$$R_\theta w = \frac{(w\cdot v)v}{|v|^2} + \left(w - \frac{(w\cdot v)v}{|v|^2}\right)\cos\theta + \left(w - \frac{(w\cdot v)v}{|v|^2}\right)\frac{v}{|v|}\sin\theta\,.$$

Using the spherical symmetry of a and β as well as the invariance properties of A and B

$$A(R_\theta w) = R_\theta w \otimes R_\theta w - \frac{1}{3}|w|^2 Id\,, \quad B(R_\theta w) = R_\theta B(w)\,,$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} A(R_\theta w)d\theta = \frac{3(v\cdot w)^2 - |v|^2|w|^2}{2|v|^4}A(v)\,,$$

$$\frac{1}{2\pi}\int_0^{2\pi} B(R_\theta w)d\theta = \frac{(|w|^2-5)w\cdot v}{(|v|^2-5)|v|^2}B(v)\,.$$

This averaging process leads then to the expected scalar integral equations, the only integration variables being $|w|$ and $\cos(v,w)$.

Then the growth estimate on a and β comes from a fixed point argument, coupled with Laplace asymptotic evaluation of the Gaussian integrals involved in the equations defining a and β.

Because of the explicit form of υ, k_1 and k_2 in the case of hard spheres, the fixed point argument can be done in the domain of $\mathcal{L}_M$, and this eventually implies that a and β have exactly the same growth as the collision frequency.

For more general hard cut-off potentials, one must use weighted spaces which are strictly smaller than the domain of $\mathcal{L}_M$, thereby degrading the growth estimate on a and β (which remains in any case polynomial). □

2 - Velocity Averaging

All kinetic models have in common the advection — or free transport — operator

$$\partial_t + v \cdot \nabla_x$$

which is the prototype of hyperbolic operators. A consequence of the formula giving the solution of the advection equation by the method of characteristics is that singularities of the initial or boundary data are propagated at finite speed $|v|$ inside the domain where the equation is posed. This is particularly annoying when dealing with nonlinear transport problems, because the interaction of the nonlinearities with the singularities coming from the data will in general be uncontrollable.

However, in the particular case of kinetic models such as the Boltzmann equation, additional structure comes to our help. With a view towards hydrodynamic limits, we anticipate that macroscopic observables — i.e. moments in v of the distribution function of the type

$$\rho_\phi(t,x) = \int_{\boldsymbol{R}^3} f(t,x,v)\phi(v)dv$$

should be of particular importance. At the level of the Boltzmann equation itself, we noticed in the presentation of the Boltzmann equation above that the collision term is local in (t,x) only, and global — i.e. some kind of convolution operator — in v. The importance of this fact should not be underestimated, and is a key to the existence of global solutions of the Boltzmann equation for initial data of arbitrary size.

Whether it be for constructing global solutions of the Boltzmann equation or for a rigorous treatment of hydrodynamic limits, one is led to seeking regularity or, better, compactness results on kinetic equations of Boltzmann type. In view of the remarks above, one cannot hope to gain regularity or compactness on the solutions of the Boltzmann equation itself — besides, since the collision integral is a convolution in the v variable, it is not needed. However, the structure of the Boltzmann equation and of hydrodynamic limits in general suggests that the regularity or compactness of moments such as ρ_ϕ should be investigated instead. This is precisely the essence of the class of results known as «Velocity Averaging», to be described below.

However, before entering the discussion of Velocity Averaging properly speaking, we recall a few basic results on the advection equation, which we shall use rather systematically in the sequel.

2.1 - *Fundamental formulas for the transport equation*

Consider first the Cauchy problem

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + a(t,x) f = S(t,x)\,, \quad t > 0\,,\ x \in \boldsymbol{R}^D \\ &f\big|_{t=0} = f^{in}(x)\,, \end{aligned} \tag{24}$$

with initial data $f^{in} \equiv f^{in}(x)$, source term S, amplification/absorption rate a, and unknown $f \equiv f(t,x)$. Here one can either assume that a, f^{in} and S are smooth (say, C^1) functions and that f is the solution of the above transport equation in the classical sense, or assume that $a \in L^\infty_{loc}$, while f^{in} and S belong to L^1_{loc}, in which case f is the solution of that same transport problem in the weak sense.

In all cases, the solution f is given by

$$
\begin{aligned}
f(t,x) = {}& f^{in}(x - tv) \exp\left(- \int_0^t a(t - s, x - sv) ds \right) \\
& + \int_0^t S(t - s, x - sv) \exp\left(- \int_0^s a(t - \sigma, x - \sigma v) d\sigma \right) ds.
\end{aligned} \tag{25}
$$

To see this, apply the *method of characteristics*: solve the transport equation above as a linear ODE in the variable t, observing that both sides of the transport equation evaluated at $(t, z + tv)$ can be recast in the form

$$
\frac{d}{dt} f(t, z + tv) + a(t, z + tv) f(t, z + tv) = S(t, z + tv).
$$

Then, in the resulting formula

$$
\begin{aligned}
f(t, z + tv) = {}& f^{in}(z) \exp\left(- \int_0^t a(s, z + sv) ds \right) \\
& + \int_0^t S(s, z + sv) \exp\left(- \int_s^t a(\sigma, z + \sigma v) d\sigma \right) ds
\end{aligned}
$$

set $z = x - tv$ and change s into $t - s$ and σ into $t - \sigma$ to arrive at (25).

Next, we shall also need the formula giving the solution of the analogous steady problem

$$
pf + v \cdot \nabla_x f + a(x) f = S(x), \quad x \in \mathbf{R}^D \tag{26}
$$

where $p > 0$, $a \equiv a(x) \in L^\infty_x$ is again the amplification/absorption rate, $S \equiv S(x) \in L^1_{loc}$ is the source term, and $f \equiv f(x)$ the unknown. The same remarks about regularity issues as in the case of the Cauchy problem (24) also apply here.

Then, the solution f to (26) is given by

$$
f(x) = \int_0^{+\infty} S(x - tv) \exp\left(-pt - \int_0^t a(x - sv) ds \right) dt. \tag{27}
$$

To see this, apply the Laplace transform to the Cauchy problem

$$\partial_t \phi + v \cdot \nabla_x \phi + a\phi = 0\,, \quad \phi\big|_{t=0} = S$$

with

$$f(x) = \int_0^{+\infty} e^{-pt}\phi(t,x)dt$$

so that

$$\int_0^{+\infty} e^{-pt}\partial_t\phi(t,x)dt = pf(x) - S(x)\,.$$

2.2 - *Velocity Averaging in L^2*

The first regularity results bearing on moments of the solution of a transport equation were obtained in the L^2 setting. Indeed, the key idea in the proof of such results is a kind of reduction to the one dimensional case, which is especially simple when expressed in terms of Fourier variables.

The setting is as follows: the advection operator operates on functions defined for a.e. $(x,v) \in \boldsymbol{R}^D \times \boldsymbol{R}^D$ where $D \geq 1$ with values in $\boldsymbol{R}$ (or $\boldsymbol{R}^N$); moments are defined in terms of m, a finite, positive Radon measure on $\boldsymbol{R}^D$ that satisfies the geometric condition

$$(GC_0) \qquad\qquad m(H) = 0 \text{ for any hyperplane } H \ni 0\,.$$

Under these assumptions, we can state the following Velocity Averaging result.

Theorem 2.1. *Let $\mathcal{F}$ be a bounded subset of $L^2(\boldsymbol{R}^D_x \times \boldsymbol{R}^D_v; dx \otimes dm(v))$ such that*

$$\{v \cdot \nabla_x f \,|\, f \in \mathcal{F}\} \text{ is bounded in } L^2(\boldsymbol{R}^D_x \times \boldsymbol{R}^D_v; dx \otimes dm(v))\,.$$

Then the set of velocity averages

$$\left\{ \int_{\boldsymbol{R}^D} f(x,v)dm(v) \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^2_{loc}(\boldsymbol{R}^D_x; dx)\,.$$

This result was stated and proved by Golse-Perthame-Sentis [**25**]. Earlier regularity remarks of the same type were reported by Agoshkov in [**1**]. Some years later, a systematic discussion of regularity and compactness results in all L^p settings was published in [**24**].

Before giving the proof of Theorem 2.1, we first recall

Rellich's compactness criterion: let $\mathcal{G}$ be a bounded subset of $L^2(\boldsymbol{R}^D)$. The set $\mathcal{G}$ is relatively compact in $L^2_{loc}(\boldsymbol{R}^D)$ iff

$$\int\limits_{|\xi|>R} |\hat{g}(\xi)|^2 d\xi \to 0 \text{ as } R \to +\infty \text{ uniformly in } g \in \mathcal{G},$$

where we denote by $\hat{g}$ the Fourier transform of g:

$$\hat{g}(\xi) = \int\limits_{\boldsymbol{R}^D} e^{-i\xi\cdot x} g(x) dx \text{ for each } g \in L^1 \cap L^2(\boldsymbol{R}^D).$$

Proof of Theorem 2.1 The argument is based upon using the partial Fourier transform of f in the x variable, henceforth denoted by

$$\hat{f}(\xi, v) = \int\limits_{\boldsymbol{R}^D} e^{-i\xi\cdot x} f(x, v) dx.$$

By Plancherel's theorem, the assumptions of Theorem 2.1 are translated into

$$\{\hat{f} \,|\, f \in \mathcal{F}\} \text{ and } \{(v\cdot\xi)\hat{f} \,|\, f \in \mathcal{F}\} \text{ are bounded in } L^2(d\xi \otimes dm(v)).$$

Equivalently, the assumptions on $\hat{f}$ can be formulated as follows

$$\{\phi = (1 + iv\cdot\xi)\hat{f} \,|\, f \in \mathcal{F}\} \text{ is bounded in } L^2(d\xi \otimes dm(v)).$$

Denote

$$\rho(x) = \int\limits_{\boldsymbol{R}^D} f(x, v) dm(v), \qquad \text{so that} \qquad \hat{\rho}(\xi) = \int\limits_{\boldsymbol{R}^D} \frac{\hat{\phi}(\xi, v) dm(v)}{1 + i\xi\cdot v}.$$

By Cauchy-Schwarz,

$$|\hat{\rho}(\xi)|^2 \le \Lambda\left(|\xi|, \frac{\xi}{|\xi|}\right) \int\limits_{\boldsymbol{R}^D} |\hat{\phi}(\xi, v)|^2 dm(v)$$

where

$$\Lambda(r, \omega) = \int \frac{dm(v)}{1 + r^2 (v\cdot\omega)^2}.$$

Since $m(\{v\cdot\omega = 0\}) = 0$ for each unit vector ω, one has, by dominated convergence,

$$\Lambda(r, \omega) \to 0 \text{ as } r \to +\infty, \qquad \text{pointwise in } \omega \in \boldsymbol{S}^{D-1}.$$

Moreover, $\Lambda(r,\cdot)$ is continuous on the unit sphere, and $\Lambda(r,\omega)\downarrow 0$ as $r\to+\infty$; by Dini's theorem,

$$\Lambda(r,\omega)\to 0 \text{ as } r\to+\infty\,,\quad \text{uniformly in } \omega\in\boldsymbol{S}^{D-1}\,.$$

Then

$$\int_{|\xi|>R}|\hat\rho(\xi)|^2d\xi\le\sup_{|\omega|=1}\Lambda(R,\omega)\iint_{\boldsymbol{R}^D\times\boldsymbol{R}^D}|\hat\phi(\xi,v)|^2d\xi dm(v)\to 0$$
$$\text{as } R\to+\infty \text{ uniformly as } f \text{ runs through } \mathcal{F}$$

and one concludes by Rellich's compactness lemma. □

Remark. Notice that the geometric condition (GC_0) on the measure m excludes in particular the case where m is a linear combination of Dirac masses, as would be the case in all discrete velocity kinetic models. As a matter of fact, the Velocity Averaging method does no apply to discrete velocity models in kinetic theory.

In the sequel, we shall need a variant of the Velocity Averaging Theorem 2.1 that applies to evolution problems. In fact, the case of the time-dependent advection operator is already included in the above result, after some suitable modification of the measure m.

Set $z=(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^D$, $w=(u,v)\in\boldsymbol{R}\times\boldsymbol{R}^D$ and

$$\mu=\delta_{u=1}\otimes m\,.$$

If $f(t,x,v)=F(t,x,u,v)\big|_{u=1}$, then we observe that the condition

$$w\cdot\nabla_zF\in L^2((\boldsymbol{R}\times\boldsymbol{R}^D)\times(\boldsymbol{R}\times\boldsymbol{R}^D);dtdx\otimes d\mu)$$

is equivalent to

$$\partial_tf+v\cdot\nabla_xf\in L^2(\boldsymbol{R}\times\boldsymbol{R}^D\times\boldsymbol{R}^D;dtdxdm(v))\,.$$

Remark. The choice of $\mu=\delta_{u=1}\otimes m$ may seem dangerous at first sight, because this measure has a Dirac component. However, the fact that μ satisfies the homogeneous geometric condition (GC_0) reduces to a very natural condition on m, as explained below.

Indeed, the *homogeneous* geometric condition (GC_0) on μ is equivalent to the following *affine* geometric condition on m:

$$(GC_a)\qquad m(H)=0 \text{ for any affine hyperplane } H\subset\boldsymbol{R}^D.$$

Based on these observations, we can state the time-dependent variant of the Velocity Averaging Theorem in the L^2 setting.

Theorem 2.2. *Assume that m satisfies (GC_a). Let $\mathcal{F}$ be a bounded subset of $L^2(\boldsymbol{R}_x^D \times \boldsymbol{R}_v^D, dxdm(v))$ and assume that $\mathcal{G}$ is a bounded subset of $L^2(\boldsymbol{R}_+ \times \boldsymbol{R}_x^D \times \boldsymbol{R}_v^D, dtdxdm(v))$.*

For each $f^{in} \in \mathcal{F}$ and each $g \in \mathcal{G}$, let f be the solution of

$$\partial_t f + v \cdot \nabla_x f = g\,, \quad f\big|_{t=0} = f^{in}.$$

Then, the set of velocity averages

$$\left\{ \int_{\boldsymbol{R}^D} f(t,x,v)dm(v) \,\middle|\, f^{in} \in \mathcal{F} \text{ and } g \in \mathcal{G} \right\}$$

is relatively compact in $L^2_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}_x^D; dtdx)$.

2.3 - *Velocity Averaging in L^1*

So far, we have been concerned with Velocity Averaging in the L^2 setting. However, for the purpose of studying the Boltzmann equation, we shall need to adapt these results to the L^1 setting. Such an extension, however, is by no means straightforward, as will be seen below. We start with a

Counterexample. Let $D \geq 1$ and $g_n \equiv g_n(x,v)$ be a bounded sequence in $L^1(\boldsymbol{R}^D \times \boldsymbol{R}^D)$ such that

$$g_n \longrightarrow \delta_{x=0} \otimes \delta_{v=v^*}$$

where $v^* \neq 0$. Let $f_n \equiv f_n(x,v)$ be the sequence of solutions of the steady transport equation

$$f_n + v \cdot \nabla_x f_n = g_n\,, \quad (x,v) \in \boldsymbol{R}^D \times \boldsymbol{R}^D\,.$$

Applying formula (27) shows that, for each $\phi \in C_c(\boldsymbol{R}^D)$

$$\int_{\boldsymbol{R}^D} \phi(x) \left(\int_{\boldsymbol{R}^D} f_n dv \right) dx = \int_0^\infty \iint_{\boldsymbol{R}^D \times \boldsymbol{R}^D} e^{-t} g_n(z,v) \phi(z+tv) dvdzdt \longrightarrow \int_0^\infty e^{-t} \phi(tv^*) dt$$

as $n \to +\infty$. Hence

$$\int_{\boldsymbol{R}^D} f_n dv \to \text{ a density carried by the half-line } \boldsymbol{R}_+ v^*$$

in the weak sense of measures as $n \to +\infty$. In particular

$$\int_{\boldsymbol{R}^D} f_n dv \text{ is not relatively compact in } L^1_{loc}(\boldsymbol{R}^D)$$

although

$$\|f_n\|_{L^1(\boldsymbol{R}^D\times\boldsymbol{R}^D)} \leq \|g_n\|_{L^1(\boldsymbol{R}^D\times\boldsymbol{R}^D)} = O(1)\,,$$
$$\|v\cdot\nabla_x f_n\|_{L^1(\boldsymbol{R}^D\times\boldsymbol{R}^D)} \leq 2\|g_n\|_{L^1(\boldsymbol{R}^D\times\boldsymbol{R}^D)} = O(1)\,.$$

This counterexample, which is taken from [**24**], shows that the analogue of Theorem 2.1 in L^1 is false. In fact, this counterexample also suggests that one should try by all means to control concentration effects. Therefore, we first collect some classical results on uniform integrability in L^1.

2.3.1 - The Dunford-Pettis criterion

A sequence of functions f_n in $L^1(\boldsymbol{R}^N)$ converges weakly to f if and only if

$$\int_{\boldsymbol{R}^N} f_n(x)\phi(x)dx \to \int_{\boldsymbol{R}^N} f(x)\phi(x)dx\,, \qquad \text{for all } \phi\in L^\infty(\boldsymbol{R}^N).$$

A bounded subset of $L^1(\boldsymbol{R}^N)$ may not be weakly relatively compact:

a) there may be *concentrations* (for instance, it may happen that $\|f_n\|_{L^1} = 1$ and $f_n \rightharpoonup \delta_0$ in the sense of Radon measures)

b) there may be *vanishing at infinity* (i.e. $\|f_n\|_{L^1} = 1$ and $f\big|_{|x|\leq R} \to 0$ in L^1 for each $R > 0$).

Exercise. It may even happen that $\|f_n\|_{L^1} = 1$, that $f_n \rightharpoonup f \in L^1$ in the sense of Radon measures but not in the weak L^1 topology.

In fact, the obstructions to weak compactness in L^1 listed above are the only possible ones, as shown by the following classical criterion.

Theorem 3.1 [Dunford-Pettis]. *A bounded subset $\mathcal{F}\subset L^1(\boldsymbol{R}^N)$ is relatively compact for the weak topology of L^1 if and only if*

- $\mathcal{F}$ *is* uniformly integrable, *meaning that*

$$\int_A |f(z)|dz \to 0 \text{ as } |A|\to 0 \text{ uniformly in } f\in\mathcal{F}$$

- $\mathcal{F}$ *is* tight, *meaning that*

$$\int_{|z|>R} |f(z)|dz \to 0 \text{ as } R \to +\infty \text{ uniformly in } f \in \mathcal{F}.$$

Hence the problem of verifying that a subset of L^1 is weakly relatively compact reduces to proving uniform integrability and tightness. There is in fact a very simple and nice way to show uniform integrability, as explained below.

First, $\mathcal{F}$ is uniformly integrable if and only if

$$\int_{|f(z)|>c} |f(z)|dz \to 0 \text{ as } c \to +\infty \text{ uniformly in } f \in \mathcal{F}$$

(the direct implication follows from observing that the set $\{|f(z)| > c\}$ has small measure, while the converse follows from dominated convergence). This last formulation of uniform integrability leads to the following natural criterion for uniform integrability.

Theorem 3.2 (De La Vallée-Poussin Criterion). *A subset $\mathcal{F}$ of $L^1(\boldsymbol{R}^N)$ is uniformly integrable if and only if there exists a function $H : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfying*

$$\frac{H(r)}{r} \to +\infty \text{ as } r \to +\infty$$

and such that

$$\sup_{f\in\mathcal{F}} \int_{\boldsymbol{R}^N} H(f(z))dz < +\infty\,.$$

Example. In kinetic theory, it is natural to choose $H(r) = r(\ln r)_+$ as the nonlinearity in the de la Vallée Poussin criterion; in other words, in the context of the kinetic theory of gases, an entropy bound implies the uniform integrability of the number densities.

Good references where to get acquainted with the Dunford-Pettis theory are for instance [**14**], [**32**], [**18**].

2.3.2 - Velocity Averaging in L^1: a first result

In this section, we begin with a first compactness statement concerning moments of a family of solutions of the transport equation that is an analogue to Theorem 2.1 in the L^1 setting. Although this result is not optimal, it is a very important step in all statements about Velocity Averaging in L^1 known so far.

We begin with the statement for the steady advection operator.

Theorem 3.3. *Let $\mathcal{F} \subset L^1(\boldsymbol{R}^D \times \boldsymbol{R}^D; dxdv)$ be weakly relatively compact and such that $\{v \cdot \nabla_x f \,|\, f \in \mathcal{F}\}$ is bounded in L^1 and uniformly integrable. Then the set*

$$\left\{ \int_{\boldsymbol{R}^D} f(x,v)dv \,|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1(\boldsymbol{R}^D)\,.$$

The analogous result for the time-dependent advection operator is as follows.

Theorem 3.4. *Let $\mathcal{F} \subset L^1([0,T] \times \boldsymbol{R}^D \times \boldsymbol{R}^D; dtdxdv)$ be weakly relatively compact and such that $\{\partial_t f + v \cdot \nabla_x f \,|\, f \in \mathcal{F}\}$ is bounded in L^1 and uniformly integrable. Then the set*

$$\left\{ \int_{\boldsymbol{R}^D} f(t,x,v)dv \,|\, f \in \mathcal{F} \right\} \textit{ is relatively compact in } L^1([0,T] \times \boldsymbol{R}^D)\,.$$

Both theorems were proved in Golse-Lions-Perthame-Sentis [**24**].

Proof for the steady case. By the Dunford-Pettis criterion, $\mathcal{F}$ is tight, and therefore one can assume without loss of generality that all the functions in $\mathcal{F}$ are supported in $\{|x| + |v| < r\}$ modulo a small error in L^1 norm.

Consider the *resolvent* of the transport operator: for $\lambda > 0$, we define $R_\lambda = (\lambda I + v \cdot \nabla_x)^{-1}$ by the formula

$$R_\lambda S(x,v) = \int_0^{+\infty} e^{-\lambda t} S(x - tv, v)dt$$

(i.e. $R_\lambda S$ is the solution $f \equiv f(x,v)$ of $\lambda f + v \cdot \nabla_x f = S$).

One checks that, for each $p \in [0, +\infty]$,

$$\begin{aligned} \|R_\lambda S\|_{L^p} &\leq \int_0^{+\infty} e^{-\lambda t} \|S(x - tv, v)\|_{L^p_{x,v}} dt \\ &= \|S\|_{L^p} \int_0^{+\infty} e^{-\lambda t} dt = \frac{\|S\|_{L^p}}{\lambda}\,. \end{aligned}$$

At this point we recall a very useful criterion to check that a subset of a Banach space is relatively compact.

Compactness Criterion. Let E be a Banach space, and $H \subset E$. The set H is *relatively compact in E*, if

$$\text{for each } \varepsilon > 0\,, \text{ there exists a compact subset } K_\varepsilon \text{ of } E \quad \text{s.t. } H \subset K_\varepsilon + B(0,\varepsilon).$$

Here is how this is applied to the situation under consideration. By assumption, the set $\mathcal{G} = \{g = f + v\cdot\nabla_x f \,|\, f \in \mathcal{F}\}$ is uniformly integrable; for each $c > 0$, decompose

$$f = f_c^< + f_c^>\,, \quad f_c^< = R_1\big(g\mathbf{1}_{|g|\le c}\big)\,,\; f_c^> = R_1\big(g\mathbf{1}_{|g|>c}\big).$$

First

$$\rho_c^>(x) = \int\limits_{|v|\le R} f_c^>(x,v)dv$$

satisfies

$$\|\rho_c^>\|_{L^1_x} \le \|f_c^>\|_{L^1_{x,v}} \le \big\|g\mathbf{1}_{|g|>c}\big\|_{L^1_{x,v}} \to 0 \text{ as } c \to +\infty \text{ uniformly in } g \in \mathcal{G}\,.$$

Then, for each $c > 0$, $g\mathbf{1}_{|g|\le c}$ is bounded in $L^2_{x,v}$ and compactly supported, and hence, by the L^2-case of Velocity Averaging (Theorem 2.1)

$$\rho_c^<(x) = \int\limits_{|v|\le R} f_c^<(x,v)dv \text{ is relatively compact in } L^1(\boldsymbol{R}^D).$$

<u>Conclusion</u>: therefore, for each $\varepsilon > 0$, we have found a compact $K_\varepsilon \subset L^1(\boldsymbol{R}^D)$ such that

$$\int\limits_{\boldsymbol{R}^D} f(x,v)dv = \rho_c^< + \rho_c^> \in K_\varepsilon + B_{L^1_x}(0,\varepsilon)\,.$$

Applying the compactness criterion above leads to the announced result. □

2.3.3 - Velocity Averaging in L^1 : a first improvement

Although a consequence thereof, the following statement significantly improves Theorem 3.3 by discarding the uniform integrability condition on $v\cdot\nabla_x f$.

Theorem 3.5. *Let $\mathcal{F} \subset L^1(\boldsymbol{R}^D \times \boldsymbol{R}^D; dxdv)$ be weakly relatively compact and such that $\{v\cdot\nabla_x f \,|\, f \in \mathcal{F}\}$ is bounded in L^1. Then the set*

$$\left\{ \int\limits_{\boldsymbol{R}^D} f(x,v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1(\boldsymbol{R}^D)\,.$$

The analogous result for the time-dependent advection operator is as follows.

Theorem 3.6. *Let $\mathcal{F} \subset L^1([0,T]\times \boldsymbol{R}^D \times \boldsymbol{R}^D; dtdxdv)$ be weakly relatively compact and such that $\{\partial_t f + v\cdot\nabla_x f \,|\, f\in\mathcal{F}\}$ is bounded in L^1. Then the set*

$$\left\{ \int_{\boldsymbol{R}^D} f(t,x,v)dv \,|\, f\in\mathcal{F} \right\} \textit{ is relatively compact in } L^1([0,T]\times\boldsymbol{R}^D)\,.$$

Proof. For each $\lambda>0$, set $R_\lambda = (\lambda I + v\cdot\nabla_x)^{-1}$. We recall that

$$\|R_\lambda\|_{\mathcal{L}(L^1_{x,v})} \le \frac{1}{\lambda}\,.$$

Write

$$f = R_\lambda(\lambda f + v\cdot\nabla_x f) = \lambda R_\lambda f + R_\lambda(v\cdot\nabla_x f)$$

so that

$$\int_{\boldsymbol{R}^D} f dv = \lambda \int_{\boldsymbol{R}^D} R_\lambda f dv + \int_{\boldsymbol{R}^D} R_\lambda(v\cdot\nabla_x f)dv.$$

Since $\{v\cdot\nabla_x f \,|\, f\in\mathcal{F}\}$ is bounded in $L^1_{x,v}$, the second term on the right hand side of the equality above can be made arbitrarily small in $L^1_{x,v}$ for some $\lambda>0$ large enough.

For such a λ, the first term on the right hand side of the equality above is relatively compact in L^1_x by Theorem 3.3.

We then conclude by applying the compactness criterion recalled in the proof of Theorem 3.3. □

2.3.4 - A uniform integrability criterion

In some cases, it may not be obvious that the the family $\mathcal{F}$ is weakly relatively compact in $L^1(\boldsymbol{R}^D\times\boldsymbol{R}^D)$. In this section, we give a useful criterion for checking the uniform integrability of a family of solutions of the transport equation.

We start with a definition of the notion of partial uniform integrability in a product space.

Definition 3.7. *Let μ and υ be two regular, positive Borel measures on $\boldsymbol{R}^D$; we say that a bounded family ϕ_ε of $L^1(\boldsymbol{R}^D_x\times\boldsymbol{R}^D_y; d\mu(x)d\upsilon(y))$ is uniformly integrable in the variable y if*

$$\int_{\boldsymbol{R}^N}\left(\sup_{\upsilon(A)<\eta}\int_A |\phi_\varepsilon(x,y)|d\upsilon(y)\right)d\mu(x)\to 0$$

as $\eta\to 0$, uniformly in ε.

Example. Assume that υ is a finite measure; then, for each $p > 1$, any bounded family in $L^1(d\mu(x); L^p(d\upsilon(y))$ is uniformly integrable in y. Indeed, if ϕ_ε is any such family, one has

$$\int_{\boldsymbol{R}^N} \left(\sup_{\upsilon(A)<\eta} \int_A |\phi_\varepsilon(x,y)| d\upsilon(y) \right) d\mu(x) \leq \eta^{1/p'} \int_{\boldsymbol{R}^N} \|\phi_\varepsilon(x,\cdot)\|_{L^p(d\upsilon)} dx$$

where $p' = \dfrac{p}{p-1}$, by applying Hölder's inequality to the inner integral.

As usual, we shall say that a family $\phi_\varepsilon \in L^1(\boldsymbol{R}^D_x \times \boldsymbol{R}^D_y; d\mu(x)d\upsilon(y))$ is locally uniformly integrable in y if, for each compact $K \subset \boldsymbol{R}^D \times \boldsymbol{R}^D$, the family $\mathbf{1}_K\phi_\varepsilon$ is uniformly integrable in y.

Theorem 3.8. *Let f_ε be a bounded family in $L^1_{loc}(\boldsymbol{R}^D \times \boldsymbol{R}^D; dxdy)$ such that*

- *$v \cdot \nabla_x f_\varepsilon$ is bounded in $L^1_{loc}(\boldsymbol{R}^D \times \boldsymbol{R}^D; dxdy)$, and*
- *the family f_ε is locally uniformly integrable in the variable v.*

Then the family f_ε is locally uniformly integrable (in both variables (x, v)).

This result was stated and proved in [**27**]; it extended an earlier remark by L. Saint-Raymond who observed in [**50**] that, under the extra assumption that f_ε is bounded in $L^1_x(L^\infty_y)$, the family of averages

$$\int_{\boldsymbol{R}^N} f_\varepsilon(x,v)\phi(v)dv$$

is locally uniformly integrable.

Sketch of the proof. We shall explain how to prove the result obtained by L. Saint-Raymond under the assumptions of the Theorem above.

<u>Step 1</u>: let $\chi \equiv \chi(t,x,v)$ be the solution of the free transport equation

$$\partial_t\chi + v \cdot \nabla_x\chi = 0\,, \qquad t > 0\,,\ (x,v) \in \boldsymbol{R}^D \times \boldsymbol{R}^D\,,$$
$$\chi(0,x,v) = \mathbf{1}_A(x)\,, \quad (x,v) \in \boldsymbol{R}^D \times \boldsymbol{R}^D\,.$$

Clearly, $\chi(t,x,v) = \mathbf{1}_A(x - tv)$: it can therefore be put in the form

$$\chi(t,x,v) = \mathbf{1}_{A_{t,x}}(v)\,, \quad t > 0\,,\ (x,v) \in \boldsymbol{R}^D \times \boldsymbol{R}^D\,,$$

where, for each $t > 0$,

$$A_{t,x} = \{v \in \boldsymbol{R}^D \,|\, x - tv \in A\}\,.$$

Further, $A_{t,x}$ is measurable and, for each $t > 0$ and $x \in \boldsymbol{R}^D$, one has

$$|A_{t,x}| = \int_{\boldsymbol{R}^D} \chi(t,x,v)dv = \int_{\boldsymbol{R}^D} \mathbf{1}_A(x - tv)dv$$
$$= \frac{1}{t^D} \int_{\boldsymbol{R}^D} \mathbf{1}_A(z)dz = \frac{|A|}{t^D} \ .$$

Step 2: without loss of generality, assume that f_ε and ϕ_ε in the statement of the theorem are nonnegative, and that all the f_ε's are supported in the same compact K of $\boldsymbol{R}^D \times \boldsymbol{R}^D$. Then

$$\begin{aligned}\int_A \int_{\boldsymbol{R}^D} f_\varepsilon(x,v)\phi(v)dvdx &= \int_{\boldsymbol{R}^D} \int_{A_{t,x}} f_\varepsilon(x,v)\phi(v)dvdx \\ &\quad - \int_0^t \iint_{\boldsymbol{R}^D \times \boldsymbol{R}^D} \chi(s,x,v) v \cdot \nabla_x f_\varepsilon(x,v)\phi(v)dxdvds\end{aligned} \tag{28}$$

as can be seen by integrating by parts the second integral on the right hand side of the equality above.

Pick $\eta > 0$ arbitrarily small; the second integral on the right hand side of (28) satisfies

$$\left| \int_0^t \iint_{\boldsymbol{R}^D \times \boldsymbol{R}^D} \chi(s,x,v) v \cdot \nabla_x f_\varepsilon(x,v)\phi(v)dxdvds \right| \le t \|v \cdot \nabla_x f_\varepsilon\|_{L^1} \|\phi\|_{L^\infty}$$

and therefore can be made less than η by choosing

$$0 < t < \frac{\eta}{1 + \|v \cdot \nabla_x f_\varepsilon\|_{L^1} \|\phi\|_{L^\infty}} \ .$$

For this $t > 0$, the first integral on the right hand side of (28) satisfies

$$\int_{\boldsymbol{R}^N} \int_{A_{t,x}} f_\varepsilon(x,v)\phi(v)dvdx \to 0$$

as $|A| \to 0$ uniformly in ε, since f_ε is uniformly integrable in v and $|A_{t,x}| = |A|/t^D$, as established in the first step above.

Therefore, for each $\eta > 0$, there exists $a > 0$ such that $|A| < a$ implies that

$$\int_A \int_{\boldsymbol{R}^D} f_\varepsilon(x,v)\phi(v)dvdx \le 2\eta$$

uniformly in $\varepsilon > 0$, which entails that the family of averages

$$\int_{\boldsymbol{R}^D} f_\varepsilon(x,v)\phi(v)dv$$

is uniformly integrable. □

By putting together Theorem 3.5 and Theorem 3.8 we arrive at the following statement.

Theorem 3.9. *Let f_ε be a bounded family in $L^1_{loc}(\boldsymbol{R}^D \times \boldsymbol{R}^D; dxdy)$ such that*

- *$v \cdot \nabla_x f_\varepsilon$ is bounded in $L^1_{loc}(\boldsymbol{R}^D \times \boldsymbol{R}^D; dxdy)$, and*
- *the family f_ε is locally uniformly integrable in the variable v.*

Then, for each compactly supported $\phi \in L^\infty(\boldsymbol{R}^D)$, the family

$$\int f_\varepsilon(x,v)\phi(v)dv \text{ is strongly relatively compact in } L^1_{loc}(\boldsymbol{R}^D).$$

As we shall see below, Theorem 3.8 is the key to one essential step in the proof of the hydrodynamic limit of the Boltzmann equation leading to the incompressible Navier-Stokes equations.

3 - Global Existence Theory for the Boltzmann Equation

Historically, the first global existence result for the (spatially inhomogeneous) Boltzmann equation is due to Ukai [**56**], who considered initial data that are perturbations of some uniform Maxwellian. He proved the global existence of a solution of the Cauchy problem for the Boltzmann equation under the assumption that this initial perturbation is smooth and small enough (in a norm that involves derivatives and weights so as to ensure decay for large v's). Subsequently, Illner and Shinbrot [**34**] considered the same Cauchy problem in the case where the initial data is a small perturbation of the vacuum state.

For the purpose of deriving hydrodynamic limits (and especially incompressible limits), it would seem that Ukai's result is exactly what is needed. However, it cannot be used as a black box, because of the potential lack of uniformity in the hydrodynamic limit on the threshold on the size of the initial perturbation that guarantees global existence.

For that reason, one has to use a global existence theory for the Boltzmann equation that holds for initial data of arbitrary sizes. This theory goes back to the late 80s and is due to R. DiPerna and P.-L. Lions. We shall give a rather detailed account of that theory, which is a crucial part in our discussion of hydrodynamic limits. Besides, our presentation of the subject deviates from the original article [**17**] and incorporates later developments of the theory of renormalized solutions.

In particular, an important part of the original proof [**17**] related the notion of renormalized solutions to that of mild solutions (a more classical object). Since the notion of mild solution is not really useful in the context of hydrodynamic limits, we have replaced that part of the original proof with a simplification of a more recent argument by P.-L. Lions [**41**] that applies to the mean-field Vlasov-Poisson-Boltzmann model.

3.1 - *Notion of renormalized solution*

The main difficulty in obtaining global solutions of the Boltzmann equation is that the collision integral is a quadratic operator that is purely local in the position variable. In other words, the Boltzmann collision integral acts as a convolution product in the v variable, and as a pointwise multiplication in the t and x variables. Since on the other hand, the natural a priori estimates satisfied by solutions of the Boltzmann equation are of the form

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|v|^2+|\ln F(t,x,v)|)F(t,x,v)dxdv \le C$$

as a consequence of the global conservation of mass and energy, and of the H Theorem, the Boltzmann collision integral is not even a well-defined distribution in the x variable — similarly f^2 is only a measurable function, which may not be locally integrable when f is a L^1 function.

P.-L. Lions and R. DiPerna made the following observation: although the loss term in the Boltzmann collision integral, i.e. $\mathcal{B}_-(F,F)$ may not be a locally integrable function in view of the remark above, the quantity

$$\frac{\mathcal{B}_-(F,F)}{1+F} = \frac{F}{1+F}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} F_* b(v-v_*,\omega)dv_*d\omega = \frac{F}{1+F}F\star_v \overline{b}\,,$$

where

$$\overline{b}(z) = \int_{\boldsymbol{S}^2} b(z,\omega)d\omega\,,$$

belongs to $L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$ provided that

$$\frac{\overline{b}}{1+|z|^2} \text{ belongs to } L^\infty$$

and F satisfies the bound

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|v|^2)F(t,x,v)dxdv \leq C$$

(that is implied by the global conservation of mass and energy). Since on the other hand the entropy production rate controls in some sense the difference between the gain and loss terms in the Boltzmann collision integral, one can hope to establish the same type of bound on

$$\frac{\mathcal{B}_+(F,F)}{1+F}\,.$$

This observation suggested that instead of considering the original Boltzmann equation

$$(\partial_t + v\cdot\nabla_x)F = \mathcal{B}(F,F)\,,$$

it might be advantageous to consider instead the equation

$$(\partial_t + v\cdot\nabla_x)\ln(1+F) = \frac{\mathcal{B}(F,F)}{1+F}\,.$$

Obviously both equalities are equivalent for rapidly decaying smooth distribution functions F.

Actually, one can make the discussion above slightly more general.

Definition 1.1. *A nonnegative function $F \in C(\boldsymbol{R}_+; L^1(\boldsymbol{R}^3\times\boldsymbol{R}^3))$ is a renormalized solution of the Boltzmann equation iff*

$$\frac{\mathcal{B}(F,F)}{\sqrt{1+F}} \in L^1_{loc}(dtdxdv)$$

and for each $\beta \in C^1(\boldsymbol{R}_+)$ s.t. $|\beta'(Z)| \leq \dfrac{C}{\sqrt{1+Z}}$ for all $Z\geq 0$, one has

$$(\partial_t + v\cdot\nabla_x)\beta(F) = \beta'(F)\mathcal{B}(F,F)$$

in the sense of distributions on $\boldsymbol{R}^*_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$.

The significance of this definition is explained by the following a priori estimate.

Lemma 1.2. *Assume that $f \equiv f(t,x,v) \geq 0$ is a measurable function such that*

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|v|^2)f(t,x,v)dxdv \leq E$$

and

$$\int_0^T \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(f)b(v-v_*,\omega)dvdv_*d\omega dxdt \leq C(T)\,,$$

with

$$d(f) = (f'f'_* - ff_*)(\ln f'f'_* - \ln ff_*). \tag{29}$$

If the average collision kernel satisfies

$$0 \leq \overline{b}(z) \leq C_b(1 + |z|^2) \text{ for each } z \in \boldsymbol{R}^3$$

then

$$\int_0^T \int_{\boldsymbol{R}^3} \int_{|v|\leq R} \frac{|\mathcal{B}(f,f)|}{\sqrt{1+f}} dvdxdt \leq \frac{1}{4} C(T) + 8\sqrt{C(T) C_b R^3 (1+R^2) ET}\,.$$

Proof. We recall the elementary inequality

$$(\sqrt{a} - \sqrt{b})^2 \leq \frac{1}{4}(a-b)(\ln a - \ln b)$$

for each $a, b > 0$. Since

$$\begin{aligned}|f'f'_* - ff_*| &= |\sqrt{f'f'_*} - \sqrt{ff_*}|(\sqrt{f'f'_*} + \sqrt{ff_*}) \\ &\leq |\sqrt{f'f'_*} - \sqrt{ff_*}|^2 + 2\sqrt{ff_*}|\sqrt{f'f'_*} - \sqrt{ff_*}|\,,\end{aligned}$$

a straightforward application of the Cauchy-Schwarz inequality shows that

$$\begin{aligned}&\int_0^T \int_{\boldsymbol{R}^3} \int_{|v|\leq R} \frac{|\mathcal{B}(f,f)|}{\sqrt{1+f}} dvdxdt \\ &\qquad \leq \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} |\sqrt{f'f'_*} - \sqrt{ff_*}|^2 b dv dv_* d\omega \\ &\qquad + 2 \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} |\sqrt{f'f'_*} - \sqrt{ff_*}|\sqrt{f_*} b dv dv_* d\omega\end{aligned}$$

so that

$$\begin{aligned}&\int_0^T \int_{\boldsymbol{R}^3} \int_{|v|\leq R} \frac{|\mathcal{B}(f,f)|}{\sqrt{1+f}} dvdxdt \leq \frac{1}{4} C(T) \\ &\quad + 2\sqrt{C(T)} \left(\int_0^T \int_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \left(\int_{|v|\leq R} \overline{b}(v - v_*) dv \right) f(t,x,v_*) dv_* dxdt \right)^{1/2}.\end{aligned}$$

Since

$$0 \le \int_{|v|\le R} \overline{b}(v-v_*)dv$$
$$\le C_b(2R)^3(1+2R^2+2|v_*|^2) \le 2C_b(2R)^3(1+R^2)(1+|v_*|^2)$$

we finally arrive at the bound

$$\int_0^T \int_{\boldsymbol{R}^3} \int_{|v|\le R} \frac{|\mathcal{B}(f,f)|}{\sqrt{1+f}} dvdxdt \le \frac{1}{4}C(T) + 8\sqrt{C(T)C_bR^3(1+R^2)ET}\,.$$

With this definition (actually a slightly more restrictive one), R. DiPerna and P.-L. Lions proved the following remarkable result in [**17**]. □

Theorem 1.3. *Let $F^{in} \ge 0$ a.e. satisfy*

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|x|^2+|v|^2+|\ln F^{in}|)F^{in}dxdv < +\infty\,,$$

and assume that the collision kernel b in Boltzmann's collision integral satisfies the weak cutoff assumption

$$b \in L^1_{loc}(\boldsymbol{R}^3\times\boldsymbol{S}^2)\,,\qquad \frac{1}{1+|z|^2}\int_{|z-w|\le R} \overline{b}(w)dw \to 0$$

as $|z| \to +\infty$ for each $R>0$.

Then, there exists a renormalized solution of the Boltzmann equation satisfying the initial condition $F\big|_{t=0} = F^{in}$. Furthermore, this renormalized solution has the following properties

- *it satisfies the continuity equation*

$$\partial_t \int_{\boldsymbol{R}^3} Fdv + \operatorname{div}_x \int_{\boldsymbol{R}^3} vFdv = 0\,,$$

and the following variant of the local conservation law of momentum:

$$\partial_t \int_{\boldsymbol{R}^3} vFdv + \operatorname{div}_x \int_{\boldsymbol{R}^3} v\otimes vFdv + \operatorname{div}_x m = 0\,,$$

where m is a nonnegative symmetric matrix whose entries belong to $L^\infty(\boldsymbol{R}_+;\mathcal{M}(\boldsymbol{R}^3))$;

• *it satisfies the total mass and momentum conservation*

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F(t)dxdv = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in}dxdv\,,$$

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} vF(t)dxdv = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} vF^{in}dxdv\,,$$

together with the following energy inequality: for each $t \geq 0$,

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \frac{1}{2}|v|^2F(t,x,v)dxdv \leq \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \frac{1}{2}|v|^2F^{in}dxdv\,,$$

more precisely, for a.e. $t \geq 0$, *one has*

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \frac{1}{2}|v|^2F(t,x,v)dxdv + \int_{\boldsymbol{R}^3} \frac{1}{2}\mathrm{trace}(m)(t) = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \frac{1}{2}|v|^2F^{in}dxdv\,;$$

• *finally, it satisfies the entropy inequality: for each* $t \geq 0$,

$$\frac{1}{4}\int_0^t ds \int_{\boldsymbol{R}^3} dx \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*)\ln\left(\frac{F'F'_*}{FF_*}\right) bd\omega\, dvdv_*$$
$$\leq \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in}\ln F^{in}dxdv - \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F\ln F(t)dxdv\,.$$

In the sequel, we shall describe the proof of Theorem 1.3, in the case where

(30) $$0 \leq b \equiv b(z,\omega) \in C_b(\boldsymbol{R}^3\times\boldsymbol{S}^2)\,.$$

3.2 - *The approximation scheme*

The renormalized solution whose existence is asserted by Theorem 1.3 will be obtained as the limit of a sequence of distribution functions that satisfy some appropriate truncation of the Boltzmann equation. The most natural truncation of the Boltzmann equation for that purpose is as follows.

(31) $$\partial_t F_n + v\cdot\nabla_x F_n = \frac{\mathcal{B}(F_n,F_n)}{1+\frac{1}{n}\int F_n dv}\,,\qquad F_n\big|_{t=0} = F^{in}\,.$$

One can show that, for each $n \geq 1$, the truncated collision integral

$$f \mapsto \frac{\mathcal{B}(f,f)}{1+\frac{1}{n}\int f dv}$$

is Lipschitz continuous on the positive cone of $L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)$. This remark, together with a few estimates that are fairly classical in spirit but too technical to be reported here, leads to the following result.

Proposition 2.1. *Let $F^{in} \equiv F^{in}(x,v) \geq 0$ a.e. be a measurable function such that*

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|x|^2+|v|^2+|\ln F^{in}(x,v)|)F^{in}(x,v)dxdv < +\infty\,. \tag{32}$$

Then, for each $n \geq 1$, the truncated Boltzmann equation has a unique solution (in the sense of distributions) $F \in C(\boldsymbol{R}_+; L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3))$. Moreover, this solution satisfies the following form of Boltzmann's H Theorem: for each $t \geq 0$, one has

$$\begin{aligned}\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F_n \ln F_n(t,x,v)dxdv + \frac{1}{4}\int_0^t\int_{\boldsymbol{R}^3} \frac{d(F_n)}{1+\frac{1}{n}\int_{\boldsymbol{R}^3} F_n dv} bdvdv_* d\omega dxds \\ = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in}\ln F^{in}(x,v)dxdv\end{aligned} \tag{33}$$

together with the a priori estimate

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|x|^2+|v|^2+|\ln F_n(t,x,v)|)F_n(t,x,v)dxdv \leq C^{in}(1+t^2) \tag{34}$$

where C^{in} is a positive constant that depends on F^{in} only — and is in particular independent of n.

See [**17**] pp. 358-361 for more details on the proof of this result.

3.3 - *A priori bounds and weak L^1 compactness*

Let F_n be the sequence of solutions of the truncated Boltzmann equation constructed in Proposition 2.1. We begin with a result that is in itself a first justification for the notion of renormalized solution.

Proposition 3.1. *For each $\delta > 0$, the sequences*

$$\frac{\mathcal{B}_-^n(F_n, F_n)}{1+\delta F_n} \quad \text{and} \quad \frac{\mathcal{B}_+^n(F_n, F_n)}{1+\delta F_n}$$

are both bounded in $L^1_{loc}(\boldsymbol{R}_+; L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3))$ and relatively weakly compact in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$.

Proof. Both the L^1 bound and uniform integrability are obvious for $\mathcal{B}_-^n$, since

$$\frac{\mathcal{B}_-^n(F_n, F_n)}{1+\delta F_n} = \mathcal{A}_n(F_n)\frac{F_n}{1+\delta F_n}$$

where we recall that

$$\mathcal{A}_n(F) = \frac{\overline{b} \star_v F}{1 + \frac{1}{n}\int_{\boldsymbol{R}^3} F dv}, \qquad \overline{b}(z) = \int_{\boldsymbol{S}^2} b(z,\omega) d\omega\,.$$

In other words

$$\mathcal{A}_n(F) = \frac{\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} F_* b(v-v_*,\omega)d\omega\, dv_*}{1 + \frac{1}{n}\int_{\boldsymbol{R}^3} F dv} = \frac{\int_{\boldsymbol{R}^3} F_* \overline{b}(v-v_*)dv_*}{1 + \frac{1}{n}\int_{\boldsymbol{R}^3} F dv}$$

so that

$$0 \le \mathcal{A}_n(F) \le \|b\|_{L^\infty} \int_{\boldsymbol{R}^3} F dv\,.$$

As for $\mathcal{B}_+^n$, pick $R >> 1$ and write

$$\begin{aligned}\frac{\mathcal{B}^n(F_n, F_n)}{1+\delta F_n} &= \frac{1}{1+\delta F_n}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{F'_n F'_{n*} - F_n F_{n*}}{1+\frac{1}{n}\int_{\boldsymbol{R}^3} F_n dv} \mathbf{1}_{F'_n F'_{n*} \le R F_n F_{n*}} b dv_* d\omega \\ &+ \frac{1}{1+\delta F_n}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{F'_n F'_{n*} - F_n F_{n*}}{1+\frac{1}{n}\int_{\boldsymbol{R}^3} F_n dv} \mathbf{1}_{F'_n F'_{n*} > R F_n F_{n*}} b dv_* d\omega.\end{aligned}$$

The first term is bounded pointwise by

$$(R-1)\frac{\mathcal{B}_-^n(F_n, F_n)}{1+\delta F_n}$$

while the $L^1([0,T]\times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$ norm of the second is bounded by the entropy production as in (14):

$$\frac{1}{\ln R}\int_0^T\int_{\boldsymbol{R}^3} dx \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{F'_nF'_{n*}-F_nF_{n*}}{1+\frac{1}{n}\int_{\boldsymbol{R}^3} F_n dv}\ln\left(\frac{F'_nF'_{n*}}{F_nF_{n*}}\right) b dv dv_* d\omega = O\left(\frac{1}{\ln R}\right).$$

Hence, for each $R >> 1$,

$$\frac{\mathcal{B}^n(F_n,F_n)}{1+\delta F_n}\in B\left(0,\frac{1}{\ln R}\right)_{L^1([0,T]\times\boldsymbol{R}^3\times\boldsymbol{R}^3)} + K_R$$

where K_R is locally uniformly integrable on $\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3$ for each finite R.

Therefore

$$\frac{\mathcal{B}^n(F_n,F_n)}{1+\delta F_n} \text{ is locally uniformly integrable on } \boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3.$$

Finally

$$\frac{\mathcal{B}^n_+(F_n,F_n)}{1+\delta F_n} = \frac{\mathcal{B}^n_-(F_n,F_n)}{1+\delta F_n} + \frac{\mathcal{B}^n(F_n,F_n)}{1+\delta F_n}$$

is uniformly integrable, being the sum of two locally uniformly integrable sequences. □

In particular, it follows from Proposition 3.1 that

$$(\partial_t + v\cdot\nabla_x)\frac{1}{\delta}\ln(1+\delta F_n) = \frac{\mathcal{B}^n(F_n,F_n)}{1+\delta F_n} = O(1)_{L^1([0,T]\times\boldsymbol{R}^3\times\boldsymbol{R}^3)}\,. \tag{35}$$

Proposition 3.2. *One has $F_n(t,\cdot,\cdot)\rightharpoonup F(t,\cdot,\cdot)$ in $L^1(\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v)$ uniformly in $t\in[0,T]$ for each $T>0$ as $n\to+\infty$, modulo extraction of a subsequence.*

Proof. For each $\delta>0$, the sequence $\frac{1}{\delta}\ln(1+\delta F_n)$ satisfies

$$0\le\frac{1}{\delta}\ln(1+\delta F_n)\le\frac{1}{\sqrt{\delta}}\sqrt{F_n}$$

so that in particular

$$(1+|x|+|v|)\frac{1}{\delta}\ln(1+\delta F_n) \text{ is bounded in } L^\infty([0,T],L^2(\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v)) \tag{36}$$

for each $T>0$. It follows from (35) that

$$\partial_t\frac{1}{\delta}\ln(1+\delta F_n) = O(1) \text{ in } L^\infty([0,T],H^{-1}(\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v)) + L^1([0,T]\times\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v)$$

for each $\delta > 0$ and each $T > 0$. By Sobolev's embedding and duality, one has $L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v) \subset H^{-s}(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)$ for $s > 3$, so that

$$\partial_t \frac{1}{\delta} \ln(1 + \delta F_n) \text{ is bounded in } L^1([0,T], H^{-4}(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)).$$

By Appendix C of [**42**], this last control and (36) imply that

$$\frac{1}{\delta} \ln(1 + \delta F_n) \text{ is relatively compact in } C([0,T], w{-}L^2(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)) \tag{37}$$

for each $T > 0$ and each $\delta > 0$.

On the other hand,

$$0 \le F_n - \frac{1}{\delta} \ln(1 + \delta F_n) \le \delta F_n^2 \mathbf{1}_{F_n \le R} + F_n \mathbf{1}_{F_n > R}$$

so that

$$\left\| F_n - \frac{1}{\delta} \ln(1 + \delta F_n) \right\|_{L^1_{x,v}} \le R\delta \|F_n\|_{L^1_{x,v}} + \frac{1}{\ln R} \iint_{\boldsymbol{R}^3 \times \boldsymbol{R}^3} F_n \ln F_n dx dv .$$

This last control clearly implies that

$$F_n - \frac{1}{\delta} \ln(1 + \delta F_n) \to 0 \text{ in } L^\infty([0,T]; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)) \tag{38}$$

as $\delta \to 0$, uniformly in n. Proposition 3.2 is a direct consequence of the compactness properties (37), (38) and of the estimate (34). □

3.4 - *Applying Velocity Averaging*

A first application of Velocity Averaging in L^1 leads to the strong convergence statement below.

Proposition 4.1. *Let $\phi \equiv \phi(t,x,v,v_*) \in C(\boldsymbol{R}_+ \times \boldsymbol{R}^3_x \times \boldsymbol{R}^3_v \times \boldsymbol{R}^3_{v_*})$ be such that*

$$\frac{|\phi(t,x,v,v_*)|}{1 + |v_*|^2} \to 0 \text{ as } |v_*| \to +\infty$$

uniformly in $(t,x,v) \in [0,T] \times \overline{B}(0,R) \times \overline{B}(0,R)$, for each $R, T > 0$. Then, modulo extraction of a subsequence, for each $p \in [1, +\infty)$

$$\int_{\boldsymbol{R}^3} F_n(t,x,v_*)\phi(t,x,v,v_*) dv_* \to \int_{\boldsymbol{R}^3} F(t,x,v_*)\phi(t,x,v,v_*) dv_*$$

in $L^p([0,T]; L^1_{loc}(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v))$ and a.e. as $n \to +\infty$.

Proof. The estimate (34) shows that the sequence F_n is both uniformly integrable and tight on $[0,T]\times \boldsymbol{R}^3\times \boldsymbol{R}^3$ for each $T>0$. This is also true of $\frac{1}{\delta}\ln(1+\delta F_n)$ for each $\delta>0$, since

$$0\le \frac{1}{\delta}\ln(1+\delta F_n)\le F_n\,.$$

By (35) and Velocity Averaging in L^1 (Theorem 3.6), one concludes that, for each $T>0$ and each $\phi\in C^1_c(\boldsymbol{R}_+\times \boldsymbol{R}^3_x\times \boldsymbol{R}^3_v\times \boldsymbol{R}^3_{v_*})$, the sequence

$$\int_{\boldsymbol{R}^3} \frac{1}{\delta}\ln(1+\delta F_n)(t,x,v_*)\phi(t,x,v,v_*)dv_*$$

is strongly relatively compact in $L^1([0,T]\times \boldsymbol{R}^3_x\times \boldsymbol{R}^3_v)$.

Because of the uniform (in n) convergence statement (38) as $\delta\to 0$, one deduces that, for each $p\in[1,+\infty)$ and modulo extraction of a subsequence,

$$\int_{\boldsymbol{R}^3} F_n(t,x,v_*)\phi(t,x,v,v_*)dv_*dv\to \int_{\boldsymbol{R}^3} F(t,x,v_*)\phi(t,x,v,v_*)dv_*$$

in $L^p([0,T];L^1_{loc}(\boldsymbol{R}^3_x\times \boldsymbol{R}^3_v))$ and a.e. as $n\to+\infty$ for each test function $\phi\in C^1_c(\boldsymbol{R}_+\times \boldsymbol{R}^3_x\times \boldsymbol{R}^3_v\times \boldsymbol{R}^3_{v_*})$. Because of (34) and by an easy density argument, this limit holds in $L^p([0,T];L^1(\boldsymbol{R}^3_x\times \boldsymbol{R}^3_v))$ for each $T>0$ and each continuous test function $\phi\equiv\phi(t,x,v,v_*)$ with subquadratic growth at infinity in the variable v_*. □

With the above convergence statement, we can pass to the limit in the Boltzmann collision integral, once it is renormalized by the macroscopic density. This average renormalization is here only to guarantee that all the quantities considered are at least locally integrable.

However, even without this average renormalization, and modulo extraction of a subsequence, the Boltzmann collision integral evaluated on F_n converges a.e. in (t,x) and weakly in v.

Here is the precise statement needed in the sequel:

Proposition 4.2. *For each $\phi\in C_c(\boldsymbol{R}_+\times \boldsymbol{R}^3\times \boldsymbol{R}^3)$*

$$\int_{\boldsymbol{R}^3} \frac{\mathcal{B}^n_\pm(F_n,F_n)}{1+\int_{\boldsymbol{R}^3} F_n dv}\phi dv\to \int_{\boldsymbol{R}^3} \frac{\mathcal{B}_\pm(F,F)}{1+\int_{\boldsymbol{R}^3} F dv}\phi dv \text{ in } L^1_{loc}(\boldsymbol{R}_+\times \boldsymbol{R}^3)$$

as $n\to+\infty$.

The proof of this proposition will use the following variant of Egorov's theorem, recalled below:

Theorem 4.3. *Assume that* $v_n \to v$ *a.e. on* $K \subset\subset \boldsymbol{R}^D$*. Then, for each* $\varepsilon > 0$*, there exists a measurable* $E \subset K$ *such that*

$$|K \setminus E| < \varepsilon\,, \qquad \text{and } v_n \to v \text{ uniformly on } E\,.$$

An interesting consequence of Egorov's theorem is the following statement on the continuity of bilinear products.

Lemma 4.4. *Assume that* $u_n \rightharpoonup u$ *in* L^1*, that* $\sup \|v_n\|_{L^\infty} < +\infty$*, and that* $v_n \to v$ *a.e.. Then* $u_n v_n \rightharpoonup uv$ *in* L^1*. If* $v = 0$*,* $u_n v_n \to 0$ *in* L^1*.*

There is a more precise version of the above statement in product spaces:

Lemma 4.5. *Assume that, for each* $\phi \in C_c(\boldsymbol{R}^D \times \boldsymbol{R}^D)$

$$u_n \rightharpoonup u \text{ in } L^1(\boldsymbol{R}^D \times \boldsymbol{R}^D), \quad \int_{\boldsymbol{R}^N} u_n \phi dv \to \int_{\boldsymbol{R}^D} u\phi dv \text{ in } L^1_{loc}(\boldsymbol{R}^D)$$

that $\sup \|v_n\|_{L^\infty(\boldsymbol{R}^D \times \boldsymbol{R}^D)} < +\infty$*, and that* $v_n \to v$ *a.e. Then*

$$\int_{\boldsymbol{R}^D} u_n v_n \phi dv \to \int_{\boldsymbol{R}^N} uv\phi dv \text{ in } L^1_{loc}(\boldsymbol{R}^D) \text{ for each } \phi \in L^\infty(\boldsymbol{R}^D \times \boldsymbol{R}^D)\,.$$

Proof of lemma 4.4. Write

$$u_n v_n - uv = u_n(v_n - v) + v(u_n - u) \tag{39}$$

since $v \in L^\infty$ and $u_n \rightharpoonup u$ in L^1, the second term converges to 0 weakly in L^1.

Without loss of generality, one can assume that $\text{supp}(u_n) \subset K$ compact; indeed, since $u_n \rightharpoonup u$ in L^1, the sequence u_n is tight. By Egorov's Theorem, given $\eta > 0$, pick $E \subset K$ such that $|K \setminus E| < \eta$ and $v_n \to v$ uniformly on E. Decompose

$$u_n(v_n - v) = u_n \mathbf{1}_{K\setminus E}(v_n - v) + u_n \mathbf{1}_E(v_n - v)\,.$$

Given $\varepsilon > 0$ arbitrarily small, the first term can be made smaller than ε in L^1 norm uniformly in n by choosing η small enough, since u_n is uniformly integrable and $v_n - v$ bounded in L^∞. With $\eta > 0$ so chosen, the second term $\to 0$ in L^1 since $v_n \to v$ uniformly on E and u_n is bounded in L^1. Hence $u_n(v_n - v) \to 0$ strongly in L^1.

Therefore, if $v = 0$, only the first term is present in the decomposition above (39), so that $u_n v_n \to 0$ strongly in L^1. In the general case where $v \neq 0$, we only have that $u_n v_n \to 0$ weakly in L^1 because of the second term in that same decomposition. □

The proof of Lemma 4.5 is very similar and left as an exercise to the interested reader.

With the two lemmas above, we can give the

Proof of Proposition 4.2. By Proposition 4.1

$$\frac{\mathcal{A}_n(F_n)}{1+\int\limits_{\boldsymbol{R}^3} F_n dv} \to \frac{\mathcal{A}(F)}{1+\int\limits_{\boldsymbol{R}^3} F dv} \text{ a.e.}$$

where we recall that

$$\mathcal{A}(F) = \iint\limits_{\boldsymbol{R}^3\times\boldsymbol{S}^2} F_* b(v-v_*,\omega) d\omega\, dv_*$$

and

$$\mathcal{A}_n(F) = \frac{\mathcal{A}(F)}{1+\frac{1}{n}\int\limits_{\boldsymbol{R}^3} F dv},$$

while

$$\left\| \frac{\mathcal{A}_n(F_n)}{1+\int\limits_{\boldsymbol{R}^3} F_n dv} \right\|_{L^\infty} \le \|b\|_{L^\infty}.$$

Applying Lemma 4.5 shows that

$$\int\limits_{\boldsymbol{R}^3} \frac{F_n \mathcal{A}_n(F_n)}{1+\int\limits_{\boldsymbol{R}^3} F_n dv} \phi dv \to \int\limits_{\boldsymbol{R}^3} \frac{F\mathcal{A}(F)}{1+\int\limits_{\boldsymbol{R}^3} F dv} \phi dv \text{ in } L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3),$$

or, in other words

$$\int\limits_{\boldsymbol{R}^3} \frac{\mathcal{B}^n_-(F_n,F_n)}{1+\int\limits_{\boldsymbol{R}^3} F_n dv} \phi dv \to \int\limits_{\boldsymbol{R}^3} \frac{\mathcal{B}_-(F,F)}{1+\int\limits_{\boldsymbol{R}^3} F dv} \phi dv \text{ in } L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3).$$

The case of $\mathcal{B}^n_+(F_n,F_n)$ is easily reduced to the case of $\mathcal{B}^n_-(F_n,F_n)$ by exchanging (v,v_*) and (v',v'_*). More precisely,

$$\int\limits_{\boldsymbol{R}^3} \frac{\mathcal{B}^n_+(F_n,F_n)}{1+\int\limits_{\boldsymbol{R}^3} F_n dv} \phi dv = \frac{1}{1+\frac{1}{n}\int\limits_{\boldsymbol{R}^3} F_n dv} \iiint\limits_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} F_n F_{n*} \phi' b(v-v_*,\omega) d\omega\, dv_*.$$

Set

$$\Phi(t,x,v,v_*) = \int\limits_{S^2} \phi' b(v - v_*, \omega) d\omega\,,$$

and define

$$\tilde{\mathcal{A}}(F) = \int\limits_{R^3} F(t,x,v_*)\Phi(t,x,v,v_*)dv_*\,,$$

while

$$\tilde{\mathcal{A}}_n(F) = \frac{\tilde{\mathcal{A}}(F)}{1 + \frac{1}{n}\int\limits_{R^3} F_n dv}\,.$$

Observe that

$$\int\limits_{R^3} \frac{\mathcal{B}^n_+(F_n,F_n)}{1+\int F_n dv}\phi dv = \int\limits_{R^3} \frac{\tilde{\mathcal{A}}_n(F_n)}{1+\int\limits_{R^3} F_n dv} F_n dv\,,$$

that

$$|\tilde{\mathcal{A}}(F_n)| \leq \|b\|_{L^\infty}\|\phi\|_{L^\infty}\int\limits_{R^3} F_n dv\,,$$

and that

$$\int\limits_{R^3} \Phi(t,x,v,v_*)F_n(t,x,v_*)dv_* \to \int\limits_{R^3} \Phi(t,x,v,v_*)F(t,x,v_*)dv_* \tag{40}$$

in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$ and a.e., possibly after extracting a subsequence of F_n. Indeed, since ϕ is a compactly supported continuous function on $\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$, Φ is also continuous and bounded on $\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$. Hence (40) follows from Proposition 4.1. □

3.5 - *Consequences of the compactness of* F_n

Before proving that F is indeed a renormalized solution of the Boltzmann equation, we first show how the entropy inequality and the variants of the conservation laws satisfied by F follow from the corresponding properties satisfied by each solution F_n of the truncated Boltzmann equation and from the convergence properties of F_n stated in Propositions 3.2 and 4.1.

3.5.1 - The entropy inequality

We begin with the following observation.

Lemma 5.1. *Under the same assumptions as in Proposition* 4.2, *one has*

$$\frac{F_n F_{n*}}{1+\delta \int_{\boldsymbol{R}^3} F_n dv} \rightharpoonup \frac{F F_*}{1+\delta \int_{\boldsymbol{R}^3} F dv}$$

$$\frac{F'_n F'_{n*}}{1+\delta \int_{\boldsymbol{R}^3} F_n dv} \rightharpoonup \frac{F' F'_*}{1+\delta \int_{\boldsymbol{R}^3} F dv}$$

in $L^1([0,T]\times \boldsymbol{R}^3_x \times \boldsymbol{R}^3_v \times \boldsymbol{R}^3_{v_*} \times \boldsymbol{S}^2_\omega)$.

Proof. Same as that of Proposition 4.2. □

Notice that the function

$$(X,Y)\mapsto (X-Y)(\ln X - \ln Y)$$

is convex on $(\boldsymbol{R}_+)^2$. Hence, by weak limit and convexity, for each $t>0$, Lemma 5.1 implies that

$$\tag{41} \begin{aligned} &\int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{d(F) b dv dv_* d\omega dx ds}{1+\delta \int_{\boldsymbol{R}^3} F dv} \\ &\leq \varliminf_{n\to+\infty} \int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{d(F_n) b dv dv_* d\omega dx ds}{1+\delta \int_{\boldsymbol{R}^3} F_n dv} . \end{aligned}$$

Likewise, Proposition 3.2 and the convexity of $z\mapsto z\ln z$ imply that, for each $t\geq 0$

$$\tag{42} \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F\ln F(t,x,v) dx dv \leq \varliminf_{n\to+\infty} \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F_n \ln F_n(t,x,v) dx dv .$$

For each $t\geq 0$ and each $n\geq \frac{1}{\delta}$, one has

$$\begin{aligned} &\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F_n \ln F_n(t,x,v) dx dv + \int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{d(F_n) b dv dv_* d\omega dx ds}{1+\delta \int_{\boldsymbol{R}^3} F_n dv} \\ &\leq \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in} \ln F^{in}(x,v) dx dv \end{aligned}$$

because of (33) in Proposition 2.1. Passing to the limit as $n \to +\infty$ in the inequality above leads to

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F\ln F(t,x,v)dxdv + \int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} \frac{d(F)bdvdv_*d\omega dxds}{1+\delta\int_{\boldsymbol{R}^3} Fdv}$$
$$\leq \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} F^{in}\ln F^{in}(x,v)dxdv$$

on account of (41) and (42), for each $\delta > 0$.

Finally, letting $\delta \to 0^+$ in this last inequality gives the entropy inequality of Theorem 1.3 by monotone convergence.

3.5.2 - The conservation laws

First $F_n(t,\cdot,\cdot) \rightharpoonup F(t,\cdot,\cdot)$ in $L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)$ uniformly in $t \in [0,T]$ for each $T > 0$ by Proposition 3.2, while

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|x|^2+|v|^2)F_n(t,x,v)dxdv \leq C(1+t^2)$$

with C independent of n by Proposition 2.1. Hence

$$\int_{\boldsymbol{R}^3} F_n\begin{pmatrix}1\\v\end{pmatrix}dv \rightharpoonup \int_{\boldsymbol{R}^3} F\begin{pmatrix}1\\v\end{pmatrix}dv$$

in $L^1([0,T]\times\boldsymbol{R}^3_x)$ as $n \to +\infty$.

Passing to the limit in the continuity equation and the global momentum conservation law for the truncated Boltzmann equation (see Proposition 2.1) leads to

$$\partial_t \int_{\boldsymbol{R}^3} Fdv + \mathrm{div}_x \int_{\boldsymbol{R}^3} vFdv = 0$$

and

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} vF(t,x,v)dv = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} vF^{in}(x,v)dv \text{ for each } t \geq 0\,.$$

The global energy conservation for the truncated Boltzmann equation (see Proposition 2.1) and the fact that $F_n \geq 0$ a.e. imply that, for each $t \geq 0$ and $R > 0$, one has

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \mathbf{1}_{|v|\le R}|v|^2 F_n(t,x,v)dvdx \le \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F_n(t,x,v)dvdx = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F^{in}(x,v)dvdx\,.$$

Hence, passing to the limit in the left hand side of the inequality above shows that

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \mathbf{1}_{|v|\le R}|v|^2 F(t,x,v)dvdx \le \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F^{in}(x,v)dvdx\,.$$

Letting then $R\to+\infty$ shows that, for each $t\ge 0$

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F(t,x,v)dvdx \le \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} |v|^2 F^{in}(x,v)dvdx$$

by monotone convergence.

Finally, observe that, for each $i,j=1,2,3$, the sequence

$$\left|\int_{\boldsymbol{R}^3} v_iv_jF_ndv\right| \le \int_{\boldsymbol{R}^3} \frac{1}{2}|v|^2F_ndv \text{ is bounded in } L^\infty(\boldsymbol{R}_+;L^1(\boldsymbol{R}^3))$$

hence, modulo extraction of a subsequence, by the Banach-Alaoglu theorem, for each $i,j=1,2,3$

$$\int_{\boldsymbol{R}^3} v_iv_jF_ndv \rightharpoonup \mu_{ij} \text{ in } L^\infty(\boldsymbol{R}_+;\mathcal{M}(\boldsymbol{R}^3)) \text{ weak}_*\,.$$

Clearly, $\mu_{ij}=\mu_{ji}$ for $i,j=1,2,3$.

Let $\xi\in\boldsymbol{R}^3$ and $\chi\equiv\chi(t,x)$ a nonnegative test function belonging to $C_c(\boldsymbol{R}_+\times\boldsymbol{R}^3)$. Then

$$\iiint_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} |\xi\cdot v|^2\chi(t,x)F_n(t,x,v)dvdxdt \to \sum_{i,j=1}^3 \iint_{\boldsymbol{R}_+\times\boldsymbol{R}^3} \xi_i\xi_j\mu_{ij}(t)dt$$

while

$$\iiint_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} |\xi\cdot v|^2\mathbf{1}_{|v|\le R}\chi(t,x)F_n(t,x,v)dvdxdt \to \iiint_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} |\xi\cdot v|^2\mathbf{1}_{|v|\le R}\chi(t,x)F(t,x,v)dvdxdt$$

as $n \to +\infty$. Hence

$$\sum_{i,j=1}^{3} \xi_i \xi_j \iint_{\boldsymbol{R}_+ \times \boldsymbol{R}^3} \chi(t,x) \left(\mu_{ij}(t) - \int_{\boldsymbol{R}^3} \mathbf{1}_{|v| \le R} v_i v_j F(t,x,v) dv \right) dt$$

$$= \lim_{n \to +\infty} \iiint_{\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3} |\xi \cdot v|^2 \mathbf{1}_{|v|>R} \chi(t,x) F_n(t,x,v) dv dx dt \ge 0 \,.$$

Letting $R \to +\infty$ in the left hand side of the inequality above, one obtains

$$\sum_{i,j=1}^{3} \xi_i \xi_j \iint_{\boldsymbol{R}_+ \times \boldsymbol{R}^3} \chi(t,x) \left(\mu_{ij}(t) - \int_{\boldsymbol{R}^3} v_i v_j F(t,x,v) dv \right) dt \ge 0,$$

by monotone convergence, which shows that

$$m_{ij}(t) = \mu_{ij}(t) - \int_{\boldsymbol{R}^3} v_i v_j F(t,x,v) dv = m_{ji}(t)$$

is a nonnegative symmetric element of $L^\infty(\boldsymbol{R}_+; \mathcal{M}(\boldsymbol{R}^3; M_3(\boldsymbol{R})))$.

The discussion above shows that

$$\int_{\boldsymbol{R}^3} v_i v_j F_n dv \rightharpoonup \int_{\boldsymbol{R}^3} v_i v_j F dv + m_{ij}$$

in $L^\infty(\boldsymbol{R}^3; \mathcal{M}(\boldsymbol{R}^3))$ weak-*, so that, by passing to the limit in the local conservation law of momentum satisfied by the solution of the truncated Boltzmann equation, one arrives at

$$\partial_t \int_{\boldsymbol{R}^3} v F dv + \operatorname{div}_x \int_{\boldsymbol{R}^3} v \otimes v F dv + \operatorname{div}_x m = 0 \,.$$

3.6 - *Passing to the limit in the renormalized equation*

In order to prove the existence part of the DiPerna-Lions theorem, it remains to prove that F_n converges to a renormalized solution of the Boltzmann equation. With the information at our disposal, and although Proposition 4.2 contains useful information on the nonlinear term, this convergence is not trivial, in particular because the only source of compactness in the problem — i.e. Velocity Averaging — gives compactness on macroscopic observables defined by F_n, i.e.

on quantities of the type

$$\rho_n^{\phi}(t,x) = \int_{\boldsymbol{R}^3} F_n(t,x,v)\phi(v)dv$$

and not on F_n itself.

3.6.1 - Some preparations

For each $\delta \in (0,1)$, define $\beta_\delta(z) = \frac{z}{1+\delta z}$; clearly $\beta_\delta \in C^\infty(\boldsymbol{R}_+)$ and satisfies

$$\beta_\delta'(z) = \frac{1}{(1+\delta z)^2} \le \frac{1}{1+\delta z}\ ,$$

as well as

$$\beta_\delta(z) \le \frac{1}{\delta}$$

for each $z \ge 0$.

Therefore, modulo extracting again a subsequence of F_n, one has

$$\beta_\delta(F_n) \rightharpoonup F_\delta \text{ in } L^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3) \text{ weak * } \tag{43}$$

for each $\delta > 0$.

On the other hand

$$0 \le \beta_\delta'(F_n)\mathcal{B}_\pm^n(F_n,F_n) \le \frac{\mathcal{B}_\pm^n(F_n,F_n)}{1+\delta F_n}$$

so that, by Proposition 3.1, modulo extracting again a subsequence of F_n, one has

$$\beta_\delta'(F_n)\mathcal{B}_\pm(F_n,F_n) \rightharpoonup \mathcal{B}_\pm^\delta \text{ in } L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)\,. \tag{44}$$

Bringing together (43) and (44) leads to

$$\begin{aligned} (\partial_t + v\cdot\nabla_x)F_\delta &= \mathcal{B}_+^\delta - \mathcal{B}_-^\delta \ \text{ on } \boldsymbol{R}_+^* \times \boldsymbol{R}^3 \times \boldsymbol{R}^3\,,\\ F_\delta\big|_{t=0} &= \beta_\delta(F^{in})\,, \qquad \text{on } \boldsymbol{R}^3 \times \boldsymbol{R}^3\,. \end{aligned} \tag{45}$$

Next consider the difference $F_n - \beta_\delta(F_n)$; one has

$$0 \le F_n - \beta_\delta(F_n) = \frac{\delta F_n^2}{1+\delta F_n} \le \delta F_n^2 \mathbf{1}_{F_n \le R} + F_n \mathbf{1}_{F_n > R}\,.$$

Pick $\varepsilon > 0$ arbitrarily small; the second term on the right hand side of the inequality above can be made less than ε in $L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3)$ uniformly in $t \in \boldsymbol{R}_+$, $n \ge 1$ and $\delta \in (0,1)$ by picking $R > 0$ large enough, since F_n is uniformly integrable and tight on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$, uniformly in t. With R so chosen, the first term is bounded by $\delta R F_n$ and vanishes in $L^\infty(\boldsymbol{R}_+;L^1(\boldsymbol{R}^3\times\boldsymbol{R}^3))$ uniformly in n as $\delta\to 0$ since F_n is bounded in $L^\infty(\boldsymbol{R}_+;L^1(\boldsymbol{R}^3\times\boldsymbol{R}^3))$. Hence

$$F_n(t,\cdot,\cdot)-\beta_\delta(F_n(t,\cdot,\cdot))\to 0 \text{ in } L^1(\boldsymbol{R}^3\times\boldsymbol{R}^3) \tag{46}$$

uniformly in $n\geq 1$ and $t\in\boldsymbol{R}_+$ as $\delta\to 0^+$, and therefore, modulo extracting a subsequence of $\delta\to 0$,

$$F_\delta(t,\cdot,\cdot)\to F(t,\cdot,\cdot) \text{ in } L^1(\boldsymbol{R}^3\times\boldsymbol{R}^3) \tag{47}$$

uniformly in $t\in\boldsymbol{R}_+$ as $\delta\to 0^+$ and on $\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3$ a.e..

The idea is now to renormalize the equation (45), which leads to

$$\begin{aligned}(\partial_t+v\cdot\nabla_x)\ln(1+F_\delta)&=\frac{\mathcal{B}_+^\delta}{1+F_\delta}-\frac{\mathcal{B}_-^\delta}{1+F_\delta}\ \text{ on } \boldsymbol{R}_+^*\times\boldsymbol{R}^3\times\boldsymbol{R}^3\,,\\ F_\delta\big|_{t=0}&=\beta_\delta(F^{in})\,,\qquad \text{on } \boldsymbol{R}^3\times\boldsymbol{R}^3\,.\end{aligned} \tag{48}$$

The advantage in considering (48) instead of the original truncated Boltzmann equation (31) is precisely the strong convergence (47). So far, we know that $F_n\rightharpoonup F$ in $L^1_{loc}(\boldsymbol{R}_+^*\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$, with strong convergence only for macroscopic observables of the type

$$\int_{\boldsymbol{R}^3} F_n(t,x,v)\phi(v)dv\,,$$

with ϕ continuous and subquadratic.

On the other hand, there is still a difficulty in considering (48), namely the rather mysterious dependence of $\mathcal{B}_\pm^\delta$ on F_δ.

3.6.2 - The loss term $\mathcal{B}_-^\delta$

Next we recall that $\mathcal{B}_-^n(F_n,F_n)=F_n\mathcal{A}_n(F_n)$ with

$$\mathcal{A}_n(F_n(t,\cdot,\cdot))\to\mathcal{A}(F(t,\cdot,\cdot)) \text{ in } L^1(\boldsymbol{R}^3\times\boldsymbol{R}^3) \tag{49}$$

uniformly in $t\in[0,T]$ as $n\to+\infty$.

Let

$$G_{n,\delta}=\frac{F_n}{(1+\delta F_n)^2}\,;$$

modulo extracting subsequences again, one has

$$G_{n,\delta}\rightharpoonup G_\delta \text{ in } L^\infty(\boldsymbol{R}_+\boldsymbol{R}^3\times\boldsymbol{R}^3) \text{ weak } *.$$

Hence

$$\mathcal{B}_-^\delta=G_\delta\mathcal{A}(F)\,. \tag{50}$$

Next observe that $G_\delta \le F_\delta$ since $G_{n,\delta} \le \beta_\delta(F_n)$ for each n; hence

$$0 \le \frac{\mathcal{B}_-^\delta}{1+F_\delta} = \frac{G_\delta}{1+F_\delta}\mathcal{A}(F) \le \mathcal{A}(F) \in L^\infty(\boldsymbol{R}_+;(L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3))\,. \tag{51}$$

Hence the family

$$\frac{\mathcal{B}_-^\delta}{1+F_\delta} \text{ is weakly relatively compact in } L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)\,.$$

On the other hand

$$G_\delta(t,\cdot,\cdot) \to F(t,\cdot,\cdot) \text{ in } L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3) \tag{52}$$

uniformly in $t \in \boldsymbol{R}_+$ as $\delta \to 0^+$, and a.e. on $\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$ (by the same argument as in the case of F_δ). Hence, possibly after extraction of a subsequence

$$\frac{\mathcal{B}_-^\delta}{1+F_\delta} = \frac{G_\delta}{1+F_\delta}\mathcal{A}(F) \to \frac{F}{1+F}\mathcal{A}(F) = \frac{\mathcal{B}_-(F,F)}{1+F} \tag{53}$$

a.e. on $\boldsymbol{R}_+^* \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$ as $\delta \to 0^+$, and hence (by the Lebesgue theorem) in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$.

3.6.3 - The gain term $\mathcal{B}_+^\delta$

As expected, this term is the most complicated one. We start with the following easy inequality.

Lemma 6.1. *For each $\delta \in (0,1)$, one has*

$$\mathcal{B}_+^\delta \le \mathcal{B}_+(F,F) \text{ a.e. on } \boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3\,.$$

Proof. First, for each $\delta \in (0,1)$ and each $n \ge 1$, one has

$$\beta_\delta'(F_n)\mathcal{B}_+(F_n,F_n) = \frac{\mathcal{B}_+(F_n,F_n)}{(1+\delta F_n)^2} \le \mathcal{B}_+(F_n,F_n)\,.$$

Denoting for simplicity

$$\int_{\boldsymbol{R}^3} F_n dv = \rho_n\,,$$

one has therefore, for each nonnegative $\phi \in C_c(\boldsymbol{R}^3)$

$$\int_{\boldsymbol{R}^3} \frac{\beta_\delta'(F_n)\mathcal{B}_+(F_n,F_n)}{1+\rho_n}\phi(v)dv \le \int_{\boldsymbol{R}^3} \frac{\mathcal{B}_+(F_n,F_n)}{1+\rho_n}\phi(v)dv\,.$$

Since

$$\beta'_\delta(F_n)\mathcal{B}_+(F_n,F_n) \rightharpoonup \mathcal{B}_+^\delta \text{ in } L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$$

and $0\le \dfrac{1}{1+\rho_n}\le 1$ while $\rho_n\to\rho$ a.e. on $\boldsymbol{R}_+\times\boldsymbol{R}^3$, Lemma 4.4 implies that

$$\frac{\beta'_\delta(F_n)\mathcal{B}_+(F_n,F_n)}{1+\rho_n} \rightharpoonup \frac{\mathcal{B}_+^\delta}{1+\rho} \text{ in } L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$$

so that

$$\int_{\boldsymbol{R}^3} \frac{\beta'_\delta(F_n)\mathcal{B}_+(F_n,F_n)}{1+\rho_n}\phi(v)dv \rightharpoonup \frac{1}{1+\rho}\int_{\boldsymbol{R}^3}\mathcal{B}_+^\delta\phi(v)dv$$

while the right hand side is known to satisfy

$$\int_{\boldsymbol{R}^3} \frac{\mathcal{B}_+(F_n,F_n)}{1+\rho_n}\phi(v)dv \to \frac{1}{1+\rho}\int_{\boldsymbol{R}^3}\mathcal{B}_+(F,F)\phi(v)dv \tag{54}$$

in $L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$. Hence

$$\frac{1}{1+\rho}\int_{\boldsymbol{R}^3}\mathcal{B}_+^\delta\phi(v)dv \le \frac{1}{1+\rho}\int_{\boldsymbol{R}^3}\mathcal{B}_+(F,F)\phi(v)dv\,;$$

since this inequality holds for each nonnegative, compactly supported continuous test function ϕ, one has

$$\mathcal{B}_+^\delta \le \mathcal{B}_+(F,F) \text{ a.e. on } \boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3\,. \tag{55}$$

□

The most important part in our argument is summarized below.

Lemma 6.2. *One has*

$$\mathcal{B}_+(F,F)\le \varliminf_{\delta\to 0^+}\mathcal{B}_+^\delta\,.$$

Proof. Let $f\equiv f(v)\ge 0$ a.e. be a measurable function; for each $K>1$, introduce as in section 3 the following decomposition

$$f'f'_* - ff_* = (f'f'_* - ff_*)\mathbf{1}_{f'f'_*>Kff_*} + (f'f'_* - ff_*)\mathbf{1}_{f'f'_*\le Kff_*} \le \frac{1}{\ln K}d(f) + (K-1)ff_*\,,$$

where $d(f)$ is the entropy production integrand given by (29). Therefore

$$\mathcal{B}_+(f,f)\le \frac{1}{\ln K}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} d(f)b(v-v_*,\omega)dvd\omega + K\mathcal{B}_-(f,f)\,. \tag{56}$$

Consider next the expression

$$\frac{\mathcal{B}_+^n(F_n,F_n)}{(1+\delta R)(1+\lambda\mathcal{A}_n(F_n))} \le \frac{\mathcal{B}_+^n(F_n,F_n)}{1+\delta F_n} + \mathbf{1}_{F_n>R}\frac{\mathcal{B}_+^n(F_n,F_n)}{1+\lambda\mathcal{A}_n(F_n)}. \tag{57}$$

We first claim that, for each $\lambda > 0$,

$$\frac{\mathcal{B}_+^n(F_n,F_n)}{1+\lambda\mathcal{A}_n(F_n)} \rightharpoonup \frac{\mathcal{B}_+(F,F)}{1+\lambda\mathcal{A}(F)} \text{ in } L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)\,. \tag{58}$$

Indeed, the inequality (56) implies that $\dfrac{\mathcal{B}_+^n(F_n,F_n)}{1+\lambda\mathcal{A}_n(F_n)}$ is weakly relatively compact in $L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$; assuming that, modulo extraction of a subsequence,

$$\frac{\mathcal{B}_+^n(F_n,F_n)}{1+\lambda\mathcal{A}_n(F_n)} \rightharpoonup L \text{ in } L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$$

one has

$$\frac{\mathcal{B}_+^n(F_n,F_n)}{(1+\int_{\boldsymbol{R}^3} F_n dv)(1+\lambda\mathcal{A}_n(F_n))} \rightharpoonup \frac{L}{1+\int_{\boldsymbol{R}^3} F dv} = \frac{\mathcal{B}_+(F,F)}{(1+\int_{\boldsymbol{R}^3} F dv)(1+\lambda\mathcal{A}(F))}$$

in $L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$ because of (54), by Lemma 4.4 since

$$\frac{1}{1+\int_{\boldsymbol{R}^3} F_n dv} \to \frac{1}{1+\int_{\boldsymbol{R}^3} F dv} \text{ a.e. and } 0 \le \frac{1}{1+\int_{\boldsymbol{R}^3} F_n dv} \le 1 \text{ a.e..}$$

Now, start from the expression (57) and the inequality (56) in the form

$$\begin{aligned}\frac{\mathcal{B}_+^n(F_n,F_n)}{(1+\delta R)(1+\lambda\mathcal{A}_n(F_n))} \le{}& \frac{\mathcal{B}_+^n(F_n,F_n)}{1+\delta F_n}\\ &+\frac{1}{\ln K}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F_n)b(v-v_*,\omega)dvd\omega + \frac{K}{\lambda}F_n\mathbf{1}_{F_n>R}\,.\end{aligned}$$

Passing to the limit in $L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$ weak in both sides of the inequality above implies that

$$\frac{\mathcal{B}^+(F,F)}{(1+\delta R)(1+\lambda\mathcal{A}(F))} \le \mathcal{B}_\delta^+ + \frac{1}{\ln K}D + \frac{K}{\lambda}F^R \tag{59}$$

where

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F_n)b(v-v_*,\omega)dvd\omega \rightharpoonup D \text{ vaguely}$$

in the sense of Radon measures on $\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$, while

$$F_n \mathbf{1}_{F_n > R} \rightharpoonup F^R \text{ in } L^1_{loc}(dtdxdv)\,.$$

First, we let $\delta \to 0^+$ and pass to the limit a.e. in (59):

$$\frac{\mathcal{B}_+(F,F)}{1+\lambda\mathcal{A}(F)} \leq \varliminf_{\delta\to 0^+} \mathcal{B}_+^\delta + \frac{1}{\ln K} D_0 + \frac{K}{\lambda} F^R$$

a.e. on $\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$, where D_0 is the L^1 part of the Radon measure D. Next, we let $R \to +\infty$, so that $F^R \to 0$ in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$ and a.e. (modulo extraction of a subsequence). Finally, we first let $K \to +\infty$ and then $\lambda \to 0^+$, which leads to the announced inequality. □

3.6.4 - Conclusion of the existence proof

On the other hand, the inequality (56) implies that

$$\beta'_\delta(F_n)\mathcal{B}^n_+(F_n,F_n) \leq \frac{1}{\ln K} \iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F_n) b(v-v_*,\omega) dv_* d\omega + K\beta'_\delta(F_n)\mathcal{B}^n_-(F_n,F_n)\,.$$

By passing to the limit in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$ weak as $n \to +\infty$, we arrive at

$$\mathcal{B}^\delta_+ \leq \frac{1}{\ln K} D_0 + K\mathcal{B}^\delta_-$$

and therefore

$$\frac{\mathcal{B}^\delta_+}{1+F_\delta} \leq \frac{1}{\ln K} D_0 + K\frac{\mathcal{B}^\delta_-}{1+F_\delta} \leq \frac{1}{\ln K} D_0 + K\mathcal{A}(F)$$

because of (51). In particular, $\dfrac{\mathcal{B}^\delta_+}{1+F_\delta}$ is weakly relatively compact in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3)$. By Lebesgue's Theorem and (55), Lemma 6.2 and (47), one has

$$\frac{\mathcal{B}^\delta_+}{1+F_\delta} \to \frac{\mathcal{B}_+(F,F)}{1+F} \text{ in } L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3) \text{ and a.e.} \tag{60}$$

as $\delta \to 0^+$.

Eventually, passing to the limit in (48) as $\delta \to 0^+$, on account of (53), (60) and (47) implies that

$$(\partial_t + v\cdot\nabla_x)\ln(1+F) = \frac{\mathcal{B}_+(F,F)}{1+F} - \frac{\mathcal{B}_-(F,F)}{1+F}$$

with the initial condition

$$F\big|_{t=0} = F^{in}$$

since the convergence $F_\delta \to F$ is uniform in $[0,T]$.

More generally, for $\beta \in C^1(\boldsymbol{R}_+)$ satisfying $|\beta'(Z)| \le \dfrac{C}{1+Z}$ for each $Z \ge 0$, one has

$$\begin{aligned}(\partial_t + v\cdot\nabla_x)\beta(F) &= (\partial_t + v\cdot\nabla_x)\beta(e^{\ln(1+F)}-1)\\ &= \beta'(e^{\ln(1+F)}-1)(1+F)\frac{\mathcal{B}(F,F)}{1+F}\\ &= \beta'(F)\mathcal{B}(F,F) \in L^1_{loc}(\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3)\,.\end{aligned}$$

To see this, write

$$(\partial_t + v\cdot\nabla_x)\beta(F)(t,x,v) = \frac{d}{dt}\beta(F)(t,x+tv,v)$$

and recall that if $f:\,[0,T]\mapsto\boldsymbol{R}$ is absolutely continuous and $\Phi\in C^1(\boldsymbol{R})$, then $\Phi\circ f$ is absolutely continuous on $[0,T]$ with $(\Phi\circ f)' = \Phi'(f)f'$ a.e.

Finally, let us prove that F is a renormalized solution of the Boltzmann equation.

First, we already know that F satisfies the entropy inequality in Theorem 1.3 as well as the bound

$$\iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} (1+|x|^2+|v|^2)F(t,x,v)dxdv \le C(1+t^2)$$

inherited from (34). Hence

$$\int_0^T\int_{\boldsymbol{R}^3}\iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F)bdvdv_*d\omega dxdt \le C'(1+T^2)$$

for some positive constant C'. Applying Lemma 1.2 shows that

$$\frac{\mathcal{B}(F,F)}{\sqrt{1+F}} \in L^1([0,T]\times\boldsymbol{R}^3\times B(0,R))$$

for each $R,T>0$.

Let then $\beta\in C^1(\boldsymbol{R}_+)$ be such that $|\beta'(Z)| \le \dfrac{C}{\sqrt{1+Z}}$ for each $Z\ge 0$. Pick a family $\beta_\delta \in C^1(\boldsymbol{R})$ such that

$$|\beta_\delta'(Z)| \le \frac{C_\delta}{1+Z} \text{ and } |\beta_\delta'(Z)| \le \frac{C'}{\sqrt{1+Z}}$$

where C_δ is a positive constant that in general depends on δ while C' is a positive constant that is independent of δ. Hence in particular

$$|\beta_\delta(Z)| \le |\beta_\delta(0)| + 2C'\sqrt{1+Z} \text{ for each } Z\ge 0 \text{ and } \delta>0\,.$$

We also require that

$$\beta_\delta(Z)\to\beta(Z) \text{ and } \beta_\delta'(Z)\to\beta'(Z) \text{ for each } Z\ge 0 \text{ as } \delta\to 0\,.$$

We already know that

$$(\partial_t + v\cdot\nabla_x)\beta_\delta(F) = \beta_\delta'(F)\mathcal{B}(F,F)$$

$$\beta_\delta(F)\big|_{t=0} = \beta_\delta(F^{in})$$

for each $\delta > 0$. In other words, for each test function $\phi \equiv \phi(t,x,v) \in C_c^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}_x^3 \times \boldsymbol{R}_v^3)$, and each $\delta > 0$, one has

$$\iiint\limits_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} \beta_\delta(F)(\partial_t + v\cdot\nabla_x)\phi dvdxdt + \iint\limits_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \beta_\delta(F^{in})\phi(0,x,v)dvdx + \iiint\limits_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} \beta_\delta'(F)\mathcal{B}(F,F)\phi dvdxdt = 0\,. \tag{61}$$

We know that $F \in C(\boldsymbol{R}_+; L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3))$ and

$$|\beta_\delta(Z)| \le |\beta_\delta(0)| + 2C'\sqrt{1+Z} \text{ for each } Z \ge 0 \text{ and } \delta > 0\,.$$

On the other hand, we also know that $F \in C(\boldsymbol{R}_+; L^1(\boldsymbol{R}^3 \times \boldsymbol{R}^3))$ and $d(F) \in L^1(\boldsymbol{R}_+ \times \boldsymbol{R}_x^3 \times \boldsymbol{R}_v^3 \times \boldsymbol{R}_{v_*}^3 \times \boldsymbol{S}^2)$, so that, by the entropy production estimate (14) together with Lemma 1.2, one has

$$|\beta_\delta'(F)\mathcal{B}(F,F)| \le C'\frac{|\mathcal{B}(F,F)|}{\sqrt{1+F}} \in L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}_x^3 \times \boldsymbol{R}_v^3)\,.$$

Passing to the limit by dominated convergence in the above equality as $\delta \to 0$ shows that

$$\iiint\limits_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} \beta(F)(\partial_t + v\cdot\nabla_x)\phi dvdxdt + \iint\limits_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \beta(F^{in})\phi(0,x,v)dvdx + \iiint\limits_{\boldsymbol{R}_+\times\boldsymbol{R}^3\times\boldsymbol{R}^3} \beta'(F)\mathcal{B}(F,F)\phi dvdxdt = 0\,.$$

In other words

$$(\partial_t + v\cdot\nabla_x)\beta(F) = \beta'(F)\mathcal{B}(F,F)$$

$$\beta(F)\big|_{t=0} = \beta(F^{in})$$

for each $\beta \in C^1(\boldsymbol{R}_+)$ such that $\beta'(Z) \le \dfrac{C}{\sqrt{1+Z}}$.

This shows that F is a renormalized solution of the Boltzmann equation, which concludes the proof of Theorem 1.3.

3.7 - *Solutions in Maxwellian equilibrium at infinity*

After [**17**], P.-L. Lions proposed some variants of the notion of renormalized solutions that are in Maxwellian equilibrium at infinity. A particular case of his theory, of paramount importance in the derivation of hydrodynamic limits, considers solutions of the Boltzmann equation posed in the Euclidian state $\boldsymbol{R}^3$ that converge to some uniform Maxwellian at infinity.

For simplicity, we consider the case where $M \equiv M(v)$ is the centered reduced Gaussian $M = \mathcal{M}_{(1,0,1)}$.

Consider the Cauchy problem

$$\begin{aligned} \partial_t F + v \cdot \nabla_x F = \mathcal{B}(F,F)\,, &\quad (x,v) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3\,,\ t > 0\,, \\ F(t,x,v) - M(v) \to 0 \text{ as } |x| \to +\infty\,, & \\ F\big|_{t=0} = F^{in}\,. & \end{aligned}$$

That $F(t,x,v) - M(v) \to 0$ as $|x| \to 0$ holds in some sense that is strong enough so that the relative entropy $H(F(t,,\cdot,\cdot)|M) < +\infty$ for each $t > 0$.

Definition 7.1. *A nonnegative function $F \in C(\boldsymbol{R}_+; L^1_{loc}(\boldsymbol{R}^3 \times \boldsymbol{R}^3))$ is a renormalized solution relatively to M of the Boltzmann equation if and only if, for each function $\Gamma \in C^1(\boldsymbol{R}_+)$ such that the function*

$$Z \mapsto \frac{\Gamma(Z)}{\sqrt{1+Z}} \text{ is bounded on } \boldsymbol{R}_+\,,$$

one has

$$M(\partial_t + v \cdot \nabla_x)\Gamma\left(\frac{F}{M}\right) = \Gamma'\left(\frac{F}{M}\right)\mathcal{B}(F,F)$$

in the sense of distributions on $\boldsymbol{R}_+^* \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$.

With this definition, P.-L. Lions proved the following variant of Theorem 1.3.

Theorem 7.2. *Let $F^{in} \geq 0$ be a measurable function such that $H(F^{in}|M) < +\infty$. There exists a renormalized solution relative to M of the Boltzmann equation such that $F\big|_{t=0} = F^{in}$, which moreover satisfies*

- *the continuity equation*

$$\partial_t \int_{\boldsymbol{R}^3} F dv + \operatorname{div}_x \int_{\boldsymbol{R}^3} vF dv = 0\,;$$

• *the momentum equation with defect measure*

$$\partial_t \int_{\boldsymbol{R}^3} vFdv + \mathrm{div}_x \int_{\boldsymbol{R}^3} v \otimes vFdv + \mathrm{div}_x m = 0$$

where m *is a Radon measure on* $\boldsymbol{R}_+ \times \boldsymbol{R}^3$ *with values in the nonnegative symmetric matrices;*

• *the entropy inequality*

$$H(F(t)|M) + \frac{1}{4}\int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F)b(v-v_*,\omega)dvdv_*d\omega dxds + \int \mathrm{trace}(m)(t) \le H(F^{in}|M) \tag{62}$$

for each $t \ge 0$.

This is precisely the notion of solution of the Boltzmann equation which is the most useful for the problem of hydrodynamic limits.

4 - Formal Hydrodynamic Limits: the Moment Method and the Hilbert and Chapman-Enskog expansions

In the present chapter, we shall present various formal computations that explain how the classical PDEs of Fluid Mechanics can be derived from the Boltzmann equation. Such computations are not rigorous proofs of these derivations; however, they give extremely valuable intuition on the subject of hydrodynamic limits of the kinetic theory of gases.

4.1 - *The dimensionless Boltzmann equation*

Before entering the subject of hydrodynamic limits properly speaking, we first describe the Boltzmann equation in dimensionless variables. In these variables, two dimensionless parameters, called the Knudsen and Strouhal numbers naturally appear in the Boltzmann equation. In this section, we consider the Boltzmann equation for general cut-off potentials.

Choose a macroscopic length scale L and time scale T, and a reference temperature Θ. This defines *two* velocity scales:

• one is the speed at which some macroscopic portion of the gas is transported over a distance L in time T, i.e.

$$U = \frac{L}{T}\,;$$

- the other one is the *thermal speed* of the molecules with energy $\frac{3}{2}k\Theta$; in fact, it is more natural to define this velocity scale as

$$c = \sqrt{\frac{5}{3}\frac{k\Theta}{m}}$$

— m being the molecular mass — which is the *speed of sound* in a monatomic gas at the temperature Θ.

Define next the dimensionless variables involved in the Boltzmann equation, i.e. the dimensionless time, space and velocity variables as

$$\hat{t} = \frac{t}{T}\,, \quad \hat{x} = \frac{x}{L}\,, \quad \text{and } \hat{v} = \frac{v}{c}\,.$$

Define also the dimensionless number density

$$\hat{F}(\hat{t},\hat{x},\hat{v}) = \frac{L^3c^3}{\mathcal{N}^3}F(t,x,v)\,,$$

where $\mathcal{N}$ is the total number of gas molecules in a volume L^3. Finally, we must rescale the collision kernel b. As mentionned earlier, $b(z,\omega)$ is the relative velocity multiplied by the scattering cross-section of the gas molecules; define

$$\hat{b}(\hat{z},\omega) = \frac{1}{c \times \pi r^2}b(z,\omega) \quad \text{with } \hat{z} = \frac{z}{c}\,,$$

where r is the molecular radius.

If f satisfies the Boltzmann equation

$$\partial_t F + v\cdot\nabla_x F = \iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} (F'F'_* - FF_*)b(v-v_*,\omega)dv_*d\omega$$

then

$$\frac{L}{cT}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{\mathcal{N}\pi r^2}{L^2}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} (\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)\hat{b}(\hat{v}-\hat{v}_*,\omega)d\hat{v}_*d\omega\,.$$

The factor multiplying the collision integral is

$$L \times \frac{\mathcal{N}\times\pi r^2}{L^3} = \frac{L}{\text{mean free path}} = \frac{1}{\text{Kn}}\,,$$

where Kn is the Knudsen number defined above. The factor multiplying the time derivative

$$\frac{\frac{1}{T}\times L}{c} =: \text{St}$$

is called the *kinetic Strouhal number* (by analogy with the notion of Strouhal number used in the dynamics of vortices). Hence the dimensionless form of the Boltzmann equation is (see § 2.9 in [**55**]):

$$\mathrm{St}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{1}{\mathrm{Kn}}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2}(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)\hat{b}(\hat{v}-\hat{v}_*,\omega)d\hat{v}_*d\omega\,. \tag{63}$$

There is some arbitrariness in the way the length, time and temperature scales L, T, Θ are chosen. The most natural thing to do is to choose these in a way that is consistent with the geometry of the domain where the gas motion takes place, the time necessary to observe significant gas motion, and the distribution function at the initial instant of time. In addition to Sone's book [**55**], we also refer to the introduction of [**5**] for a more detailed presentation of the Boltzmann equation in dimensionless variables.

All hydrodynamic limits of the Boltzmann equation correspond to situations where the Knudsen number Kn satisfies

$$\mathrm{Kn} \ll 1\,.$$

In other words, the Knudsen number governs the transition from the kinetic theory of gases to hydrodynamic models, just as the Reynolds number in Fluid Mechanics governs the transition from laminar to turbulent flows — except that the hydrodynamic limit is much better understood than the latter situation.

But there is no universal prescription for the Strouhal number in the context of the hydrodynamic limit; as we shall see below, various hydrodynamic regimes can be derived from the Boltzmann equation by appropriately tuning the Strouhal number.

4.2 - *Compressible Euler limit of the Boltzmann equation: the moment method*

The compressible Euler limit is the easiest of all hydrodynamic limits of the Boltzmann equation at the formal level — and the one for which obtaining a complete mathematical proof is the most challenging at the time of this writing.

The compressible Euler equations are obtained as a scaling limit of the Boltzmann equation in the case where

$$\mathrm{St} = 1\,, \qquad \mathrm{Kn} = \varepsilon \ll 1\,.$$

With these notations, the dimensionless Boltzmann equation (63) is put in the form

$$\begin{aligned}\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon &= \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon)\\ &= \frac{1}{\varepsilon}\iint (F'_\varepsilon F'_{\varepsilon *} - F_\varepsilon F_{\varepsilon *})b(v-v_*,\omega)d\omega\, dv_*.\end{aligned} \tag{64}$$

Obviously, one expects that, as $\varepsilon \to 0$,

$$F_\varepsilon \to F\,, \qquad \text{while } \mathcal{B}(F_\varepsilon, F_\varepsilon) \to \mathcal{B}(F,F) = 0\,.$$

Because of Boltzmann's H Theorem (Theorem 3.1 above) this implies that $F(t,x,\cdot)$ is a Maxwellian for each (t,x), i.e.

$$F(t,x,v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}(v) = \frac{\rho(t,x)}{(2\pi\theta(t,x))^{3/2}}\, e^{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}} \tag{65}$$

for some $\rho \equiv \rho(t,x) > 0$, $\theta \equiv \theta(t,x) > 0$ and $u \equiv u(t,x) \in \boldsymbol{R}^3$. Such a number density is called a *local Maxwellian*, and indeed, since Maxwellians are the thermodynamic equilibria in the kinetic theory of gases, we see that the limit $\mathrm{Kn} \to 0$ naturally leads to number densities that are local thermodynamic equilibria. In other words, this limit agrees with the description of fluid regimes.

It only remains to find the governing equations for the unknowns ρ, u and θ. For simplicity, we choose to discuss the case where the spatial domain is the periodic box — i.e. the torus $\boldsymbol{T}^3$.

Theorem 2.1. *Let $\rho^{in} \equiv \rho^{in}(x) > 0$, $\theta^{in} \equiv \theta^{in}(x) > 0$ and $u^{in} \equiv u^{in}(x) \in \boldsymbol{R}^3$ be continuous on $\boldsymbol{T}^3$. Assume that, for each $\varepsilon > 0$, the Boltzmann equation* (64) *has a solution F_ε such that*

$$F_\varepsilon\big|_{t=0} = \mathcal{M}_{(\rho^{in},u^{in},\theta^{in})}\,.$$

Assume that

$$\int_{\boldsymbol{T}^3} \left[(\rho^{in}\ln\rho^{in} - \rho^{in} - 1) + \frac{1}{2}\rho^{in}|u^{in}|^2 + \frac{3}{2}\rho^{in}(\theta^{in} - 1 - \ln\theta^{in})\right]dx$$
$$= H(\mathcal{M}_{(\rho^{in},u^{in},\theta^{in})}|\mathcal{M}_{(1,0,1)}) < +\infty$$

that F_ε is rapidly decaying and such that $\ln F_\varepsilon$ has polynomial growth as $|v| \to +\infty$; assume further that $F_\varepsilon \to F$ a.e., and that the decay properties above are uniform in this limit.

Then F is a local Maxwellian of the form (65), *whose parameters (ρ, u, θ) satisfy the compressible Euler system*

$$\begin{aligned} \partial_t\rho + \mathrm{div}_x(\rho u) &= 0\,,\\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) &= 0\,,\\ \partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \mathrm{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) &= 0\,, \end{aligned} \tag{66}$$

together with the initial condition

$$(\rho,u,\theta)\big|_{t=0} = (\rho^{in}, u^{in}, \theta^{in})\,.$$

They also satisfy the differential entropy inequality

$$\partial_t\left(\rho \ln \frac{\rho}{\theta^{3/2}}\right) + \operatorname{div}_x\left(\rho u \ln \frac{\rho}{\theta^{3/2}}\right) \le 0\,. \tag{67}$$

The proof given below follows the analysis in [**2**].

Proof. Without loss of generality, assume that

$$\int_{\boldsymbol{T}^3} \rho^{in}(x)dx = 1\,.$$

Proceeding as in Corollary 3.3, multiply each side of the scaled Boltzmann equation (64) by $\ln F_\varepsilon + \frac{1}{2}|v|^2$; by using the H Theorem and Proposition 2.1, we see that

$$\begin{aligned}\varepsilon &\iint_{\boldsymbol{T}^3\times\boldsymbol{R}^3} \left[F_\varepsilon \ln\left(\frac{F_\varepsilon}{M}\right) - F_\varepsilon + M\right](t,x,v)dxdv \\ &\qquad + \frac{1}{4}\int_0^t \int_{\boldsymbol{T}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F_\varepsilon) b(v-v_*,\omega)d\omega\, dvdv_* dxds \\ &\qquad \le \varepsilon \iint_{\boldsymbol{T}^3\times\boldsymbol{R}^3} \left[F_\varepsilon \ln\left(\frac{F_\varepsilon}{M}\right) - F_\varepsilon + M\right](0,x,v)dxdv\end{aligned}$$

where

$$d(F_\varepsilon) = (F'_\varepsilon F'_{\varepsilon *} - F_\varepsilon F_{\varepsilon *})\ln\left(\frac{F'_\varepsilon F'_{\varepsilon *}}{F_\varepsilon F_{\varepsilon *}}\right)$$

and $M = \mathcal{M}_{(1,0,1)}$ is the centered reduced Gaussian. Because of the elementary inequality

$$a\ln(a/b) - a + b \ge 0\,, \qquad a \ge 0\,,\ b > 0\,,$$

the first integral in the left hand side of the inequality above is nonnegative. As for the integral in the right hand side, it can be computed explicitly since

$$F_\varepsilon\big|_{t=0} = \mathcal{M}_{(\rho^{in},u^{in},\theta^{in})}\,.$$

Eventually, the inequality above implies that

$$\begin{aligned}&\frac{1}{4}\iiint_{\boldsymbol{T}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F_\varepsilon) b(v-v_*,\omega)d\omega\, dvdv_* \\ &\le \varepsilon \int_{\boldsymbol{T}^3}\left[\rho^{in}\ln\rho^{in} + \frac{1}{2}\rho^{in}|u^{in}|^2 + \frac{3}{2}\rho^{in}(\theta^{in} - \ln\theta^{in} - 1)\right]dx = O(\varepsilon)\,.\end{aligned}$$

By Fatou's lemma, this implies that

$$d(F) = 0\,,$$

which implies in turn that F is a local Maxwellian because of Boltzmann's H Theorem.

Then, by passing to the limit as $\varepsilon \to 0$ in the local conservation laws (8) satisfied by F_ε for each positive ε, it is found that

$$\begin{aligned}
\partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0\,,\\
\partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v \otimes vF dv = 0\,,\\
\partial_t \int_{\mathbf{R}^3} \frac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\frac{1}{2}|v|^2 F dv = 0\,.
\end{aligned}$$

Setting

$$F(t,x,v) \equiv \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}(v)$$

in the above system, and observing that

$$\begin{aligned}
\int_{\mathbf{R}^3} \mathcal{M}_{(\rho,u,\theta)} dv =& \rho\,,\\
\int_{\mathbf{R}^3} v\mathcal{M}_{(\rho,u,\theta)} dv =& \rho u\,,\\
\int_{\mathbf{R}^3} \frac{1}{2}|v|^2 \mathcal{M}_{(\rho,u,\theta)} dv =& \rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)
\end{aligned}$$

while

$$\begin{aligned}
\int_{\mathbf{R}^3} v \otimes v\mathcal{M}_{(\rho,u,\theta)} dv = \rho(u \otimes u + \theta I)\,,\\
\int_{\mathbf{R}^3} \frac{1}{2} v|v|^2 \mathcal{M}_{(\rho,u,\theta)} dv = \rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right),
\end{aligned}$$

we see that (ρ, u, θ) satisfy the compressible Euler system (66).

Finally, the differential inequality (10) is satisfied by F_ε for each $\varepsilon > 0$; passing to the limit as $\varepsilon \to 0$ leads to

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF \ln F dv \le 0\,.$$

Since

$$\int_{\boldsymbol{R}^3} \mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} dv = \rho \ln \left(\frac{\rho}{(2\pi\theta)^{3/2}} \right) - \frac{3}{2}\rho$$

while

$$\int_{\boldsymbol{R}^3} v\mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} dv = \rho u \ln \left(\frac{\rho}{(2\pi\theta)^{3/2}} \right) - \frac{3}{2}\rho u \,,$$

the differential inequality above, combined with the continuity equation — the first equation in (66) — eventually leads to (67). □

The basic mechanism of this proof can be summarized as follows: first, analyzing the entropy production term in Boltzmann's H Theorem shows that the limiting number density is a local Maxwellian.

Then, passing to the limit in the local conservation laws that result from the symmetries of the Boltzmann collision integral leads to the desired hydrodynamic equations, once we know that the limiting number density is a local Maxwellian.

As we shall see in the sequel, this basic principle lies at the heart of each of the hydrodynamic limits of the Boltzmann equation — or of other kinetic models of the same kind — that are discussed in this course.

4.3 - *Asymptotic expansions*

One can go beyond the compressible Euler limit as stated in Theorem 2.1, in several directions.

For practical purposes, it is of course interesting to know by how much the number deviates from the local Maxwellian governed by the compressible Euler system. A second question, obviously related to the first one, is to determine higher order hydrodynamic corrections to the compressible Euler system.

Both problems obviously require using asymptotic expansions of the number density in terms of some appropriate small parameter (the Knudsen number, in this case).

We therefore consider the same scaling as the one leading to the compressible Euler system, but for later use assume that the Strouhal number takes some fixed value, i.e.

$$\text{Kn} = \varepsilon \ll 1 \,, \qquad \text{and St} = \tau > 0 \,. \tag{68}$$

The scaled Boltzmann equation is

$$\tau\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon) \,, \tag{69}$$

and will be studied asymptotically for fixed Strouhal number τ and in the vanishing Knudsen number limit $\varepsilon \to 0$.

All methods for hydrodynamic limits based on asymptotic expansions seek solutions of the scaled Boltzmann equation as formal power series in ε

$$F_\varepsilon(t,x,v) = \sum_{n\geq 0} \varepsilon^n F_n(t,x,v)\,, \tag{70}$$

with coefficients F_n that are smooth in (t,x,v) and rapidly decaying as $|v| \to +\infty$.

The leading order approximation F_0 is expected to be the limiting hydrodynamic distribution function, while the successive corrections F_n account for the finite Knudsen effects. These coefficients F_n are found by plugging ansatz (70) in the scaled equation (69), and balancing the resulting coefficients of the successive powers of ε on each side of (69):

Order ε^{-1}:

$$\mathcal{B}(F_0,F_0) = 0\,,$$

Order ε^{-0}:

$$\tau\partial_t F_0 + v\cdot\nabla_x F_0 = 2\mathcal{B}(F_0,F_1)\,,$$

Order ε:

$$\tau\partial_t F_0 + v\cdot\nabla_x F_0 = 2\mathcal{B}(F_0,F_2) + \mathcal{B}(F_1,F_1)\,,$$

....

Order ε^{-n}:

$$\tau\partial_t F_n + v\cdot\nabla_x F_n = \sum_{i+j=n+1} \mathcal{B}(F_i,F_j)\,.$$

We therefore expect the v-dependence of the functions F_n to be determined by solving an integral equation (defined by the collision integral) at the order ε^{n-1}, while its (t,x)-dependence will be determined by evolution PDEs resulting from the equation at order ε^n.

While this strategy is conceptually very simple, it leads to several difficulties that one should not underestimate.

Depending on the exact form of the ansatz, this identification leads to different hierarchies of PDEs. A first difficulty is to decide at the formal level whether the infinite system of equations so obtained is well-posed, in other words to determine compatibility conditions between the various equations.

These compatibility conditions are in general not sufficient for a rigorous mathematical result. For instance, the Cauchy problems for the coefficients F_n may be not well-posed in the same functional framework. Even if such a functional framework exists, the Cauchy problems for the F_n may be not well-posed on a uniform time. It could also happen that the convergence radius of the formal power series is 0.

More serious problems related to this approach based on asymptotic expansion will be discussed at the end of the present section.

Essentially two kinds of asymptotic expansions have been used in the context of the Boltzmann equation, the Hilbert expansion and the Chapman-Enskog expansion.

4.3.1 - Hilbert's expansion

Hilbert's expansion is historically the older and goes back to Hilbert's fundamental paper [**33**] on the kinetic theory of gases.

In view of the discussion above, it is natural to seek the solution of the scaled Boltzmann equation as the local Maxwellian whose parameters are governed by the compressible Euler system, plus a fluctuation.

$$F_\varepsilon(t,x,v) = F_0\left(1 + \sum_{n\geq 1} \varepsilon^n g_n(t,x,v)\right). \tag{71}$$

In other words,

$$F_0(t,x,v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}$$

where ρ, u and θ satisfy the compressible Euler system

$$\begin{aligned} \tau\partial_t\rho + \mathrm{div}_x(\rho u) &= 0\,,\\ \tau\partial_t(\rho u) + \mathrm{div}_x(\rho u\otimes u) + \nabla_x(\rho\theta) &= 0\,,\\ \tau\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \mathrm{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) &= 0\,. \end{aligned} \tag{72}$$

We recall from chapter 1, section 6 the definition of Boltzmann's linearized collision integral about the (local) Maxwellian F_0:

$$\mathcal{L}_{F_0} g = -2F_0^{-1}\mathcal{B}(F_0, F_0 g)\,;$$

likewise, we define the quadratic part in the expansion of the collision integral about F_0

$$\mathcal{Q}_{F_0}(g_i, g_j) = F_0^{-1}\mathcal{B}(F_0 g_i, F_0 g_j)\,.$$

Then, for each $n \geq 1$, identifying the coefficients of the successive powers of ε on each side of the scaled Boltzmann equation leads to the sequence of equalities:

$$\tau\partial_t(F_0 g_n) + v\cdot\nabla_x(F_0 g_n) = -F_0\mathcal{L}_{F_0} g_{n+1} + F_0 \sum_{\substack{i+j=n+1\\ 1\leq i,j\leq n}} \mathcal{Q}_{F_0}(g_i, g_j)\,.$$

By Theorem 6.1 in chapter 1, section 6, one can define the pseudo-inverse $\mathcal{L}_{F_0}^{-1}$ of the linearized collision operator $\mathcal{L}_{F_0}$ on $(\ker \mathcal{L}_{F_0})^{\perp} \subset L^2(F_0 dv)$.

Once g_n is known, one computes g_{n+1} by solving the Fredholm integral equation

$$g_{n+1} = \Pi_{F_0} g_{n+1} + \mathcal{L}_{F_0}^{-1}\Bigg(-F_0^{-1}(\tau\partial_t(F_0 g_n) + v\cdot\nabla_x(F_0 g_n)) + \sum_{\substack{i+j=n+1\\ 1\le i,j\le n}} \mathcal{Q}_{F_0}(g_i,g_j)\Bigg) \tag{73}$$

where Π_{F_0} denotes the orthogonal projection on $\ker \mathcal{L}_{F_0}$ in $L^2(F_0 dv)$. The compatibility condition at order $n+1$ is therefore

$$\Bigg(F_0^{-1}(\tau\partial_t(F_0 g_n) + v\cdot\nabla_x(F_0 g_n)) - \sum_{\substack{i+j=n+1\\ 1\le i,j\le n}} \mathcal{Q}_{F_0}(g_i,g_j)\Bigg) \perp \ker(\mathcal{L}_{F_0})$$

for the inner product of $L^2(F_0 dv)$, in other words,

$$\tau\partial_t \int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_0 \Pi_{F_0} g_n dv + \mathrm{div}_x \int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} v F_0 g_n dv = 0\,. \tag{74}$$

This relation means that, at each order $n \ge 1$, the part $\Pi_{F_0} g_n$ of g_n that belongs to the nullspace of $\mathcal{L}_{F_0}$ — i.e. the hydrodynamic part of g_n — satisfies the linearized compressible Euler equations (with source terms depending on g_{n-j}, for $j = 1, \dots, n-1$).

4.3.2 - Chapman-Enskog's expansion

The expansion known today as Chapman-Enskog's expansion was found independently by Chapman (1916) and Enskog (1917). It is a variant of Hilbert's expansion, where the main idea is to collect all the contributions to the local thermodynamic equilibrium at leading order.

It takes the form

$$F_\varepsilon(t,x,v) = M_{F_\varepsilon}(t,x,v)\Bigg(1 + \sum_{n\ge 1} \varepsilon^n \tilde{g}_n(\varepsilon,t,x,v)\Bigg) \tag{75}$$

where M_{F_ε} is the Maxwellian with the same moments of order ≤ 2 as F_ε, i.e.

$$M_{F_\varepsilon}(t,x,v) = \mathcal{M}_{(\rho_\varepsilon(t,x), u_\varepsilon(t,x), \theta_\varepsilon(t,x))}\,, \tag{76}$$

with

$$\rho_\varepsilon = \int F_\varepsilon dv = \int M_{F_\varepsilon} dv, \quad \rho_\varepsilon u_\varepsilon = \int F_\varepsilon v dv = \int M_{F_\varepsilon} v dv\,,$$

$$\rho_\varepsilon(|u_\varepsilon|^2 + 3\theta_\varepsilon) = \int F_\varepsilon |v|^2 dv = \int M_{F_\varepsilon} |v|^2 dv\,.$$

The relative number density fluctuation of order ε^n, i.e. $\tilde{g}_n$ is of the form:

$$\tilde{g}_n(\varepsilon, t, x, v) = \gamma_n[\rho_\varepsilon(t,x), u_\varepsilon(t,x), \theta_\varepsilon(t,x), v]\,,$$

where the notation $\gamma_n[\rho_\varepsilon(t,x), u_\varepsilon(t,x), \theta_\varepsilon(t,x), v]$ designates a function of v that depends on (t,x) through $\rho_\varepsilon, u_\varepsilon$ and θ_ε and their successive partial x-derivatives evaluated at the point (t,x).

By definition of the Maxwellian M_{F_ε},

$$\int \gamma_n[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, v] M_{F_\varepsilon} dv = \int \gamma_n[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, v] |v|^2 M_{F_\varepsilon} dv = 0\,,$$

$$\int \gamma_n[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, v] v M_{F_\varepsilon} dv = 0\,.$$

At variance with Hilbert's expansion, the Chapman-Enskog ansatz requires knowing in advance that the successive corrections to within any order in ε to the compressible Euler system are systems of local conservation laws.

These conservation laws are truncations of the formal local conservation laws deduced from the local conservation of mass, momentum and energy that hold for (classical) solutions of the Boltzmann equation, and assume the form

$$\tau\partial_t \begin{pmatrix} \rho_\varepsilon \\ \rho_\varepsilon u_\varepsilon \\ \rho_\varepsilon(|u_\varepsilon|^2 + 3\theta_\varepsilon) \end{pmatrix} + \sum_{n\geq 0} \varepsilon^n \operatorname{div}_x \phi_n(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) = 0\,, \tag{77}$$

where the formal fluxes ϕ_n are defined by

$$\phi_n(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) = \int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \gamma_n[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, v] v M_{F_\varepsilon} dv\,.$$

At leading order, we obtain as usual the compressible Euler system in the form (72).

The first correction to the compressible Euler equations is then given by

$$\tau\partial_t M_{F_\varepsilon} + v\cdot\nabla_x M_{F_\varepsilon} = -M_{F_\varepsilon}\mathcal{L}_\varepsilon(\gamma_1[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, \cdot])$$

or equivalently

$$\begin{aligned} &\tau\partial_t\left(\log\rho_\varepsilon - \frac{3}{2}\log\theta_\varepsilon - \frac{1}{2\theta_\varepsilon}|v-u_\varepsilon|^2\right) \\ &\qquad + \operatorname{div}_x\left(v\log\rho_\varepsilon - \frac{3}{2}v\log\theta_\varepsilon - \frac{1}{2\theta_\varepsilon}v|v-u_\varepsilon|^2\right) \\ &\qquad = -\mathcal{L}_\varepsilon(\gamma_1[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, \cdot])\,, \end{aligned} \tag{78}$$

where $\mathcal{L}_\varepsilon$ denotes the linearization of the collision operator at the local Maxwellian state M_{F_ε}.

Using the compressible Euler system to eliminate time derivatives leads to

$$A\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) : D(u_\varepsilon) + 2B\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) \cdot \nabla_x \sqrt{\theta_\varepsilon} = -\mathcal{L}_\varepsilon(\gamma_1[\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, v]) \tag{79}$$

where $D(u)$ is the traceless part of the deformation tensor of u

$$D(u) = \frac{1}{2}(\nabla_x u + (\nabla_x u)^T) - \frac{1}{3}\mathrm{div}_x u\,,$$

while we recall from (21)

$$A(z) = z \otimes z - \frac{1}{3}|z|^2 Id\,, \quad B(z) = \frac{1}{2}z(|z|^2 - 5)\,.$$

By Theorem 6.1 in chapter 1, $\mathcal{L}_\varepsilon$ is a Fredholm operator, so that, in view of (22), equation (79) has a unique solution $\gamma_1 \perp \ker(\mathcal{L}_\varepsilon)$ (in the sense of $L^2(M_{F_\varepsilon}dv)$) whose expression is

$$\gamma_1(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) = -\tilde{A}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) : D(u_\varepsilon) - 2\tilde{B}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) \cdot \nabla_x \sqrt{\theta_\varepsilon}\,, \tag{80}$$

where the tensor field $\tilde{A}$ and the vector field $\tilde{B}$ are defined as in (23):

$$\tilde{A}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) = -\mathcal{L}_\varepsilon^{-1}\left(A\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right)\right),$$

$$\tilde{B}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) = -\mathcal{L}_\varepsilon^{-1}\left(B\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right)\right).$$

By Proposition 6.5 in chapter 1, there exist two scalar, radial functions α and β such that

$$\tilde{A}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) = \frac{1}{\rho_\varepsilon\sqrt{\theta_\varepsilon}}\alpha\left(\frac{|v-u_\varepsilon|}{\sqrt{\theta_\varepsilon}}\right)A\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right)$$

$$\tilde{B}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right) = \frac{1}{\rho_\varepsilon\sqrt{\theta_\varepsilon}}\beta\left(\frac{|v-u_\varepsilon|}{\sqrt{\theta_\varepsilon}}\right)B\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right).$$

Since the term γ_1 involves first derivatives of the moments ρ_ε, u_ε and θ_ε, second derivatives of these moments appear in the flux at the next to leading order in ε.

By truncating the formal conservation law (77) at the order $O(\varepsilon)$, one obtains the compressible Navier-Stokes system with $O(\varepsilon)$ dissipation terms:

$$\begin{aligned}&\partial_t\begin{pmatrix}\rho_\varepsilon\\ \rho_\varepsilon u_\varepsilon\\ \rho_\varepsilon\left(\frac{1}{2}|u_\varepsilon|^2+\frac{3}{2}\theta_\varepsilon\right)\end{pmatrix}\\ &+\operatorname{div}_x\begin{pmatrix}\rho_\varepsilon u_\varepsilon\\ \rho_\varepsilon u_\varepsilon^{\otimes 2}+\rho_\varepsilon\theta_\varepsilon Id-\varepsilon\mu_\varepsilon D(u_\varepsilon)\\ \rho_\varepsilon u_\varepsilon\left(\frac{1}{2}|u_\varepsilon|^2+\frac{5}{2}\theta_\varepsilon\right)-\varepsilon\kappa_\varepsilon\nabla_x\theta_\varepsilon-\varepsilon\mu_\varepsilon D(u_\varepsilon)\cdot u_\varepsilon\end{pmatrix}=0\,.\end{aligned}\tag{81}$$

The viscosity and heat conductivity are given by the formulas

$$\mu_\varepsilon=\frac{1}{10}\frac{\sqrt{\theta_\varepsilon}}{\rho_\varepsilon}\int A\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right):\tilde{A}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right)M_{F_\varepsilon}dv\,,$$

$$\kappa_\varepsilon=\frac{1}{3}\frac{\sqrt{\theta_\varepsilon}}{\rho_\varepsilon}\int B\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right)\cdot\tilde{B}\left(\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\right)M_{F_\varepsilon}dv\,,$$

which, after expressing the integrand in terms of the new variable

$$V=\frac{v-u_\varepsilon}{\sqrt{\theta_\varepsilon}}$$

reduce to

$$\begin{aligned}\mu_\varepsilon&=\frac{1}{10}\sqrt{\theta_\varepsilon}\int A(V):\tilde{A}(V)\mathcal{M}_{(1,0,1)}(V)dV\,,\\ \kappa_\varepsilon&=\frac{1}{3}\sqrt{\theta_\varepsilon}\int B(V)\cdot\tilde{B}(V)\mathcal{M}_{(1,0,1)}(V)dV\,.\end{aligned}\tag{82}$$

Since $\mathcal{L}_{M_{(1,0,1)}}$ is a nonnegative self-adjoint operator in $L^2(\mathcal{M}_{(1,0,1)}dv)$ and $\tilde{A}$ and $\tilde{B}\perp\ker\mathcal{L}_{M_{(1,0,1)}}$ in $L^2(\mathcal{M}_{(1,0,1)}dv)$, the integrals above are positive scalars.

Their expression in terms of the scalar functions a and β defined in Proposition 6.5 in chapter 1 is

$$\begin{aligned}\mu_\varepsilon&=\frac{2}{15}\sqrt{\theta_\varepsilon}\int_0^{+\infty}a(r)r^6e^{-\frac{1}{2}r^2}\frac{dr}{\sqrt{2\pi}}\,,\\ \kappa_\varepsilon&=\frac{1}{6}\sqrt{\theta_\varepsilon}\int_0^{+\infty}\beta(r)r^2(r^2-5)^2e^{-\frac{1}{2}r^2}\frac{dr}{\sqrt{2\pi}}\,.\end{aligned}\tag{83}$$

Notice the dependence of the viscosity and heat conductivity upon the temperature θ_ε; that these coefficients are proportional to the square-root of θ_ε is characteristic of the hard sphere gas.

There are further corrections to the compressible Euler and Navier-Stokes system, obtained by truncating the Chapman-Enskog expansion at order 2 or more. These corrections were proposed by D. Burnett (1935); however, the corresponding fluid dynamical models are in general ill-posed. Recently, D. Levermore and A. Bobylev have proposed modifications of the expansion method — more precisely, of the truncation algorithm — that would lead to well-posed hydrodynamic models; these recent developments have unfortunately not yet been published.

4.3.3 - Miscellaneous remarks on the Hilbert or Chapman-Enskog expansions

Even at formal level, both expansions are not equivalent.

Indeed the compressible Euler system is not known to be stable in L^∞ norm with respect to viscous perturbations. In particular, in the case of bounded domains, the set of boundary conditions adapted to the compressible Navier-Stokes system is not compatible with the Euler system (therefore leading to viscous boundary layers in the inviscid limit). This seems to indicate that one cannot expect the formal series to converge.

Although there is no difficulty in defining them, truncated expansions do not provide an entirely satisfying alternative to considering the complete power series since

-- they may not be nonnegative for all t, x and v; and

-- they fail to justify the hydrodynamic limit after the instant of time when singularities appear in the limiting solution; for instance, it is known that this first singular time is in general finite in the case of the compressible Euler system (see [**54**]).

However, many of the early mathematical justifications of hydrodynamic limits of the Boltzmann equation are based on truncated asymptotic expansions. For instance, R. Caflisch gave a rigorous justification of the compressible Euler limit up to the first singular time for the solution of the Euler system: see [**10**]. Later, M. Lachowicz [**35**] completed Caflisch's analysis by including initial layers in the truncated expansion, thereby dealing with more general initial data than in Caflisch's original paper. By the same method, A. DeMasi, R. Esposito and J. Lebowitz [**15**] justified the hydrodynamic limit of the Boltzmann equation leading to the incompressible Navier-Stokes equations. Like Caflisch's, their proof holds for as long as the solution of the Navier-Stokes equations is smooth; besides the solution of the Boltzmann equation so constructed that converges to a local equilibrium governed by the Navier-Stokes equation fails to be nonnegative. It could be that this problem can be solved by the same method as in Lachowicz's paper [**35**]; however, there is no written account of this so far.

4.4 - *Formal Incompressible Limits*

The compressible Euler sytem has been extensively studied by generations of mathematicians, and yet, at the time of this writing, there is no global existence result for initial data of arbitrary size. Probably the best result in that direction is the global existence of BV solutions in one space dimension, for initial data of small enough BV norm, a remarkable result by T.-P. Liu [**45**] based on Glimm's pioneering article [**21**].

More is known in the case of the compressible Navier-Stokes system; see in particular the lucid exposition of the subject by E. Feireisl [**20**]. But our discussion of the Chapman-Enskog expansion shows that the dissipation terms are small and of the same order as the Knudsen number order Kn in the compressible Navier-Stokes system derived from the Boltzmann equation. Therefore these dissipation terms vanish in the hydrodynamic limit. Since almost nothing is known about the uniformity of the solutions of the compressible Navier-Stokes system in the vanishing viscosity regime, the compressible Navier-Stokes system is not a realistic target for rigorous hydrodynamic limits.

As an alternative, we shall discuss hydrodynamic limits of the Boltzmann equation leading to incompressible fluid flows, about which much more is known. Specifically, we shall explain how

- the Stokes equations,
- the incompressible Euler equations, and
- the incompressible Navier-Stokes equations

can be derived from the Boltzmann equation. Our formal derivations of these hydrodynamic models follows the same strategy as the moment method of section 2, but involves computations similar to the treatment of the first order correction in the Chapman-Enskog expansion (especially in the computation of dissipation terms).

All hydrodynamic limits of the Boltzmann equation leading to incompressible fluid models start from the scaled Boltzmann equation (69), and use a scaling where both the Strouhal number τ and the Knudsen number ε are small. From now on, we choose ε as the master parameter, and let $\tau = \tau_\varepsilon$ depend on ε.

Furthermore, in all incompressible limits, the number density F will be sought in the form of a perturbation of some uniform Maxwellian state, which, by Galilean and scaling invariance can be chosen to be the centered, reduced Gaussian distribution

$$M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}|v|^2} .$$

In other words,

$$F_\varepsilon = M(1 + \delta_\varepsilon g_\varepsilon) \tag{84}$$

where $g_\varepsilon = O(1)$ in some norm, while δ_ε represents the order of magnitude of the Mach number Ma. To see that δ_ε really matches the usual notion of Mach number (i.e. the ratio of the velocity of the fluid to the speed of sound), consider the special case where

$$F_\varepsilon = \mathcal{M}_{(1,\delta_\varepsilon u,\theta)} \text{ with } u = O(1)\,. \tag{85}$$

By a Taylor expansion of the Maxwellian about $\mathcal{M}_{(1,0,\theta)}$, one finds that

$$\mathcal{M}_{(1,U,\theta)} = \mathcal{M}_{(1,0,\theta)}\left(1 + \frac{U\cdot v}{\theta}\right) + O(|U|^2/\theta)\,.$$

In other words, in (85),

$$\delta_\varepsilon \text{ is the size of } \frac{|U|}{\sqrt{\frac{5}{3}\theta}}$$

which is precisely the local Mach number. Indeed, we recall that in a perfect monatomic gas at temperature θ, the speed of sound is $\sqrt{\frac{5}{3}\theta}$.

In [4], one can find a systematic discussion of all possible incompressible hydrodynamic limits of the Boltzmann equation that can be attained by varying the parameters ε, δ_ε, and τ_ε. Specifically

- if $\varepsilon = \delta_\varepsilon = \tau_\varepsilon$, the Boltzmann equation converges to the incompressible Navier-Stokes equations;
- if $\varepsilon \ll \delta_\varepsilon = \tau_\varepsilon$, the Boltzmann equation converges to the incompressible Euler equations; and
- if $\delta_\varepsilon \ll \varepsilon = \tau_\varepsilon$, the Boltzmann equation converges to the Stokes equations.

The Stokes equations are

$$\begin{aligned} &\partial_t u + \nabla_x p = \mu \Delta_x u\,, \quad \operatorname{div}_x u = 0 \quad u(0,x) = u^{in}(x)\,,\\ &\tfrac{5}{2}\partial_t\theta = \kappa\Delta_x\theta\,, \qquad\qquad\qquad\quad \theta(0,x) = \theta^{in}(x)\,, \end{aligned} \tag{86}$$

where $\mu > 0$ is the kinematic viscosity and $\kappa > 0$ is the thermal diffusivity. Notice that the Stokes system is one of the simplest systems of fluid dynamical equations imaginable, being essentially a system of linear heat equations.

The Navier-Stokes equations — more precisely the Navier-Stokes-Fourier system is

$$\begin{aligned} &\partial_t u + (u\cdot\nabla_x)u + \nabla_x p = \mu\Delta_x u\,, \quad \operatorname{div}_x u = 0, \quad u(0,x) = u^{in}(x)\,,\\ &\tfrac{5}{2}\left(\partial_t\theta + u\cdot\nabla_x\theta\right) = \kappa\Delta_x\theta\,, \qquad\qquad\qquad \theta(0,x) = \theta^{in}(x)\,, \end{aligned} \tag{87}$$

where the kinematic viscosity μ and the thermal diffusivity κ have the same values as in the Stokes system.

The incompressible Euler equations are

$$\begin{aligned} &\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = 0\,, \quad \mathrm{div}_x u = 0, \quad u(0,x) = u^{in}(x)\,, \\ &\frac{5}{2}\big(\partial_t \theta + u \cdot \nabla_x \theta\big) = 0\,, \qquad\qquad \theta(0,x) = \theta^{in}(x)\,. \end{aligned} \tag{88}$$

The starting point in our analysis is the scaled Boltzmann equation written below in terms of the relative number density

$$G_\varepsilon = \frac{F_\varepsilon}{M}\,. \tag{89}$$

By using the notation

$$\mathcal{Q}(\phi,\psi) = M^{-1}\mathcal{B}(M\phi, M\psi)\,, \tag{90}$$

the scaled Boltzmann equation reads

$$\tau_\varepsilon \partial_t G_\varepsilon + v \cdot \nabla_x G_\varepsilon = \frac{1}{\varepsilon}\,\mathcal{Q}(G_\varepsilon, G_\varepsilon)\,, \qquad G_\varepsilon(0,x,v) = G_\varepsilon^{in}(x,v)\,. \tag{91}$$

In the sequel, the following notation for moments will be particularly convenient

$$\langle \phi \rangle = \int_{\boldsymbol{R}^3} \phi(v) M(v) dv\,.$$

With this notation for moments, the local conservation laws are written in terms of the relative number density fluctuation g_ε defined by (84) as

$$\begin{aligned} \tau_\varepsilon \partial_t \langle g_\varepsilon \rangle + \mathrm{div}_x \langle v g_\varepsilon \rangle &= 0\,, \\ \tau_\varepsilon \partial_t \langle v g_\varepsilon \rangle + \mathrm{div}_x \langle v \otimes v g_\varepsilon \rangle &= 0\,, \\ \tau_\varepsilon \partial_t \Big\langle \frac{1}{2}|v|^2 g_\varepsilon \Big\rangle + \mathrm{div}_x \Big\langle v \frac{1}{2}|v|^2 g_\varepsilon \Big\rangle &= 0\,. \end{aligned} \tag{92}$$

Theorem 4.1 (Bardos-Golse-Levermore [3], [4]). *Let G_ε be a family of distribution solutions of the scaled Boltzmann initial-value problem* (91) *posed on $\boldsymbol{R}_x^3 \times \boldsymbol{R}_v^3$ with initial data G_ε^{in} that satisfies the initial relative entropy constraint*

$$H(MG_\varepsilon^{in}|M) \le C^{in}\delta_\varepsilon^2\,.$$

Let $G_\varepsilon^{in} = 1 + \delta_\varepsilon g_\varepsilon^{in}$ and $G_\varepsilon = 1 + \delta_\varepsilon g_\varepsilon$ where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, and where the fluctuations g_ε^{in} and g_ε are bounded in $L^\infty(dt; L^2(Mdvdx))$.

Moreover:

(1) *Assume that in the sense of distributions the family* g_ε^{in} *satisfies*

$$\lim_{\varepsilon\to 0}\Big(P\langle v g_\varepsilon^{in}\rangle, \Big\langle \Big(\frac{1}{5}|v|^2 - 1\Big) g_\varepsilon^{in}\Big\rangle\Big) = (u^{in}, \theta^{in})$$

for some $(u^{in}, \theta^{in}) \in L^2(dx; \boldsymbol{R}^3 \times \boldsymbol{R})$, *where* P *denotes the Leray projection onto divergence-free vector fields;*

(2) *Assume that the local conservation laws* (92) *are also satisfied in the sense of distributions for every* g_ε*;*

(3) *Assume that the family* g_ε *converges in the sense of distributions as* $\varepsilon \to 0$ *to* $g \in L^\infty(dt; L^2(Mdvdx))$. *Furthermore, assume that* $\mathcal{L}g_\varepsilon \to \mathcal{L}g$, *while for each* $\xi \in L^2(Mdv)$ *the moments* $\langle \xi g_\varepsilon\rangle$ *converge to* $\langle \xi g\rangle$, *and that every formally small term vanishes, all in the sense of distributions as* $\varepsilon \to 0$.

Then g *is the unique local infinitesimal Maxwellian*

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}(|v|^2 - 5) \tag{93}$$

determined by the solution (u, θ) *of*

- *the Stokes system* (86) *when* $\delta_\varepsilon = o(\varepsilon)$ *and* $\tau_\varepsilon = \varepsilon$,
- *the Navier-Stokes system* (87) *when* $\delta_\varepsilon/\varepsilon \to 1$ *(or any finite value) and* $\tau_\varepsilon = \varepsilon$,
- *the Euler system* (88) *when* $\delta_\varepsilon/\tau_\varepsilon \to 1$ *(or any finite value) and* $\varepsilon = o(\tau_\varepsilon)$,

with initial data (u^{in}, θ^{in}). *In the viscous systems* (86) *and* (87), μ *and* κ *are given by formulas* (83) *with* $\theta_\varepsilon = 1$.

We shall sketch below the formal derivation of the Navier-Stokes system, which has all the possible terms (convection and diffusion) at the leading order. The formal derivations of the Euler or Stokes systems are similar.

Proof. In terms of g_ε, the Boltzmann equation (91) becomes

$$\varepsilon\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon}\mathcal{L}g_\varepsilon = \mathcal{Q}(g_\varepsilon, g_\varepsilon)\,, \tag{94}$$

where $\mathcal{L}$ is the linearization of Boltzmann's collision integral at the Maxwellian state M, i.e.

$$\mathcal{L}\phi = -2M^{-1}\mathcal{B}(M, M\phi)\,.$$

Here we have used the fact that $\delta_\varepsilon = \tau_\varepsilon = \varepsilon$.

<u>Step 1: Asymptotic fluctuations</u>

First, we seek the asymptotic form of the number density fluctuations g_ε in the vanishing ε limit.

Multiplying the Boltzmann equation (94) by ε and letting $\varepsilon \to 0$ suggests that

$$g_\varepsilon \to g \text{ in the sense of distributions on } \boldsymbol{R}_+ \times \boldsymbol{R}^3 \times \boldsymbol{R}^3 \text{ with } \mathcal{L}g = 0\,.$$

By Theorem 6.3 of chapter 1, g is an *infinitesimal Maxwellian*, i.e., is of the form

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - 3)\,. \tag{95}$$

Notice that g is parametrized by its own moments, since

$$\rho = \langle g\rangle\,, \quad u = \langle vg\rangle\,, \quad \text{and } \theta = \left\langle \left(\frac{1}{3}|v|^2 - 1\right) g\right\rangle .$$

<u>Step 2: Incompressibility and Boussinesq's relations</u>

Passing to the limit in the sense of distributions in the continuity equation, i.e. the first equality in (92), we obtain

$$\mathrm{div}_x\langle vg\rangle = 0\,, \quad \text{or equivalently } \mathrm{div}_x\, u = 0\,, \tag{96}$$

which is the incompressibility condition in the Navier-Stokes equations.

Passing to the limit in the sense of distributions in the momentum equation, i.e. the second equality in (92), we obtain

$$\mathrm{div}_x\langle v\otimes vg\rangle = 0\,, \quad \text{or equivalently } \nabla_x(\rho+\theta) = 0\,.$$

Since $g \in L^\infty(dt; L^2(Mdvdx))$, $\rho+\theta \in L^\infty(dt; L^2(dx))$; the only a.e. constant function in $L^2(\boldsymbol{R}^3)$ being 0, one finds that

$$\rho + \theta = 0\,. \tag{97}$$

This is sometimes called the Boussinesq relation (although this terminology usually refers to the Navier-Stokes equation for liquids). Together with (95), it implies that g is of the form (93)

<u>Step 3: The motion and heat equations</u>

The momentum equation in (92) is recast as

$$\partial_t\langle vg_\varepsilon\rangle + \mathrm{div}_x \frac{1}{\varepsilon}\langle A(v)g_\varepsilon\rangle + \nabla_x\frac{1}{\varepsilon}\left\langle \frac{1}{3}|v|^2 g_\varepsilon\right\rangle = 0\,. \tag{98}$$

Since $\mathcal{L}$ is self-adjoint on $L^2(Mdv)$,

$$\begin{aligned}\frac{1}{\varepsilon}\langle A(v)g_\varepsilon\rangle &= \frac{1}{\varepsilon}\langle(\mathcal{L}\tilde{A})(v)g_\varepsilon\rangle\\ &= \left\langle \tilde{A}(v)\frac{1}{\varepsilon}\mathcal{L}g_\varepsilon\right\rangle = \langle\tilde{A}\,\mathcal{Q}(g_\varepsilon,g_\varepsilon)\rangle - \langle\tilde{A}(\varepsilon\partial_t + v\cdot\nabla_x)g_\varepsilon\rangle\,.\end{aligned} \tag{99}$$

Let Π be the orthogonal projection on $\ker\mathcal{L}$ in $L^2(\boldsymbol{R}^3; Mdv)$: for each

$\phi \in L^2(\boldsymbol{R}^3; Mdv)$, one has

$$\Pi\phi = \langle\phi\rangle + v\cdot\langle v\phi\rangle + \frac{1}{2}(|v|^2-3)\left\langle\left(\frac{1}{3}|v|^2-1\right)\phi\right\rangle .$$

Because of step 1, one expects that g_ε can be replaced by Πg_ε as $\varepsilon\to 0$ in the right-hand side of (99). Hence

$$\begin{aligned}\frac{1}{\varepsilon}\langle A(v)g_\varepsilon\rangle &\simeq \langle\tilde{A}\mathcal{Q}(\Pi g_\varepsilon,\Pi g_\varepsilon)\rangle - \langle\tilde{A}v\cdot\nabla_x\Pi g_\varepsilon\rangle\\ &= \langle\tilde{A}\mathcal{Q}(\Pi g_\varepsilon,\Pi g_\varepsilon)\rangle - \langle\tilde{A}\otimes A\rangle : \nabla_x\langle vg_\varepsilon\rangle\end{aligned}$$

in some sense as $\varepsilon\to 0$. The contraction in the last term of the right-hand side of the equality above bears on the indices of A and $\nabla_x\langle vg_\varepsilon\rangle$; in other words, with the convention of repeated indices,

$$(\langle\tilde{A}\otimes A\rangle : \nabla_x\langle vg_\varepsilon\rangle)_{ij} = \langle\tilde{A}_{ij}A_{kl}\rangle\partial_{x_k}\langle v_l g_\varepsilon\rangle .$$

The nonlinear term is simplified as follows.

Lemma 4.2. *For each* $\phi\in\ker\mathcal{L}$, *one has*

$$\mathcal{Q}(\phi,\phi) = \frac{1}{2}\mathcal{L}(\phi^2) .$$

Proof. Differentiate twice the relation

$$\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)},\mathcal{M}_{(\rho,u,\theta)}) = 0$$

with respect to the parameters ρ, u and θ. See [4] for a complete argument. □

Eventually, we arrive at the formula

$$\begin{aligned}\frac{1}{\varepsilon}\langle A(v)g_\varepsilon\rangle &\simeq \frac{1}{2}\langle\tilde{A}\mathcal{L}((\Pi g_\varepsilon)^2)\rangle - \langle\tilde{A}\otimes A\rangle : \nabla_x\langle vg_\varepsilon\rangle\\ &= \frac{1}{2}\langle A|\Pi g_\varepsilon|^2\rangle - \langle\tilde{A}\otimes A\rangle : \nabla_x\langle vg_\varepsilon\rangle\\ &= \langle vg_\varepsilon\rangle\otimes\langle vg_\varepsilon\rangle - \frac{1}{3}|\langle vg_\varepsilon\rangle|^2 I - \mu D(\langle vg_\varepsilon\rangle) ,\end{aligned} \tag{100}$$

where

$$\mu = \frac{1}{10}\langle\tilde{A}:A\rangle$$

and, for each vector field $\xi\equiv\xi(x)\in\boldsymbol{R}^3$

$$D(\xi) = \nabla_x\xi + (\nabla_x\xi)^T - \frac{2}{3}(\mathrm{div}_x\xi)I .$$

Substituting the formula (100) for the momentum flux in (98), and taking into account the incompressibility condition (96), we arrive at the motion equation in the Navier-Stokes system (87).

The heat equation is obtained in a very similar way, starting from the relation

$$\partial_t\left\langle\frac{1}{2}(|v|^2-5)\,g_\varepsilon\right\rangle+\mathrm{div}_x\frac{1}{\varepsilon}\langle B(v)\,g_\varepsilon\rangle=0\,. \tag{101}$$

□

5 - Mathematical Tools for the Incompressible Hydrodynamic Limits of the Boltzmann Equation

In the present chapter, we shall discuss various mathematical approaches to the problem of obtaining rigorous justifications of the incompressible hydrodynamic limits of the Boltzmann equation for a hard sphere gas.

More precisely, we shall first discuss an approach of the incompressible limits based on the spectral analysis of the Boltzmann equation linearized about the uniform Maxwellian equilibrium that sets the scale of the speed of sound.

In a second part, we present several uniform a priori estimates resulting from Boltzmann's H Theorem; these estimates will be used in the last two chapters of this survey, devoted respectively to the incompressible Euler and Navier-Stokes limits of the Boltzmann equation.

5.1 - *Spectral analysis of the linearized Boltzmann equation*

Because all incompressible limits are in particular small Mach number limits, they correspond to weakly nonlinear regimes. In such a context, one expects that the linearization of the Boltzmann equation about the uniform background Maxwellian state that defines the speed of sound at leading order should play an important role.

In particular in order to obtain some convergence rate measuring the accuracy of the hydrodynamic limit, one must understand how the free-transport operator $v\cdot\nabla_x$ modifies the properties of $\mathcal{L}_M$.

We start with a simple property which will not be used as it is, but helps understanding a crucial mechanism that occurs in fluid regimes, namely the transfer of regularity from the v variable to the x variable.

Proposition 1.1. *Assume that the collision kernel is that of a hard sphere gas. Let T be the unbounded operator from $L^2(\boldsymbol{T}^3_x\times\boldsymbol{R}^3_v;(1+|v|)Mdxdv)$ to $L^2(\boldsymbol{T}^3_x\times\boldsymbol{R}^3_v;(1+|v|)^{-1}Mdxdv)$ defined by*

$$T=v\cdot\nabla_x+\mathcal{L}_M\,,$$

with domain

$$\mathcal{D}(T) = \{\phi \in L^2(\boldsymbol{T}^3_x \times \boldsymbol{R}^3_v; (1+|v|)Mdxdv) \,|\, v \cdot \nabla_x \phi \in L^2(\boldsymbol{T}^3_x \times \boldsymbol{R}^3_v; (1+|v|)^{-1}Mdxdv)\}\,.$$

Then the nullspace of T is

$$\ker T = \mathrm{span}_{\boldsymbol{R}}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

Furthermore the operator T is coercive on $(\ker T)^\perp$*: there exists $C > 0$ such that*

$$\|T\phi\|_{L^2((1+|v|)^{-1}Mdvdx)} \geq C\|\phi\|_{L^2((1+|v|)Mdvdx)}\,,$$

whenever $\phi \in \mathcal{D}(T)$ satisfies

$$\iint \phi \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} Mdvdx = 0\,.$$

Proof. Let ϕ be any element of $\ker T$. Then,

$$\iint \phi \mathcal{L}_M \phi Mdxdv = \iint \phi T \phi Mdxdv = 0\,,$$

from which we deduce by the relative coercivity of $\mathcal{L}_M$ (see Theorem 6.3 in chapter 1) that

$$\phi(x,\cdot) \in \ker\mathcal{L}_M \text{ for almost every } x \in \boldsymbol{T}^3\,.$$

In other words, there exist some functions ρ, u, and $\theta \in L^2(\boldsymbol{T}^3)$ such that

$$\phi(x,v) = \rho(x) + u(x)\cdot v + \theta(x)\frac{1}{2}(|v|^2 - 3)\,.$$

That $T\phi = 0$ is then equivalent to

$$v \cdot \nabla_x \left(\rho(x) + u(x)\cdot v + \theta(x)\frac{1}{2}(|v|^2-3)\right) = 0\,,$$

from which we deduce that the functions ρ, u, and θ are necessarily independent of x.

As for the second statement, observe that

$$T = \upsilon(|v|) + v\cdot\nabla_x - \mathcal{K}$$

and that $\mathcal{K}$ is $(\upsilon(|v|) + v\cdot\nabla_x)$-compact on $L^2(\upsilon Mdvdx)$ by velocity averaging (see chapter 2). Besides, $(\upsilon(|v|) + v\cdot\nabla_x)$ is closed and bounded-invertible from $L^2(\upsilon Mdvdx)$ to $L^2(\upsilon^{-1}Mdvdx)$. Thus T is a Fredholm operator from $L^2(\boldsymbol{T}^3_x \times \boldsymbol{R}^3_v; (1+|v|)Mdxdv)$ to $L^2(\boldsymbol{T}^3_x \times \boldsymbol{R}^3_v; (1+|v|)^{-1}Mdxdv)$, and therefore is coercive on $(\ker T)^\perp$. □

Of course the previous proposition does not suffice to obtain a precise description of the asymptotic behavior of solutions to the Boltzmann equation in the hydrodynamic limit. This requires refined results on the spectral representation of T, which give in particular the dependence of the first eigenmodes and eigenprojections with respect to the spatial scale.

Theorem 1.2 (Ellis & Pinsky [**19**]]). *Assume that the collision kernel is that of a hard sphere gas, and let*

$$T = v \cdot \nabla_x + \mathcal{L}_M$$

be the linearized Boltzmann operator. Let $\hat{T}(\xi)$ be its Fourier representation

$$\hat{T}(\xi) = iv \cdot \xi + \mathcal{L}_M \,.$$

Denote by $\hat{U}(t,\xi)$ the semi-group generated by $\hat{T}(\xi)$.

Then

$$\hat{U}(t,\xi) = \sum_{j=1}^{4} \mathbf{1}_{|\xi|\leq\kappa} e^{t\lambda_j(\xi)} P_j(\xi) + R(t,\xi)$$

where $\lambda_j(\xi)$ are eigenvalues of $\hat{B}(\xi)$ with eigenprojections $P_j(\xi)$, three of which are simple and one of which has multiplicity 2.

These eigenvalues have the following asymptotic expansion

$$\lambda_j(\xi) = ia_j|\xi| - \beta_j|\xi|^2 + O(|\xi|^3) \text{ as } |\xi| \to 0$$

where

$$\beta_j > 0\,, \quad |\lambda_j(\xi) - ia_j|\xi| + \beta_j|\xi|^2| \leq \beta_j|\xi|/2 \text{ for } |\xi| \leq \kappa\,,$$

while

$$P_j(\xi) = P_{0j}(\xi/|\xi|) + |\xi| P_{1j}(\xi/|\xi|) + |\xi|^2 P_{2j}(\xi)\,.$$

Moreover

$$\|R(t,\xi)\| \leq C \exp(-\sigma t) \textit{ for some positive } \sigma \textit{ and } C\,.$$

The proof of such a theorem is extremly technical, and is based on an argument of analytic continuation in the ξ variable.

In other words, while the nature of the spectrum and the spectral gap are given by velocity averaging plus an abstract argument, the precise study of the analytic continuation of the 5-uple eigenvalue 0 for $\xi = 0$ requires precise asymptotic evaluations. As we shall see below, knowing the first coefficients in the asymptotic expansion of these eigenvalues at large scale (i.e. for small ξ) is of paramount importance in the justification of the hydrodynamic limit of the Boltzmann equation.

5.2 - *Applications to hydrodynamic limits of the spectral theory of the linearized Boltzmann equation*

The first mathematical proof of the compressible Euler limit of the Boltzmann equation was proposed by Nishida [**47**]; his argument used the above description of the spectrum of the linearized Boltzmann equation by Ellis and Pinsky, together with an abstract variant of the Cauchy-Kovalevski theorem due to Nirenberg and Ovsyannikov [**48**]. We shall not describe his result in this course. Rather, we shall explain how the spectral analysis of the linearized Boltzmann equation «à la Ellis-Pinsky» allows constructing global solutions of the (nonlinear) Boltzmann equation, with estimates that are uniform in the incompressible Navier-Stokes limit. Although in the same spirit as Nishida's, this result, due to C. Bardos and S. Ukai [**6**], puts less severe restrictions on the regularity of the hydrodynamic solutions. Indeed, Nishida's analysis considered analytic solutions of the compressible Euler system, and therefore was only local in time; on the contrary, the work of Bardos and Ukai considered global solutions of the Navier-Stokes equations, corresponding to initial velocity fields that are «small» in some appropriate Sobolev norm.

5.2.1 - Notation and background

We therefore consider the scaled Boltzmann equation for a hard sphere gas with Strouhal and Knudsen numbers satisfying

$$\mathrm{St} = \mathrm{Kn} = \varepsilon \ll 1\,.$$

We are concerned with solutions of that scaled Boltzmann equation in the vicinity of the uniform Maxwellian state

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}|v|^2\right)$$

(in the notation of chapter 1, section 3). It will be convenient to use the relative number density and relative number density fluctuation

$$G_\varepsilon = \frac{F_\varepsilon}{M} \text{ and } g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}\,,$$

instead of the number density F_ε, solution of the scaled Boltzmann equation. In terms of g_ε, the scaled Boltzmann equation takes the form

$$\varepsilon\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon}\mathcal{L}g_\varepsilon = \mathcal{Q}(g_\varepsilon, g_\varepsilon) \tag{102}$$

where $\mathcal{L}$ and $\mathcal{Q}$ denote respectively the operators defined by

$$\mathcal{L}g = -2M^{-1}\mathcal{B}(M, Mg)\,,\quad \mathcal{Q}(g,g) = M^{-1}\mathcal{B}(Mg, Mg)\,. \tag{103}$$

The formal analysis of the incompressible Navier-Stokes limit in chapter 4, section 4.4 suggests that g_ε should converge to some infinitesimal Maxwellian

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}(|v|^2 - 5)\,,$$

whose moments u and θ satisfy the Navier-Stokes-Fourier system.

5.2.2 - Application to the incompressible Navier-Stokes limit

Before stating the Bardos-Ukai theorem, which is the main result in this section, we need a few notations regarding functional spaces. We shall use the spaces

$$L^{\infty,\beta} = \{g \equiv g(v)\,|\, \sup_v\,(1+|v|^\beta)|M^{1/2}g(v)| < +\infty\}\,,$$

$$H_{l,\beta} = \{g \equiv g(x,v)\,|\, \|g\|_{l,\beta} = \sup_v (1+|v|^\beta)\|M^{1/2}g(\cdot,v)\|_{H^l_x} < +\infty\}\,,$$

and

$$X_{l,\beta} = C(\boldsymbol{R}_+; H_{l,\beta}) \cap L^\infty(\boldsymbol{R}_+; H_{l,\beta})\,,$$

where the subscripts l and β will be chosen so as to ensure that $\mathcal{Q}$ maps $H_{l,\beta}\times H_{l,\beta}$ into $H_{l,\beta}$. Note that this property is a stronger constraint for $H_{l,\beta}$ than that of being an algebra because of the weight $|v-v^*|$ in the collision integrand.

Theorem 2.1 (Bardos & Ukai [**6**]) *Assume that the collision kernel is that of a hard sphere gas. Let $g_0 \in H_{l,\beta}$ for $l > 3/2$ and $\beta > 5/2$ such that*

$$\|g_0\|_{l,\beta} \le a_0 \tag{104}$$

for some a_0 sufficiently small.

Then, for any $\varepsilon \in (0,1]$ there exists a unique global solution $g_\varepsilon \in X_{l,\beta}$ to the scaled Boltzmann equation (102)-(103) *with initial data*

$$g_\varepsilon\big|_{t=0} = g_0\,.$$

Moreover, the following convergences hold as $\varepsilon \to 0$:

$$g_\varepsilon \rightharpoonup g \text{ weakly} * \text{ in } L^\infty(\boldsymbol{R}_+; H_{l,\beta})\,,$$
$$g_\varepsilon \to g \text{ strongly in } C_{loc}(\boldsymbol{R}_+\times \boldsymbol{R}^3; L^{\infty,\beta})\,.$$

And the limit point g is the infinitesimal Maxwellian

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}(|v|^2 - 5)\,,$$

whose moments u and θ are the unique solution to the incompressible Navier-

Stokes-Fourier equations

$$\begin{aligned} \partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p &= \mu \Delta_x u\,, \quad \mathrm{div}_x u = 0\,,\\ \partial_t \theta + u \cdot \nabla_x \theta &= \kappa \Delta_x \theta\,,\\ u\big|_{t=0} &= P u_0\,,\\ \theta\big|_{t=0} &= \frac{3}{5}\theta_0 - \frac{2}{5}\rho_0 \end{aligned}$$

where P designates the Leray projection on divergence free vector-fields.

Before discussing the proof, let us start with a few comments on this result.

The first remark is about the specificity of the incompressible Navier-Stokes asymptotics : the formal study in chapter 4, section 4 shows that this is the only macroscopic limit corresponding to some finite Reynolds number regime, and therefore the only case where global solutions are known to exist for the limiting system. The perturbative method presented here uses the existence of classical solutions in the Sobolev space H^l for $l > \frac{3}{2}$ with initial data small enough.

The main idea by Ukai [**57**] is to prove that a similar theory holds for the scaled Boltzmann equation in incompressible viscous regime: it has indeed global classical solutions in the weighted Sobolev space $H_{l,\beta}$ for $l > 3/2, \beta > 5/2$ provided that initial data are small enough. If, in addition $g_0 \in L^1(L^2(Mdv))$, the solution also satisfies a dispersion estimate

$$\|g_\varepsilon(t)\|_{l,\beta} \leq a_1(1+t)^{-3/4}(\|g_0\|_{l,\beta} + \|g_0\|_{L^1_x(L^2)})\,.$$

The derivation of the Navier-Stokes limit for the Boltzmann equation [**6**] relies on a rigorous proof of the relation between these two theories. The point to be stressed is that exactly the same type of assumptions are made on the initial data. The Bardos-Ukai result results from sharp bounds on the linearized operator.

The main restrictions in such a result are therefore the regularity and the smallness conditions on the initial data. Such assumptions are not expected to be necessary, working with Leray solutions of the incompressible Navier-Stokes equations and with renormalized solutions to the scaled Botzmann equation. The particularity of both types of solutions is to depend only on estimates with intrinsic physical meaning. The analogy between both types of solutions is the key point in the compactness method to be presented in the last chapter.

The last remark bears on the breakdown of the uniform convergence near $t = 0$, which corresponds to the appearance of an initial layer associated with the scaled Boltzmann equation. The necessary and sufficient condition for the uniform convergence up to $t = 0$ is that the initial g_0 is well-prepared, meaning that it is of the form

$$g_0(v) = u_0 \cdot v + \frac{1}{2}\theta_0(|v|^2 - 5) \text{ with } \mathrm{div}_x u_0 = 0\,.$$

The profile condition in the v-variable ensures indeed that the gas is initially in a state of local thermodynamic equilibrium and therefore that no further relaxation is to be expected. On the other hand, as the constraints on the macroscopic parameters, i.e. the Boussinesq and incompressibility relations, are satisfied initially, acoustic waves are not expected to arise in the system, and to disturb the convergence process in the initial layer before being dispersed.

5.2.3 - Proof of the Bardos-Ukai Theorem

We are going to describe the main ideas in the Bardos-Ukai analysis; however, we shall not give all the details since the proof is extremely technical. The general strategy is as follows:

– as a first step, one constructs global smooth solutions to the scaled Boltzmann equation (102) under a uniform smallness condition, which is done by a standard fixed point argument, coupled with uniform continuity estimates in $H_{l,\beta}$ for $\mathcal{L}$ and $\mathcal{Q}$;

– the second step is the proof of the convergence towards the solution of the incompressible Navier-Stokes equations, by using estimates on the spectral gap of the linearized collision operator perturbed by the free transport, and convenient approximations for the semi-group generated by the part coming from the zero-eigenvalue. This second part is more complicated than the first one since it requires obtaining higher order approximations, but the strategy is quite similar.

5.2.3.1 - Uniform regularity estimates

In order to prove the global existence of a unique strong solution to the scaled Boltzmann equation, we use a fairly standard method based on the Duhamel formula and on the Picard fixed point theorem.

Denote by U_ε the semi-group generated by

$$T_\varepsilon = \frac{1}{\varepsilon^2}(\varepsilon v \cdot \nabla_x + \mathcal{L})\,. \tag{105}$$

The scaled Boltzmann equation can be reduced to the integral equation

$$g_\varepsilon = N_\varepsilon[g_\varepsilon], \tag{106}$$

where the functional N_ε is defined by

$$
\begin{aligned}
&N_\varepsilon[g](t) = U_\varepsilon(t)g_0 + \psi_\varepsilon[g,g](t)\,,\\
&\psi_\varepsilon[g,g](t) = \frac{1}{\varepsilon}\int_0^t U_\varepsilon(t-s)\mathcal{Q}(g(s),g(s))ds\,.
\end{aligned}
\tag{107}
$$

The global well-posedness of the Cauchy problem for (102) is established by proving that N_ε is a contraction in a ball of $X_{l,\beta}$.

Furthermore the smallness condition is uniform with respect to ε if the radius of this ball is independent of ε. The crucial point is therefore to obtain uniform continuity estimates on U_ε and ψ_ε in $X_{l,\beta}$.

The uniform continuity of the linear semi-group U_ε is given by the following Lemma:

Lemma 2.2. *Denote by U_ε the semi-group generated by the operator T_ε defined by* (105).

Then, if $\beta > \frac{3}{2}$, there exists a nonnegative constant C_1 such that

$$\|U_\varepsilon(t)g_0\|_{l,\beta} \le C_1\|g_0\|_{l,\beta}\,.$$

Sketch of the proof. The continuity property of U_ε is obtained using its spectral representation and the spectral estimates stated in Theorem 1.2. A scaling argument gives indeed

$$\hat{U}_\varepsilon(t,\xi) = \hat{U}\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right)$$

which, together with the spectral estimates on U, leads to

$$\|\hat{U}_\varepsilon(t,\xi)\| \le C\left(\exp\left(-\frac{1}{2}(\min_j \beta_j)|\xi|^2 t\right) + \exp\left(-\frac{1}{\varepsilon^2}\sigma t\right)\right). \tag{108}$$

This first estimate provides the uniform continuity of U_ε in $H^l_x(L^2_v)$.

Notice that one should get a more precise estimate, splitting the part corresponding to the zero-eigenvalue. Indeed, as

$$\Pi = \sum_{j=1}^{4} P_{0j}(\xi/|\xi|)$$

is the orthogonal projection on $\ker(\mathcal{L})$, and in particular as Π does not depend on ξ

$$\left\|\frac{1}{\varepsilon}\hat{U}_\varepsilon(Id-\Pi)(t,\xi)\right\| \le Ct^{-1/2}\left(\exp\left(-\frac{1}{2}(\min_j \beta_j)|\xi|^2 t\right) + \exp\left(-\frac{1}{\varepsilon^2}\sigma t\right)\right), \tag{109}$$

using standard estimates on $z \mapsto z\exp(-z^2)$.

In order to obtain refined estimates, and especially to gain integrability with respect to the v-variable, one has to use more about the structure of the linearized collision operator. Indeed, we have seen in the first chapter that $\mathcal{L}$ can be split as

$$\mathcal{L} = \upsilon - \mathcal{K}$$

where the frequency part satisfies the lower bound (Theorem 6.1)

$$\upsilon(|v|) \geq \upsilon_0(1 + |v|)\,,$$

while the integral part $\mathcal{K}$ improves integrability in the v variable (Theorem 6.2):

$$\mathcal{K} :\ H^l_x(L^2_v) \to H_{l,0}\,,$$

and

$$\mathcal{K} :\ H_{l,\beta} \to H_{l,\beta+1}\,.$$

From the explicit formula for the semi-group $\bar{U}_\varepsilon$ generated by $(v \cdot \nabla_x + \upsilon)$ and Duhamel's formula

$$U_\varepsilon(t) = \bar{U}_\varepsilon(t) + \int_0^t \bar{U}_\varepsilon(t-s)\mathcal{K}U_\varepsilon(s)ds\,,$$

we deduce that

$$\|U_\varepsilon(t)u\|_Y \leq \exp\left(-\frac{1}{\varepsilon^2}\upsilon_0 t\right)\|u\|_Y + C_{X\to Y}\varepsilon^{-2}\int_0^t \exp\left(-\frac{1}{\varepsilon^2}\upsilon_0(t-s)\right)\|U_\varepsilon(s)u\|_X ds \tag{110}$$

where $\mathcal{K}$ maps X into Y.

Starting from the spectral estimate (108) on U_ε, and iterating on the inequality (110) gives finally

$$\|U_\varepsilon(t)g_0\|_{l,\beta} \leq C_1\|g_0\|_{l,\beta}\,,$$

provided that $H_{l,\beta} \subset H^l_x(L^2_v)$, that is for $\beta > \dfrac{3}{2}$. □

The uniform continuity of the bilinear operator ψ_ε is obtained in a very similar way.

Lemma 2.3. *Denote by ψ_ε the symmetric bilinear operator defined by* (107). *Then, if $\beta > \dfrac{5}{2}$ and $l > \dfrac{3}{2}$, there exists a nonnegative constant C_2 such that*

$$\|\psi_\varepsilon[g,h]\|_{l,\beta} \leq C_2\|g\|_{l,\beta}\|h\|_{l,\beta}\,.$$

Proof. Since 1, v_j for $j = 1, 2, 3$ and $|v|^2$ are collision invariants,

$$\Pi\mathcal{Q}(g,h) = 0 \text{ for all } g, h \in L^2(\nu M(v)dv)\,.$$

Then

$$\begin{aligned}\psi_\varepsilon[g,h](t) &= \int_0^t \frac{1}{\varepsilon} U_\varepsilon(t-s)(I-\Pi)\mathcal{Q}[g,h](s)ds \\ &= \phi_\varepsilon\left(\frac{\mathcal{Q}[g,h]}{\nu}\right).\end{aligned}$$

Standard continuity estimates for $\mathcal{Q}$ shows that

$$\|\nu^{-1}\mathcal{Q}(g,h)\|_{l,\beta} + \|\mathcal{Q}(g,h)\|_{L^1_x(L^2_v)} \le C\|g\|_{l,\beta}\|h\|_{l,\beta} \tag{111}$$

for $\beta > 3/2$ and $l > 3/2$.

It remains to get a uniform continuity estimate on the linear operator ϕ_ε. The spectral estimate (109) on $\frac{1}{\varepsilon}U_\varepsilon(I - P_0)$ implies that

$$\left\|\frac{1}{\varepsilon}U_\varepsilon(t)(I-\Pi)q\right\|_{H^l_x(L^2_v)} \le Ct^{-1/2}(1+t)^{-3/4}\big(\|q\|_{H^l_x(L^2_v)} + \|q\|_{L^1_x(L^2_v)}\big)\,,$$

which allows initializing some iterating process as for U_ε:

$$\|\phi_\varepsilon(q)\|_{H^l_x(L^2_v)} \le C\big(\|\nu q\|_{H^l_x(L^2_v)} + \|\nu q\|_{L^1_x(L^2_v)}\big)\,. \tag{112}$$

By using Hilbert's decomposition of $\mathcal{L}$, we gain integrability in the v variable:

$$\phi_\varepsilon(q)(t) = \frac{1}{\varepsilon}\int_0^t \bar{U}_\varepsilon(I-P_0)\nu q(s)ds + \frac{1}{\varepsilon^2}\int_0^t \bar{U}_\varepsilon(t-s)\mathcal{K}\phi_\varepsilon q(s)ds$$

thus

$$\|\phi_\varepsilon(q)(t)\|_Y \le C\varepsilon\|\nu q\|_Y + C_{X\to Y}\int_0^t \|\phi_\varepsilon(q)(s)\|_X ds\,. \tag{113}$$

Starting from (112) and iterating inequality (113) leads to

$$\|\phi_\varepsilon(q)\|_{l,\beta} \le C\big(\|q\|_{l,\beta} + \|\nu q\|_{H^l_x(L^2_v)} + \|\nu q\|_{L^1_x(L^2_v)}\big)\,,$$

which, together with (111), gives the expected continuity property

$$\|\psi_\varepsilon[g,h]\|_{l,\beta} \le C_2\|g\|_{l,\beta}\|h\|_{l,\beta}\,,$$

provided that $\nu H_{l,\beta} \subset H^l_x(L^2_v)$, or equivalently $\beta > 5/2$. □

Proof of existence. Equipped with these preliminary results, we get immediately the global existence of a unique solution to (102). Indeed, we deduce from Lemmas 2.2 and 2.3 the estimates below on the operator N_ε defined in (107):

$$\|N_\varepsilon[g]\|_{l,\beta} \leq C_1\|g_0\|_{l,\beta} + C_2\|g\|^2_{l,\beta}$$

and

$$\|N_\varepsilon[g] - N_\varepsilon[h]\|_{l,\beta} \leq C_2\big(\|g\|_{l,\beta} + \|h\|_{l,\beta}\big)\|g-h\|_{l,\beta}\,.$$

Choosing a_0 and a_1 such that

$$C_2 a_1 < 1 \text{ and } C_1 a_0 + C_2 a_1^2 \leq a_1\,,$$

we get that N_ε is a contraction on the ball of radius a_1 as soon as

$$\|g_0\|_{l,\beta} \leq a_0\,,$$

meaning in particular that the smallness condition is independent of ε.

5.2.3.2 - The incompressible Navier-Stokes limit

At this point, under the smallness condition (104) on the initial data g_0, for each $\varepsilon > 0$, we are able to construct a global smooth solution g_ε to the scaled Boltzmann equation (102). It remains to study the asymptotic behavior of the family (g_ε) as $\varepsilon \to 0$. Weak convergence (modulo extraction of a subsequence) is obvious since the fixed point argument above gives uniform bounds. But strong convergence is needed in order to pass to the limit in the nonlinear terms.

This strong convergence properties depends on the existence of limits for U_ε and ψ_ε as $\varepsilon \to 0$, and on a compactness argument. The basic idea is indeed to come back to the previous study replacing the uniform continuity estimates on U_ε and ψ_ε given in Lemmas 2.2 and 2.3 with a convenient choice of equivalents V and ψ as $\varepsilon \to 0$ that are defined in the following lemmas.

Lemma 2.4. *Denote by U_ε the semi-group generated by the operator T_ε defined by* (105).

Then, there exists a linear operator $V(t)$ satisfying the same continuity estimate in $X_{l,\beta}$ for $\beta > \dfrac{3}{2}$ as $U_\varepsilon(t)$

$$\|V(t)g_0\|_{l,\beta} \leq C_1\|g_0\|_{l,\beta}\,,$$

such that

$$\forall g_0 \in H_{l,\beta}, \quad U_\varepsilon(t)g_0 \to V(t)g_0 \text{ strongly in } C(\boldsymbol{R}_+^* \times \boldsymbol{R}^3, L^{\infty,\beta})$$

and uniformly in time (near $t=0$) if and only if $g_0 = V(0)g_0 = \Pi g_0$.

Lemma 2.5. *Denote by ψ_ε the symmetric bilinear operator defined by* (107). *Then, there exists a bilinear symmetric operator ψ satisfying the same continuity estimate in $X_{l,\beta}$ for $\beta > \frac{5}{2}$ and $l > \frac{3}{2}$ as ψ_ε, i.e.*

$$\|\psi[g,h]\|_{l,\beta} \le C_2 \|g\|_{l,\beta} \|h\|_{l,\beta} \,,$$

and such that

$$\forall g, h \in X_{l,\beta}, \quad \psi_\varepsilon[g,h] \to \psi[g,h] \text{ strongly in } C(\boldsymbol{R}_+ \times \boldsymbol{R}^3, L^{\infty,\beta}) \,,$$

while

$$g \in X_{l,\beta} \mapsto \psi[g,g] \in C(\boldsymbol{R}_+ \times \boldsymbol{R}^3, L^{\infty,\beta}) \text{ is locally compact} \,.$$

Furthermore, if g_k converges to g strongly in $C(\boldsymbol{R}_+^* \times \boldsymbol{R}^3; L^{\infty,\beta})$ and weakly- in $L^\infty(\boldsymbol{R}_+; H_{l,\beta})$, then*

$$\psi[g_k, g_k] \to \psi[g,g] \text{ weakly} - * \text{ in } L^\infty(\boldsymbol{R}_+; H_{l,\beta}) \,.$$

The proof of both lemmas relies on a precise study of the following approximations of U_ε and $\frac{1}{\varepsilon} U_\varepsilon(I - \Pi)$

$$\hat{V}_\varepsilon(t,\xi) = \sum_{j=1}^{4} \exp\left(\left(\frac{i a_j |\xi|}{\varepsilon} - \beta_j |\xi|^2\right) t\right) P_{0j}(\xi/|\xi|) \tag{114}$$

and

$$\hat{Z}_\varepsilon(t,\xi) = \sum_{j=1}^{4} \exp\left(\left(\frac{i a_j |\xi|}{\varepsilon} - \beta_j |\xi|^2\right) t\right) |\xi| P_{1j}(\xi/|\xi|)(Id - \Pi) \tag{115}$$

and is established by a stationary phase method applied to the spectral representation of U_ε as will be explained later in the next paragraph.

Proof of convergence. Taking both lemmas for granted, we conclude the proof of convergence.

Start from the decomposition

$$\begin{aligned} g_\varepsilon &= V(t) g_0 + (U_\varepsilon(t) - V(t)) g_0 + \psi[g_\varepsilon, g_\varepsilon] + (\psi_\varepsilon[g_\varepsilon, g_\varepsilon] - \psi[g_\varepsilon, g_\varepsilon]) \\ &= V(t) g_0 + r_{1\varepsilon} + \psi[g_\varepsilon, g_\varepsilon] + r_{2\varepsilon} \,. \end{aligned} \tag{116}$$

By Lemma 2.4,

$$r_{1\varepsilon} \to 0 \text{ in } C(\boldsymbol{R}_+^* \times \boldsymbol{R}^3; L^{\infty,\beta}) \,.$$

By Lemma 2.5,

$$\psi[g_\varepsilon, g_\varepsilon] \text{ is relatively compact and } r_{2\varepsilon} \to 0 \text{ in } C(\boldsymbol{R}_+ \times \boldsymbol{R}^3, L^{\infty,\beta}) \,.$$

Then,

$$g_\varepsilon \text{ is strongly relatively compact in } C(\boldsymbol{R}_+^* \times \boldsymbol{R}^3, L^{\infty,\beta}).$$

Applying again Lemma 2.5, we see that

$$g_\varepsilon \to g \text{ strongly in } C(\boldsymbol{R}_+^* \times \boldsymbol{R}^3, L^{\infty,\beta})$$

while

$$g_\varepsilon \rightharpoonup g \text{ weakly-* in } L^\infty(\boldsymbol{R}_+, H_{l,\beta}) \text{ and } \psi[g_\varepsilon, g_\varepsilon] \to \psi[g,g]$$

modulo extraction of a subsequence, so that eventually

$$g(t) = V(t)g_0 + \psi[g,g](t).$$

Since V and ψ satisfy the same continuity properties as U_ε and ψ_ε, the limit g is the unique solution of the limiting integral equation, which implies that the whole family g_ε converges to g in the vanishing ε limit.

The convergence of the moments is an easy consequence of the previous results. The formal study shows that these moments must satisfy the incompressible Navier-Stokes equations. These equations have to be supplemented by initial conditions, obtained from the identity $g(0) = V(0)g_0 + \psi[g,g](0) = \Pi g_0$.

5.2.3.3 - The stationary phase method

In order to construct the limiting semi-group $V(t)$ and the limiting bilinear operator ψ defined in Lemmas 2.4 and 2.5, we need to know more about the spectral decomposition of U_ε than the result of analytic continuation stated in Theorem 1.2.

In particular, precise estimates on the leading order approximation $ia_j|\xi|$ of each eigenvalue $\lambda_j(\xi)$ are required.

If $a_j = 0$, the corresponding contribution to the semigroup is expected to be essentially independent of ε, while if $a_j \neq 0$, it is expected to produce high frequency oscillations and to vanish in the limit.

The crucial point in the proofs of Lemmas 2.4 and 2.5 is therefore the following computation which improves the results in Theorem 1.2:

Theorem 2.6 (Ellis & Pinsky [**19**]). *Let T be the linear operator*

$$T = v \cdot \nabla_x + \mathcal{L}_M,$$

or equivalently, in Fourier variables,

$$\hat{T}(\xi) = iv \cdot \xi + \mathcal{L}_M.$$

Denote by $\lambda_j(\xi)$ the eigenvalues of $\hat{T}(\xi)$ that vanish for $\xi = 0$, and by $P_j(\xi)$ the corresponding eigenprojections.

Then, λ_j is a C^∞ function of $|\xi|$. The first order approximation of $\lambda_j(\xi)$ near $\xi = 0$ is given by

$$ia_j \neq 0 \text{ if } j = 1, 2\,, \text{ and } ia_j = 0 \text{ if } j = 3, 4\,.$$

Furthermore, the eigenprojections are C^∞ functions of $|\xi| \leq \kappa$, and are defined at leading order in $|\xi|$ by

$$\Im(P_{01}(\tilde{\xi})) = \operatorname{span}\left\{1 - \tilde{\xi} \cdot v + \frac{1}{2}(|v|^2 - 3)\right\},$$

$$\Im(P_{02}(\tilde{\xi})) = \operatorname{span}\left\{1 + \tilde{\xi} \cdot v + \frac{1}{2}(|v|^2 - 3)\right\},$$

$$\Im(P_{03}(\tilde{\xi})) = \operatorname{span}\left\{-1 + (|v|^2 - 3)\right\},$$

and

$$\Im(P_{04}(\tilde{\xi})) = \operatorname{span}\{v \cdot \tilde{\xi}^\perp, v \cdot (\tilde{\xi} \times \tilde{\xi}^\perp\}$$

where $\tilde{\xi} = \dfrac{\xi}{|\xi|} \in \boldsymbol{S}^2$, and $\tilde{\xi}^\perp$ is any unit vector such that $\tilde{\xi} \cdot \tilde{\xi}^\perp = 0$.

Taking this result for granted, we return to the approximations V_ε and Z_ε of U_ε and $\frac{1}{\varepsilon} U_\varepsilon(I - \Pi)$, using the stationary phase method. For the sake of simplicity, we will only sketch the proof of Lemma 2.4. Indeed Lemma 2.5 requires to keep one further order in the expansions with respect to ε of the eigenvalues and eigenprojections of U_ε.

P r o o f o f L e m m a 2.4. Start from the decomposition:

$$\begin{aligned} \hat{U}_\varepsilon &= \sum_{j=1}^{4} \mathbf{1}_{|\varepsilon\xi| \leq \kappa} \exp\left(\frac{t}{\varepsilon^2} \lambda_j(\varepsilon\xi)\right) P_j(\varepsilon\xi) + R\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right) \\ &= \hat{V}_\varepsilon(t, \xi) + h_\varepsilon(t, \xi) + R\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right), \end{aligned} \tag{117}$$

where $\hat{V}_\varepsilon$ is defined as previously by

$$\hat{V}_\varepsilon(t, \xi) = \sum_{j=1}^{4} \exp\left(\left(\frac{ia_j|\xi|}{\varepsilon} - \beta_j|\xi|^2\right)t\right) P_{0j}(\xi/|\xi|)\,.$$

• The oscillating parts of $U_\varepsilon(t)g_0$ (corresponding to the first two terms in V_ε) are controlled by a nonstationary phase argument. In view of Theorem 2.6, we only have to study integrals of the form

$$I(t,x,v) = b(v)\int \exp\left(i\xi\cdot x + ia_j|\xi|t/\varepsilon - \beta_j|\xi|^2 t\right)a(\xi/|\xi|)\hat{\gamma}_0(\xi)d\xi$$

where $a \in C^\infty(\boldsymbol{S}^2)$, $b \in L^{\infty,\beta}$ for arbitrary $\beta \geq 0$, and γ_0 denotes some average of g_0 against an element of $L^{\infty,\beta}$.

Since $a_j \neq 0$, for $l > 3/2$

$$|I(t,x,v)| \leq C\left(\frac{\varepsilon}{t}\right)^\delta b(v)\left(\|\tilde{\gamma}_0(t)\|_{L^1_x} + \|\tilde{\gamma}_0(t)\|_{H^l_x}\right)$$

for some $\delta \in (0,1)$, where

$$\tilde{\gamma}_0(t,x) = \int \exp\left(i\xi\cdot x - \beta_j|\xi|^2 t\right)\hat{\gamma}_0(\xi)d\xi\,.$$

Classical estimates on the heat kernel and the definition of γ_0 imply that

$$\|V_{j\varepsilon}(t)g_0\|_{L^\infty_x(L^{\infty,\beta})} \leq C\left(\frac{\varepsilon}{t}\right)^\gamma\left(\|g_0\|_{L^1_x(L^2(Mdv))} + \|g_0\|_{l,\beta}\right) \tag{118}$$

• The convergence of the higher order terms h_ε lies on the following decomposition : for $j \in \{1,2,3,4\}$

$$\begin{aligned}\hat{h}_{j\varepsilon}(t,\xi) &= \mathbf{1}_{\varepsilon|\xi|\leq\kappa}\exp\left(\frac{1}{\varepsilon^2}\lambda_j(\varepsilon\xi)t\right)[P_j(\varepsilon\xi) - P_{0j}(\xi/|\xi|)]\\ &\quad + \mathbf{1}_{\varepsilon|\xi|\leq\kappa}\left[\exp\left(\frac{1}{\varepsilon^2}\lambda_j(\varepsilon\xi)t\right) - \exp\left(ia_j\frac{t}{\varepsilon} - \beta_j|\xi|^2\right)\right]P_{0j}(\xi/|\xi|)\\ &\quad + (1 - \mathbf{1}_{\varepsilon\xi\leq\kappa})\left(\exp\left(\frac{ia_j t}{\varepsilon} - \beta_j|\xi|^2\right)P_{0j}(\xi/|\xi|)\right.\,.\end{aligned}$$

The fundamental results on the spectral representation of U_ε stated in Theorem 1.2 lead then to

$$|\hat{h}_{j\varepsilon}(t,\xi)| \leq C\varepsilon|\xi|\exp\left(-\frac{1}{2}\beta_j|\xi|^2 t\right).$$

As U_ε and V_ε are uniformly continuous in $H_{s,\beta}$ for any $s \in \boldsymbol{R}$, by interpolation, we get for $s = l - \gamma > 3/2$

$$\|h_{j\varepsilon}(t)u\|_{L^\infty(L^{\infty,\beta})} \leq \|h_{j\varepsilon}(t)u\|_{s,\beta} \leq C\varepsilon^\gamma\|u\|_{l,\beta}\,. \tag{119}$$

• The convergence of the remainder R is obtained in a very similar way.

$$R\left(\frac{t}{\varepsilon},\varepsilon\xi\right) = \hat{U}_\varepsilon(t)\left(Id - \mathbf{1}_{\varepsilon|\xi|\leq\kappa}\sum_{j=1}^4 P_j(\varepsilon\xi)\right)$$

is uniformly continuous in $H_{l,\beta}$. And, by Sobolev embedding arguments,

$$\|R_\varepsilon(t)u\|_{L^\infty_x(L^{\infty,\beta}_v)} \leq C\exp\left(-\frac{\sigma t}{\varepsilon^2}\right)\|u\|_{l,\beta}\,. \tag{120}$$

In order to obtain uniform convergence, if $g_0 = \Pi g_0$, notice that

$$\left(Id - \mathbf{1}_{\varepsilon|\xi|\leq\kappa}\sum_{j=1}^{4}P_j(\varepsilon\xi)\right)\Pi = \sum_{j=1}^{4}(P_{0j}(\xi/|\xi|) - \mathbf{1}_{\varepsilon|\xi|\leq\kappa}P_j(\varepsilon\xi))\Pi\,,$$

which gives by the same techniques as previously

$$\|R_\varepsilon(t)\Pi u\|_{L^\infty_x(L^{\infty,\beta}_v)} \leq C\varepsilon^\gamma\exp\left(-\frac{\sigma t}{\varepsilon^2}\right)\|u\|_{l,\beta}\,. \tag{121}$$

Combining (117) with (118), (119) and (120) or (121) gives respectively the convergence on $\boldsymbol{R}^*_+$ or the uniform convergence under the well-posedness condition

$$U_\varepsilon(t)g_0 \to V(t)g_0\,,$$

where

$$\hat{V}(t,\xi) = \sum_{j=3,4}\exp\left(-\beta_j|\xi|^2t\right)P_{0j}(\xi/|\xi|)$$

is the non-oscillating part of $\hat{V}_\varepsilon$.

A similar argument allows to establish Lemma (2.5), and thus to conclude the convergence proof leading to Theorem 2.1. Therefore, the core of the argument is the precise description of the spectral representation of the linear operator T given in Theorems 1.2 and 2.6.

This means first that we consider the problem of hydrodynamic limits essentially in a linear (or perturbative) framework. The disadvantage inherent to that strategy is the need for a deep result of spectral theory used as a black box . In particular, this approach fails to provide a real understanding of the coupling between relaxation and hydrodynamic modes in the full nonlinear Boltzmann equation.

Furthermore one cannot expect to extend such a result to classes of initial data with less regularity; in other words, this method justifies the incompressible Navier-Stokes limit of the Boltzmann equation for a meager subset of all physically admissible initial data.

5.3 - *Uniform a priori estimates on the scaled Boltzmann equation*

To avoid restrictions on the class of initial data that are inherent to methods similar to the Bardos-Ukai argument, one needs to abandon the detailed spectral analysis of the Boltzmann linearized equation and use instead the available a priori estimates on the scaled Boltzmann equation that are uniform in the Strouhal and Knudsen numbers.

The only such a priori estimates are consequences of the Boltzmann H Theorem; they are the common basis to the derivations of either the incompressible Euler or Navier-Stokes(-Fourier) equations from renormalized solutions of the Boltzmann equation. We shall therefore present these estimates in the case where

$$\mathrm{Ma} = \mathrm{St} = \varepsilon \text{ and } \mathrm{Kn} = \varepsilon^q \text{ with } q \geq 1$$

that encompasses both limits.

We therefore consider the Cauchy problem for the scaled Boltzmann equation

$$\begin{aligned} &\varepsilon \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q} \mathcal{B}(F_\varepsilon, F_\varepsilon)\,, \quad (t,x,v) \in \boldsymbol{R}_+^* \times \boldsymbol{R}^3 \times \boldsymbol{R}^3 \,. \\ &F_\varepsilon\big|_{t=0} = F_\varepsilon^{in}\,. \end{aligned} \tag{122}$$

Because we are interested in a regime where the number density F_ε stays in the vicinity of a uniform Maxwellian M, it will be convenient to use the relative number density and relative number density fluctuation, defined as

$$\begin{aligned} G_\varepsilon &= \frac{F_\varepsilon}{M} \qquad \text{and } g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}\,, \\ G_\varepsilon^{in} &= \frac{F_\varepsilon^{in}}{M} \qquad \text{and } g_\varepsilon^{in} = \frac{F_\varepsilon^{in} - M}{\varepsilon M}\,. \end{aligned} \tag{123}$$

Here, M denotes the same uniform Maxwellian state as earlier in this chapter, namely

$$M(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2}|v|^2\right).$$

In terms of g_ε, the scaled Boltzmann equation (122) takes the form

$$\begin{aligned} &\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon^q} \mathcal{L} g_\varepsilon = \frac{1}{\varepsilon^{q-1}} \mathcal{Q}(g_\varepsilon, g_\varepsilon)\,, \\ &g_\varepsilon\big|_{t=0} = g_\varepsilon^{in}\,. \end{aligned} \tag{124}$$

where $\mathcal{L}$ and $\mathcal{Q}$ are the operators defined in (103).

We recall from chapter 1, section 4 the definition of the relative entropy of a number density with respect to the uniform Maxwellian M:

$$H(F_\varepsilon|M)(t) = \iint \left(F_\varepsilon \ln\left(\frac{F_\varepsilon}{M}\right) - F_\varepsilon + M \right)(t,x,v)\,dxdv\,. \tag{125}$$

We have seen in chapter 3, section 7 that renormalized solutions to the Boltzmann equation relatively to M can be defined for initial data whose relative entropy with respect to M is finite, and that such solutions satisfy the DiPerna-Lions entropy inequality

$$H(F_\varepsilon(t)|M) + \frac{1}{4}\frac{1}{\varepsilon^{q+1}}\int_0^t \int_{\mathbf{R}^3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\varepsilon)|(v-v_*)\cdot\omega|dvdv_*d\omega dxds \le H(F_\varepsilon^{in}|M)\,, \tag{126}$$

where the dissipation integrand $d(F)$ is defined by (29).

We shall only consider initial data that are close to M in the sense that

$$H(F_\varepsilon^{in}|M) \le C^{in}\varepsilon^2\,. \tag{127}$$

This corresponds indeed to an initial relative number density fluctuation that is of order 1 in some sense — or equivalently, to an initial number density that is to within $O(\varepsilon)$ of the uniform Maxwellian state M.

Because of the DiPerna-Lions entropy inequality, the initial entropy bound (127) implies

- the relative entropy bound

$$H(F_\varepsilon(t)|M) \le C^{in}\varepsilon^2\,, \tag{128}$$

- the entropy production bound

$$\int_0^{+\infty} \int_{\mathbf{R}^3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\varepsilon)|(v-v_*)\cdot\omega|dvdv_*d\omega dxdt \le 4C^{in}\varepsilon^{q+3}\,. \tag{129}$$

In the sequel, we shall discuss the various implications of both these controls.

5.3.1 - Implications of the relative entropy bound

The implications of the relative entropy bound that we shall discuss here are straightforward consequences of pointwise inequalities satisfied by the nonlinearity that defines the relative entropy, i.e.

$$h(z) = (1+z)\log(1+z) - z\,. \tag{130}$$

Notice that

$$H(F_\varepsilon|M) = \iint h(\varepsilon g_\varepsilon)Mdvdx\,;$$

since

$$h(z) \sim \frac{1}{2}z^2\,, \quad z \to 0\,, \tag{131}$$

one could think that the relative entropy bound (128) is more or less equivalent to an L^2 estimate of the type

$$\iint |g_\varepsilon(t,x,v)|^2 M dv dx \le 2C^{in}.$$

However, this is not entirely correct, since g_ε can take values $>> 1/\varepsilon$, for which replacing $h(z)$ by $\frac{1}{2}z^2$ is not justified.

Instead of (131), we must use global properties of h. First, h satisfies Young's inequality

$$pz \le h^*(p) + h(z), \quad p, z \ge 0,$$

where h^* is the Legendre dual of h:

$$h^*(p) = e^p - p - 1. \tag{132}$$

Notice that h^* is super-quadratic (as can be seen from the Taylor series that defines h^*): in other words,

$$h^*(\lambda p) \le \lambda^2 h^*(p), \quad p \ge 0, \ \lambda \in [0,1].$$

Also, notice that

$$h(|z|) \le h(z), \quad z > -1.$$

Putting all these inequalities together, we arrive at the following improvement of the Young inequality above:

$$p|z| \le \lambda h^*(p) + \frac{1}{\lambda} h(z), \quad p \ge 0, \ z \ge -1, \ \lambda \in (0,1]. \tag{133}$$

Lemma 3.1. *Let $F_\varepsilon^{in} \equiv F_\varepsilon^{in}(x,v)$ be a family of mesurable, a.e. nonnegative initial number densities that satisfies* (127). *For each $\varepsilon > 0$, let F_ε be a renormalized solution of the scaled Boltzmann equation* (122) *relatively to M.*

Then, for each sequence $\varepsilon_n \to 0$, the family of fluctuations (g_{ε_n}) associated to F_{ε_n} as in (123) *is weakly relatively compact in $L^1_{loc}(dtdx, L^1(M(1+|v|^2)dv))$.*

Proof. Pick $\theta \in (0,1|$; Young's inequality (133) implies that, for each n such that $\varepsilon_n \in (0,\theta)$, i.e. for all but a finite number of n's, one has

$$(1+|v|^2)|g_{\varepsilon_n}| \le \frac{4\theta}{\varepsilon_n^2} h(\varepsilon_n g_{\varepsilon_n}) + \frac{4}{\theta} h^* \left(\frac{1+|v|^2}{4} \right)$$

by taking $z = \varepsilon_n g_{\varepsilon_n}$, $p = \frac{1}{4}(1+|v|^2)$ and $\lambda = \frac{\varepsilon_n}{\theta}$.

Consider first the case $\theta = 1$; hence, for each measurable set $E \subset \boldsymbol{R}^3$ of finite measure

$$\iint_E (1+|v|^2)|g_{\varepsilon_n}(t)|Mdvdx \le 4C^{in} + 4|E| \int \exp\left(\frac{1}{4}(1+|v|^2)\right)Mdv\,.$$

Hence

$$(1+|v|^2)g_{\varepsilon_n} \text{ is bounded in } L^\infty(dt; L^1_{loc}(dx : L^1(Mdv)))\,.$$

For general $\theta \in (0,1)$,

$$\iint_E (1+|v|^2)|g_{\varepsilon_n}(t)|Mdvdx \le 4\theta C^{in} + \frac{4}{\theta}|E| \int \exp\left(\frac{1}{4}(1+|v|^2)\right)Mdv\,.$$

Letting $|E| \to 0$ and $\theta = |E|^{1/2}$ shows that

$$\iint_E (1+|v|^2)|g_{\varepsilon_n}(t)|Mdvdx \le 4(C^{in} + e^{1/4})|E|^{1/2}$$

for each n such that $\varepsilon_n < |E|^{1/2}$, i.e. for all but a finite number of n's. This shows that the family

$$(1+|v|^2)g_{\varepsilon_n} \text{ is locally uniformly integrable on } \boldsymbol{R}^3 \times \boldsymbol{R}^3$$

uniformly in $t \ge 0$. By the Dunford-Pettis criterion (see chapter 2, section 3), this implies the announced weak compactness for the sequence g_{ε_n} of relative number density fluctuations. □

The formal argument given at the beginning of this paragraph suggests that, in the vanishing ε limit, the limiting points of g_ε belong to $L^2(Mdvdx)$ uniformly in t. Hence the weighted L^1-bound implied Lemma 3.1 is certainly not optimal.

Instead, we propose to consider the following renormalized fluctuation

$$\hat{g}_\varepsilon = \frac{2}{\varepsilon}\left(\sqrt{G_\varepsilon} - 1\right). \tag{134}$$

The advantage of this renormalized fluctuation over the original one is explained in the next lemma.

Lemma 3.2. *Let $F^{in}_\varepsilon \equiv F^{in}_\varepsilon(x,v)$ be a family of mesurable, a.e. nonnegative initial number densities that satisfies* (127). *For each $\varepsilon > 0$, let F_ε be a renormalized solution of the scaled Boltzmann equation* (122) *relatively to M. Let G_ε be the relative number density as in* (123). *Then, the renormalized fluctuation $\hat{g}_\varepsilon$ defined in* (134) *is bounded in $L^\infty(\boldsymbol{R}_+; L^2(Mdvdx))$.*

Proof. The elementary inequality

$$h(z) \geq (\sqrt{1+z}-1)^2\,, \quad z>-1 \tag{135}$$

implies that

$$\iint \hat{g}_\varepsilon^2(t,x,v)M(v)dxdv \leq \frac{2}{\varepsilon^2}H(F_\varepsilon|M)(t) \leq 2C^{in}\,,$$

which is the announced result. □

A natural application of this refined a priori estimate is to decompose

$$g_\varepsilon = \hat{g}_\varepsilon + \frac{1}{4}\,\hat{g}_\varepsilon^2. \tag{136}$$

Therefore, we see that the fluctuation g_ε is bounded in $L^2(Mdvdx)$, up to a remainder of order ε in $L^1(Mdvdx)$, uniformly in $t \geq 0$.

5.3.2 - Continuity properties of the collision integral

Before going further, we give an elementary continuity property of the Boltzmann collision integral for a hard sphere gas, which can be easily extended to all hard cut-off potentials.

Lemma 3.3 (Golse-Perthame-Sulem [**26**]). *Let $\mathcal{B}$ denote the Boltzmann collision integral in the hard sphere case, let*

$$M(v) = \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}|v|^2\right)$$

be the centered reduced Gaussian and let $\mathcal{Q}_M$ be the quadratic operator defined by

$$\mathcal{Q}_M(\phi,\phi) = M^{-1}\mathcal{B}(M\phi, M\phi)\,.$$

Then, there exists $C>0$ such that

$$\|\mathcal{Q}_M(\phi,\phi)\|_{L^2((1+|v|)^{-1}Mdv)} \leq C\|\phi\|_{L^2(Mdv)}\|\phi\|_{L^2((1+|v|)Mdv)}\,.$$

Proof. Consider for instance the gain part in the collision integral

$$\mathcal{Q}_M^+(\phi,\phi) = \iint \phi'\phi'_*|(v_*-v)\cdot\omega|M_*dv_*d\omega\,;$$

then

$$\|\mathcal{Q}_M^+(\phi,\phi)\|^2_{L^2((1+|v|)^{-1}Mdv)} = \int\left(\iint \phi'\phi'_*|(v_*-v)\cdot\omega|M_*dv_*d\omega\right)^2(1+|v|)^{-1}Mdv\,.$$

Let

$$C = \sup \frac{\iint |(v - v_*) \cdot \omega| M_* dv_* d\omega}{1 + |v|} .$$

By Cauchy-Schwarz inequality

$$\begin{aligned} &\| \mathcal{Q}_M^+(\phi, \phi) \|^2_{L^2((1+|v|)^{-1} M dv)} \\ &\qquad \le C \iiint \phi^2 \phi_*^2 |(v_* - v) \cdot \omega| M M_* dv dv_* d\omega \\ &\qquad \le 4\pi C \iint \phi^2 \phi_*^2 (|v| + |v_*|) M M_* dv dv_* , \end{aligned}$$

which implies the announced continuity for $\mathcal{Q}_M^+$. □

5.3.3 - Implications of the entropy production bound

The relative entropy bound (128) controls the distance of the number density F_ε to the background Maxwellian state M (by controling the number density fluctuation g_ε or its renormalized variant $\hat{g}_\varepsilon$).

By analogy, we expect that the entropy production bound (129) controls the distance between the number density F_ε and the set of all local Maxwellian distributions.

Lemma 3.4. *Let $F_\varepsilon^{in} \equiv F_\varepsilon^{in}(x, v)$ be a family of mesurable, a.e. nonnegative initial number densities that satisfies* (127). *For each $\varepsilon > 0$, let F_ε be a renormalized solution of the scaled Boltzmann equation* (122) *relatively to M. Let g_ε be the number density fluctuation associated to F_ε as in* (123), *while $\hat{g}_\varepsilon$ designates the renormalized fluctuation defined by* (134).

Then

$$\hat{g}_\varepsilon - \Pi \hat{g}_\varepsilon \to 0 \text{ strongly in } L^1_{loc}(dtdx, L^2(Mdv)) .$$

Proof. Replace the hard sphere collision kernel with

$$b(v - v_*, \omega) = \frac{|(v - v_*) \cdot \omega|}{1 + |v - v_*|} .$$

Obviously the entropy production bound (129) implies that

$$\int_0^{+\infty} \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{S}^2} d(F_\varepsilon) b(v - v_*, \omega) dv dv_* d\omega dx dt \le 4 C^{in} \varepsilon^{q+3} .$$

Henceforth, we denote $\mathcal{Q}^b$ and $\mathcal{L}^b$ the operators defined in (103) with the new collision kernel b. First, the dissipation integrand $d(F_\varepsilon)$ is estimated by using the elementary inequality

$$(z-y)\ln\left(\frac{z}{y}\right) \geq 4(\sqrt{z}-\sqrt{y})^2 . \tag{137}$$

Hence, by the Cauchy-Schwarz inequality

$$\int_0^{+\infty}\iint \mathcal{Q}^b(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon})^2(t,x,v)Mdvdxdt$$
$$\leq 4\pi\int_0^t\int_{\boldsymbol{R}^3}\iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(F_\varepsilon)b(v-v_*,\omega)dvdv_*d\omega dxds \leq 4\pi C^{in}\varepsilon^{q+3} .$$

Next consider the decomposition

$$\frac{1}{\varepsilon}\mathcal{L}\hat{g}_\varepsilon = \frac{1}{2}\mathcal{Q}^b(\hat{g}_\varepsilon,\hat{g}_\varepsilon) - \frac{1}{\varepsilon^2}\mathcal{Q}^b(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon}) . \tag{138}$$

The second term on the right hand side of the equality above is $O(\varepsilon^{q-1})$ in $L^2(Mdvdxdt)$ because of the previous inequality. On the other hand, since the collision kernel b is bounded by 1, using the Cauchy-Schwarz inequality as in the proof of Lemma 3.3 shows that

$$\|\mathcal{Q}^b(\hat{g}_\varepsilon,\hat{g}_\varepsilon)\|_{L^2(Mdv)} \leq C\|\hat{g}_\varepsilon\|^2_{L^2(Mdv)} .$$

Hence, by Lemma 3.2 above, one has

$$\mathcal{Q}^b(\hat{g}_\varepsilon,\hat{g}_\varepsilon) = O(1)_{L^\infty(dt;L^1(dx;L^2(Mdv)))} .$$

Finally, by an argument similar to Hilbert's in the proof of Theorem 6.1 in chapter 1, $\mathcal{L}^b$ is a bounded self-adjoint Fredholm operator on $L^2(Mdv)$ with nullspace $\operatorname{span}\{1,v_1,v_2,v_3,|v|^2\}$ so that, for some positive constant c, the following relative coercivity estimate holds:

$$\|\mathcal{L}^b\hat{g}_\varepsilon\|_{L^2(Mdv)} \geq c\|\hat{g}_\varepsilon - \Pi\hat{g}_\varepsilon\|_{L^2(Mdv)} .$$

These last two estimates together with (138) imply the announced result. □

6 - From the Boltzmann Equation to the Incompressible Euler System: Convergence Proof

In this chapter, we describe the hydrodynamic limit of the Boltzmann equation to the incompressible Euler equation, for initial data that are compatible with any solenoidal velocity field of finite kinetic energy. We shall also describe in some detail the method used in the proof, the relative entropy method, which is remarkably versatile, and has been applied to other asymptotic problems.

6.1 - *Statement of the incompressible Euler limit*

The incompressible Euler limit of the Boltzmann equation corresponds to the scaling

$$\mathrm{Ma} = \mathrm{St} = \varepsilon \text{ and } \mathrm{Kn} = \varepsilon^q \text{ with } q > 1\,.$$

We therefore consider the scaled Boltzmann equation

$$(139)\qquad \begin{aligned} &\varepsilon\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon^q}\mathcal{B}(F_\varepsilon, F_\varepsilon)\,,\quad (t,x,v)\in \boldsymbol{R}_+^*\times\boldsymbol{R}^3\times\boldsymbol{R}^3\,,\\ &F_\varepsilon\big|_{t=0} = F_\varepsilon^{in}\,.\end{aligned}$$

We shall assume that

$$F_\varepsilon^{in}(x,v) = \mathcal{M}_{(1,\varepsilon u^{in}(x),1)}(v) + o(\varepsilon)\,,$$

where u^{in} is a solenoidal vector field and $\mathcal{M}_{(1,\varepsilon w,1)}$ is the Gaussian distribution centered at εw with covariance matrix I:

$$\mathcal{M}_{(1,\varepsilon w,1)}(v) = \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}|v-\varepsilon w|^2\right)\,,$$

and the error term $o(\varepsilon)$ is measured in terms of the relative entropy (see below). We shall also denote $M = \mathcal{M}_{(1,0,1)}$.

Throughout this chapter, the entropy production rate will play an essential role in measuring the distance from the number density F_ε to the manifold of local Maxwellian states; we shall denote it

$$D(F) = \frac{1}{4}\iint (F'F'_* - FF_*)\ln\left(\frac{F'F'_*}{FF_*}\right)|(v-v_*)\cdot\omega|dv_*d\omega\,.$$

As before, we shall use the notion of relative number density and of relative number density fluctuation

$$(140)\qquad \begin{aligned} &G_\varepsilon = \frac{F_\varepsilon}{M}\,,\quad \text{and } g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}\,,\\ &G_\varepsilon^{in} = \frac{F_\varepsilon^{in}}{M}\,,\quad \text{and } g_\varepsilon^{in} = \frac{F_\varepsilon^{in} - M}{\varepsilon M}\,.\end{aligned}$$

In terms of g_ε, the scaled Boltzmann equation (139) takes the form

$$\varepsilon\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon^q}\mathcal{L}g_\varepsilon = \frac{1}{\varepsilon^{q-1}}\mathcal{Q}(g_\varepsilon, g_\varepsilon)\,. \tag{141}$$

where $\mathcal{L}$ and $\mathcal{Q}$ are the operators defined in (103).

According to the formal analysis, g_ε is then expected to converge to the infinitesimal Maxwellian

$$g(t,x,v) = u(t,x)\cdot v\,,$$

where u satisfies the incompressible Euler equations.

Under suitable conditions on the initial data, we are actually able to prove [8], [**44**], [**53**] that the distance between any solution of the scaled Boltzmann equation (139) and the local Maxwellian $\mathcal{M}_{(1,\varepsilon u,1)}$ where u satisfies the incompressible Euler equations is $o(\varepsilon)$ as $\varepsilon\to 0$.

Theorem 1.1 (Saint-Raymond [**53**]). *Let $F_\varepsilon^{in}\in L^1_{loc}(\boldsymbol{R}^3\times\boldsymbol{R}^3)$ be a family of initial fluctuations satisfying*

$$\frac{1}{\varepsilon^2}H(F_\varepsilon^{in}|\mathcal{M}_{1,\varepsilon u^{in},1})\to 0\,, \tag{142}$$

where u^{in} is a solenoidal vector field such that $u^{in}\in H^s(\boldsymbol{R}^3)$ for some $s>\frac{5}{2}$.

Let F_ε be, for each $\varepsilon>0$, a renormalized solution relatively to M of the Boltzmann equation (139) *with $q>1$.*

Then, modulo extraction of a subsequence,

$$g_\varepsilon \rightharpoonup g \text{ weakly in } L^1_{loc}([0,T)\times\boldsymbol{R}^3, L^1(M(1+|v|^2)dv))$$

in the limit as $\varepsilon\to 0$, with

$$g(t,x,v) = u(t,x)\cdot v\,,$$

where u is the unique maximal classical solution of the incompressible Euler equations

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0\,,\quad \operatorname{div}_x u = 0\,,$$
$$u_{|t=0} = u^{in}\,,$$

that belongs to $L^\infty_{loc}((0,T);W^{1,\infty}(\boldsymbol{R}^3))$.

Notice that this result deals with renormalized (therefore weak) solutions of the Boltzmann equation, and with classical solutions of the incompressible Euler equations. This particular feature is characteristic of the relative entropy method used in the proof. As we shall see, the relative entropy method is based on the stability properties of the target equations — in the present case, of the incompressible Euler equations.

6.2 - *Properties of the incompressible Euler equations*

We first recall the basic existence result for the incompressible Euler equations.

Theorem 2.1 (Beale, Kato & Majda [7]). *Let $u^{in} \equiv u^{in}(x) \in H^s(\boldsymbol{R}^3)$ for $s > \frac{5}{2}$ be a solenoidal vector field. Then there exist a unique $T^* \in (0, +\infty]$ and a unique vector field $u \in L^\infty_{loc}([0,T^*), H^s(\boldsymbol{R}^3))$ that is a solution of the incompressible Euler equations*

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0\,, \quad \mathrm{div}_x u = 0\,, \tag{143}$$

with initial data $u_{|t=0} = u^{in}$, which satisfies in addition

$$\int_0^{T^*} \|\mathrm{curl}_x u(t,x)\|_{L^\infty(\boldsymbol{R}^3)} dt = +\infty\,.$$

Of course, as the proof of existence is based on the Cauchy-Lipschitz theorem, one easily obtains the continuous dependence of the solution with respect to the initial data $u^{in} \in H^s(\boldsymbol{R}^3)$.

This stability result is the key point of the convergence proof for classical solutions. In order to consider a larger class of solutions, one would need a more robust stability principle, yet to be discovered.

Theorem 2.2 ([42]) *Let $u \in L^\infty(\boldsymbol{R}^+; L^2(\boldsymbol{R}^3) \cap C(\boldsymbol{R}_+; w - L^2(\boldsymbol{R}^3))$ satisfy the incompressible Euler equations* (143) *in the distribution sense.*

Then, for each solenoidal vector field $w \in L^1_{loc}(\boldsymbol{R}_+; W^{1,\infty} \cap H^1(\boldsymbol{R}^3))$ and each $t \geq 0$,

$$\begin{aligned} \|u - w\|^2_{L^2}(t) &\leq \|u^{in} - w^{in}\|^2_{L^2}\, e^{2\int_0^t \|X(w)(\tau)\|_\infty d\tau} \\ &+ 2\int_0^t \left(\int (\partial_t w + w \cdot \nabla_x w) \cdot (w - u)(s,x) dx \right) e^{2\int_s^t \|X(w)(\tau)\|_\infty d\tau} ds\,, \end{aligned} \tag{144}$$

where $X(w) = \frac{1}{2}(\nabla w + (\nabla w)^T)$ is the stress tensor associated with w.

In particular, if w is a smooth solution to (143) *on* $[0,T]$

$$\textit{for each } t \in [0,T]\,, \quad \|u - w\|^2_{L^2}(t) \leq \|u^{in} - w^{in}\|^2_{L^2} e^{2\int_0^t \|X(w)(\tau)\|_\infty d\tau}\,.$$

Notice an important feature of this stability result: while the stability is measured in the L^2 norm, it requires that at least one of the solutions be Lipschitz continuous.

This inequality can be used to define dissipative solutions of the incompressible Euler equations [**42**]. Such solutions always exist globally in time, but are not known to be weak solutions of the equation in conservative form. They coincide with the unique smooth solution having the same initial data as long as the latter does exist.

In particular, the convergence result stated in Theorem 1.1 can be naturally extended into a global convergence result towards these dissipative solutions.

6.3 - *The relative entropy method*

Equipped with these basic results on the limiting system, we now explain the strategy of the proof. We proceed actually by analogy : we have to build some functional that is the analogue for the scaled Boltzmann equation of the L^2 norm for the incompressible Euler system.

The functional which measures the stability for the scaled Boltzmann equation is obtained naturally from the relative entropy $H(F_\varepsilon|M)$ that is a nonnegative Lyapunov functional for the Boltzmann equation (see chapter 1), and controls the size of the fluctuation in incompressible regimes (see chapter 5).

The modulated entropy is then defined for each solenoidal vector field $w \in C_c^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}^3)$ by

$$
\begin{aligned}
H(F_\varepsilon(t,\cdot,\cdot))|\mathcal{M}_{(1,\varepsilon w(t,\cdot),1)}) = H(F_\varepsilon(t,\cdot,\cdot))|M) \\
+\frac{1}{2}\iint (|v-\varepsilon w|^2 - |v|^2) F_\varepsilon(t,x,v)dvdx
\end{aligned}
\tag{145}
$$

The core of the proof is therefore to establish a stability inequality on the modulated entropy of the same type as (144). This will provide the convergence of the modulated entropy to zero as $\varepsilon \to 0$ under the assumption (142). Finally, we conclude by proving that the relative entropy $H(f\,|\,g)$ controls the L^1 norm of the difference $f-g$.

The specific nature of the limiting system does not occur in the proof, except in defining the maximal time interval over which the hydrodynamic limit is established: the length of that interval is indeed the life time of the classical solution of the limiting equation defined by the initial data chosen at the beginning.

The idea of using the notion of relative entropy for this kind of problems comes from the notion of entropic convergence developped by C. Bardos, F. Golse and C.D.

Levermore in [**5**], and on the other hand from Yau's elegant derivation of the hydrodynamic limit of the Ginzburg-Landau lattice model [**58**].

Applying the relative entropy method to the case of the Boltzmann equation, the convergence of renormalized solutions of the scaled Boltzmann equation to solutions of the incompressible Euler equations for well-prepared data is established in [**8**] assuming

(i) the local conservation of momentum which is not guaranteed for renormalized solutions of the Boltzmann equation — see chapter 3 for a discussion of this particular point; and

(ii) some control on the decay of the relative number density fluctuations at large velocities, in the form of a nonlinear weak compactness estimate.

Under the same assumptions, a similar analysis leads to a convergence result in the Navier-Stokes regime [**23**]. The problems one encounters when trying to implement the principle described in the previous paragraph are indeed essentially due to the structure of the Boltzmann equation, especially to the lack of a priori bounds on particle velocities and to the shortcomings in the notion of renormalized solution.

In any case, assumption (i) was removed by Lions and Masmoudi in [**44**]; their argument uses the matrix-valued defect measure in the local momentum conservation satisfied by renormalized solutions of the Boltzmann equation (see chapter 3). That this defect measure vanishes in the incompressible Euler limit follows from the DiPerna-Lions entropy inequality presented in chapter 3.

Another, more serious difficulty is to circumvent the need for assumption (ii). This requires estimates on large velocities using in a crucial way the dissipation control given by the H-theorem [**53**]. Notice that we still do not know whether the weak compactness statement assumed in [**8**] is true. Instead, we introduce a suitable decomposition of the momentum flux, and estimate each term in that decomposition either by the modulated entropy, or by the entropy production. In other words, the argument is based on loop estimates instead of a priori estimates, and the conclusion follows from Gronwall's inequality. This strategy based on the Gronwall inequality has been first used in the framework of the BGK equation [**51**], and then adapted to the original Boltzmann equation [**53**] using refined dissipation estimates from [**28**] briefly discussed in chapter 5 (Lemma 3.4).

6.3.1 - Derivation of the modulated entropy

The starting point is therefore the entropy inequality with defect measure (62) satisfied by renormalized solutions relatively to M of the scaled Boltzmann equation:

$$H(F_\varepsilon(t)|M) + \int_{\boldsymbol{R}^3} \operatorname{trace}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \iint D(F_\varepsilon)(s,x,v)dsdxdv \leq H(F_\varepsilon^{in}|M) \tag{146}$$

where $m_\varepsilon \in L^\infty(\boldsymbol{R}^+, \mathcal{M}(\boldsymbol{R}^3, M_3(\boldsymbol{R})))$ is the momentum defect measure.

By definition of the modulated entropy (145), we then have

$$\begin{aligned} H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1})(t) &+ \int_{\boldsymbol{R}^3} \operatorname{trace}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \iint D(F_\varepsilon)dsdxdv \\ &\leq H(F_\varepsilon^{in}|\mathcal{M}_{1,\varepsilon w^{in},1}) + \int_0^t \frac{d}{dt} \iint \frac{1}{2}(\varepsilon^2 w^2 - 2\varepsilon v\cdot w)F_\varepsilon dvdx. \end{aligned}$$

From the continuity equation

$$\partial_t \int F_\varepsilon dv + \operatorname{div}_x \frac{1}{\varepsilon}\int vF_\varepsilon dv = 0\,, \tag{147}$$

and the conservation of momentum with defect measure

$$\partial_t \int vF_\varepsilon dv + \operatorname{div}_x \frac{1}{\varepsilon}\int v\otimes vF_\varepsilon dv + \frac{1}{\varepsilon}\operatorname{div}_x m_\varepsilon = 0\,, \tag{148}$$

we deduce that

$$\begin{aligned} H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1})(t) &+ \int_{\boldsymbol{R}^3} \operatorname{trace}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \iint D(F_\varepsilon)dsdxdv \\ &\leq H(F_\varepsilon^{in}|\mathcal{M}_{1,\varepsilon w^{in},1}) + \int_0^t \iint \varepsilon\partial_t w\cdot(\varepsilon w - v)F_\varepsilon(s,x,v)dvdxds \\ &- \frac{1}{2}\int_0^t \int \varepsilon w^2(s,x)\operatorname{div}_x\left(\int vF_\varepsilon(s,x,v)dv\right)dxds \\ &+ \int_0^t \int w\cdot \operatorname{div}_x\left(\int v\otimes vF_\varepsilon(s,x,v)dv + m_\varepsilon(s,x)\right)dxds. \end{aligned}$$

Integrating by parts and rewriting $v = v - \varepsilon w + \varepsilon w$, we obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon^2} H(F_\varepsilon | \mathcal{M}_{1,\varepsilon w,1})(t) + \frac{1}{\varepsilon^2} \int_{\mathbf{R}^3} \operatorname{trace}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+3}} \int_0^t \iint D(F_\varepsilon) ds dx dv \\
&\qquad \le \frac{1}{\varepsilon^2} H(F_\varepsilon^{in} | \mathcal{M}_{1,\varepsilon w^{in},1}) - \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbf{R}^3} X(w) : m_\varepsilon ds \\
&\qquad + \frac{1}{\varepsilon} \int_0^t \iint (\partial_t w + w \cdot \nabla_x w) \cdot (\varepsilon w - v) F_\varepsilon(s,x,v) dv dx ds \\
&\qquad - \frac{1}{\varepsilon^2} \int_0^t \iint X(w) : (v - \varepsilon w)^{\otimes 2} F_\varepsilon(s,x,v) dv dx ds,
\end{aligned}
\tag{149}
$$

where we recall that $X(w) = \frac{1}{2}(\nabla w + (\nabla w)^T)$ is the stress tensor associated to the vector field w.

This inequality satisfied by renormalized solutions of the scaled Boltzmann equation (139) relatively to M is very similar to (144). In the sequel, we shall explain how to use this analogy.

6.3.2 - Constraints on the weak limit

In order to deal with the acceleration term

$$\frac{1}{\varepsilon} \iint (\partial_t w + w \cdot \nabla_x w) \cdot (\varepsilon w - v) F_\varepsilon(s,x,v) dx dv \,,$$

we use the weak compactness implied by the uniform entropy bound

$$\frac{1}{\varepsilon^2} H(F_\varepsilon | M)(t) \le \frac{1}{\varepsilon^2} H(F_\varepsilon^{in} | M) \le C^{in} \,.$$

By Lemma 3.1 in chapter 5, we have

$$\int F_\varepsilon dv \to 1 \text{ strongly in } L^1_{loc}(dtdx) \,,$$

and

$$\frac{1}{\varepsilon} \int v (F_\varepsilon - M) dv = \frac{1}{\varepsilon} \int v F_\varepsilon dv \rightharpoonup \bar{u} \text{ weakly in } L^1_{loc}(dtdx) \,,$$

possibly after extraction of a subsequence.

Hence the acceleration term satisfies

$$\tag{150} \begin{aligned} &\frac{1}{\varepsilon}\iint(\partial_t w + w\cdot\nabla_x w)\cdot(\varepsilon w - v)F_\varepsilon(s,x,v)dxdv \\ &\qquad \rightharpoonup \int(\partial_t w + w\cdot\nabla_x w)\cdot(w-\bar u)(s,x)dx \end{aligned}$$

in $L^1_{loc}(\boldsymbol{R}_+)$.

Besides, taking limits in the continuity equation (147), we see that $\bar u$ satisfies the incompressibility condition

$$\mathrm{div}_x \bar u = 0\,.$$

6.3.3 - Characterization of the limit by the stability inequality

It remains to handle the flux term, namely with

$$\begin{aligned} &-\frac{1}{\varepsilon^2}\int_0^t\iint X(w):(v-\varepsilon w)^{\otimes 2}F_\varepsilon(s,x,v)dvdxds \\ &= -\frac{1}{\varepsilon^2}\int_0^t\iint X(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon - \mathcal{M}_{1,\varepsilon w,1})(s,x,v)dvdxds\,. \end{aligned}$$

Passing to the limit directly in this term requires a priori estimates that control the effect of large velocities and entail equiintegrability with respect to the x variable. Such estimates were referred to as assumption (ii) above. However there is little hope to establish such a claim : with this type of scaling, we do not know how to obtain a priori compactness with respect to space variables.

The key new idea is therefore to estimate the flux in terms of the modulated entropy and of the entropy dissipation, in the following manner.

Lemma 3.1. *Under the assumptions of Theorem* 1.1, *for each* $K>0$, *there exists some nonnegative constant* $C_K \le C\sqrt{K}$ *such that, for all solenoidal vector field* $w\in C^\infty_c(\boldsymbol{R}\times\boldsymbol{R}^3)$,

$$\tag{151} \begin{aligned} &-\frac{1}{\varepsilon^2}\int_0^t\iint X(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon - \mathcal{M}_{1,\varepsilon w,1})(s,x,v)dvdxds \\ &\qquad \le \frac{C_K}{\varepsilon^2}\int_0^t \|X(w)\|_\infty H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1})(s)ds \\ &\qquad + O(C_K\varepsilon^{(q-1)/2}) + O\left(\sqrt{K}\exp\left(-\frac{K}{4}\right)\right). \end{aligned}$$

The proof of this lemma is given in the last part of this chapter. It is based on some convenient decompositions of $(F_\varepsilon - \mathcal{M}_{1,\varepsilon w,1})$, using a square-root renormalization as in (134)-(136), and on the fact that the collision integral $\mathcal{B}(F_\varepsilon, F_\varepsilon)$ is $O(\varepsilon^{(q+3)/2})$ in some sense as $\varepsilon \to 0$.

Taking this lemma for granted, it is easy to finish the proof of Theorem 1.1.

Indeed we deduce from (149) and (151) that

$$\begin{aligned}
&\frac{1}{\varepsilon^2} H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1})(t) + \frac{1}{\varepsilon^2}\int_{\mathbf{R}^3} \operatorname{trace}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+3}}\int_0^t \iint D(F_\varepsilon) dv dx ds \\
&\le \frac{1}{\varepsilon^2} H(F_\varepsilon^{in}|\mathcal{M}_{1,\varepsilon w^{in},1}) + \frac{1}{\varepsilon^2}\int_0^t \|X(w)\|_\infty \int_{\mathbf{R}^3} \operatorname{trace}(m_\varepsilon)(s) ds \\
&+ \frac{1}{\varepsilon}\int_0^t \iint (\partial_t w + w\cdot\nabla_x w)\cdot(\varepsilon w - v) F_\varepsilon(s,x,v) dv dx ds \\
&+ \frac{C_K}{\varepsilon^2}\int_0^t \|X(w)\|_\infty H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1})(s) ds + O(C_K \varepsilon^{(q-1)/2}) \\
&+ O\left(\sqrt{K}\exp\left(-\frac{K}{4}\right)\right).
\end{aligned}$$

Integrating next this differential inequality leads to

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\left(H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1})(t) + \int_{\mathbf{R}^3} \operatorname{trace}(m_\varepsilon)(t) \right) \\
&\le \frac{1}{\varepsilon^2} H(F_\varepsilon^{in}|\mathcal{M}_{1,\varepsilon w^{in},1}) \exp\left(C_K \int_0^t \|X(w)\|_\infty (s) ds \right) \\
&+ \frac{1}{\varepsilon}\int_0^t \iint (\partial_t w + w\cdot\nabla_x w)\cdot(\varepsilon w - v) F_\varepsilon(s,x,v) \exp\left(C_K \int_s^t \|X(w)\|_\infty(\sigma) d\sigma \right) dv dx ds \qquad (152) \\
&+ O\left(C_K \varepsilon^{(q-1)/2} \exp\left(C_K \int_s^t \|X(w)\|_\infty(\sigma) d\sigma \right) dv dx ds \right) \\
&+ O\left(\sqrt{K}\exp\left(-\frac{K}{4}\right) \exp\left(C_K \int_s^t \|X(w)\|_\infty(\sigma) d\sigma \right) dv dx ds \right)
\end{aligned}$$

which extends to all solenoidal $w \in L^\infty([0,T]; W^{1,\infty} \cap H^1(\mathbf{R}^3))$, in particular to all smooth solutions of the incompressible Euler equations.

By the convergence property (150) and the assumption (142) on the initial data, we see that, for each $K > 0$

$$\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{1,\varepsilon u,1}) = O\left(\sqrt{K}\exp\left(-\frac{K}{4}\right)\exp\left(C_K\int_0^t \|X(w)\|_\infty(s)ds\right)\right)$$

as $\varepsilon \to 0$, from which we deduce that

$$\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{1,\varepsilon u,1}) \to 0 \text{ as } \varepsilon \to 0\,.$$

This allows to identify the weak limit $\bar{u}$ of the scaled bulk velocity. Simple computations give indeed

$$\begin{aligned}\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{1,\varepsilon u,1}) &= \frac{1}{\varepsilon^2}H(M_{F_\varepsilon}|\mathcal{M}_{1,\varepsilon u,1}) + \frac{1}{\varepsilon^2}H(F_\varepsilon|M_{F_\varepsilon})\\ &\geq \frac{1}{\varepsilon^2}H(M_{F_\varepsilon}|\mathcal{M}_{1,\varepsilon u,1}) \geq \frac{1}{2}\int \rho_\varepsilon|u_\varepsilon - u|^2dx\,,\end{aligned}$$

with

$$\rho_\varepsilon = \int F_\varepsilon dv \text{ and } \rho_\varepsilon u_\varepsilon = \frac{1}{\varepsilon}\int vF_\varepsilon dv\,,$$

from which we deduce that $\bar{u} = u$.

6.4 - *Estimates on the flux term*

In order to estimate the flux in terms of the modulated entropy and of the entropy production as stated in Lemma 3.1, we need a suitable decomposition well adapted to the structure of the collision operator.

The main idea is that the local Maxwellian M_{F_ε}, uniquely defined as the local Maxwellian such that

$$\int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} M_{F_\varepsilon} dv = \int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_\varepsilon dv\,, \tag{153}$$

is expected to give a good approximation of the distribution F_ε, provided that its moments remain bounded.

Now, for Maxwellian distributions, the flux term

$$-\frac{1}{\varepsilon^2}\iint X(w) : (v - \varepsilon w)^{\otimes 2} M_{F_\varepsilon}(s,x,v)dvdx$$

can be computed explicitly in terms of the density, mean velocity and temperature of M_{F_ε} and estimated by the modulated entropy $H(M_{F_\varepsilon}|\mathcal{M}_{1,\varepsilon w,1})$ that is more or less equivalent to the L^2 norm of the moments of $F_\varepsilon - \mathcal{M}_{(1,2w,1)}$ — see [**51**].

The first difficulty to apply this strategy is to obtain a control on the relaxation to local Maxwellians of the number density. In the case of the Boltzmann equation, the entropy production is not known to measure the distance between F_ε and M_{F_ε}.

A second difficulty to be addressed is related to cases where moments are far from their asymptotic values (i.e. become very large pointwise), or to singular cases where the macroscopic density or temperature vanish.

6.4.1 - Decomposition of the flux term

The second difficulty is handled by a macroscopic truncation as follows: let

$$\chi_\varepsilon = \mathbf{1}\left\{(t,x)\,\Big|\,\int \mathcal{M}_{(1,\varepsilon w,1)} h\left(\frac{F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)}}{\mathcal{M}_{(1,\varepsilon w,1)}}\right) dv \le \eta\right\}. \tag{154}$$

If the moments are far from their asymptotic values in the sense that $\chi_\varepsilon = 0$, the flux term is estimated directly by the modulated entropy, using both the Young and Bienaymé-Chebyshev inequalities:

$$\begin{aligned} &\frac{1}{\varepsilon^2}\iint (1-\chi_\varepsilon)|v-\varepsilon w|^2 |F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)}| dx dv \\ &\le \frac{4}{\varepsilon^2}\iint \mathcal{M}_{(1,\varepsilon w,1)}\left(h\left(\frac{F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)}}{\mathcal{M}_{(1,\varepsilon w,1)}}\right) + (1-\chi_\varepsilon)h^*\left(\frac{|v-\varepsilon w|^2}{4}\right)\right) dv dx \\ &\le \frac{C(1+\eta^{-1})}{\varepsilon^2} H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon w,1)}). \end{aligned} \tag{155}$$

If the moments are bounded, i.e. if $\chi_\varepsilon = 1$, we proceed as explained at the beginning of this section, namely using the fact that F_ε is expected to behave asymptotically as the corresponding Maxwellian M_{F_ε}. In order to make up for the lack of control on the relaxation, we replace the decomposition

$$F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)} = F_\varepsilon - M_{F_\varepsilon} + M_{F_\varepsilon} - \mathcal{M}_{(1,\varepsilon w,1)}$$

used to study the flux term of the BGK equation [**52**] with a decomposition using the orthogonal projection Π_w on the nullspace of the collision integral linearized at $\mathcal{M}_{(1,\varepsilon w,1)}$:

$$\Pi_w : L^2(\mathcal{M}_{(1,\varepsilon w,1)} dv) \to \ker(\mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}}).$$

Consider the decomposition

$$\frac{1}{\varepsilon^2}(F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)}) = \mathcal{M}_{(1,\varepsilon w,1)}\left(\frac{1}{\varepsilon}f_\varepsilon + \frac{1}{4}f_\varepsilon^2\right)$$
$$\text{with } f_\varepsilon = \frac{2}{\varepsilon}\left(\sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}} - 1\right). \tag{156}$$

By Lemma 3.2 in chapter 5, we expect the L^1 norm of the second term to be controlled by the modulated entropy.

The second step is to understand the relaxation mechanism which allows getting control on the first term (which is unbounded but has a bounded momentum flux). Using the Fredholm property of $\mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}}$ (obtained from Theorem 6.3 in chapter 1 by the transformation (20)), we check that the kinetic flux A_w defined by

$$A_w(v) = (v - \varepsilon w)^{\otimes 2} - \frac{1}{2}|v - \varepsilon w|^2 I \tag{157}$$

is orthogonal to the nullspace $\ker(\mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}})$, so that there exists a unique $\tilde{A}_w \in \ker(\mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}})^\perp$ such that

$$\mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}}(\tilde{A}_w) = A_w .$$

In particular, as $\mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}}$ is symmetric,

$$\frac{1}{\varepsilon}\int \mathcal{M}_{(1,\varepsilon w,1)} A_w f_\varepsilon dv = \frac{1}{\varepsilon}\int \mathcal{M}_{(1,\varepsilon w,1)} \tilde{A}_w \mathcal{L}_{\mathcal{M}_{(1,\varepsilon w,1)}}(f_\varepsilon) dv .$$

From now on, we use the notation

$$\mathcal{L}_w g = -2\mathcal{M}^{-1}_{(1,\varepsilon w,1)}\mathcal{B}(\mathcal{M}_{(1,\varepsilon w,1)}, \mathcal{M}_{(1,\varepsilon w,1)}g),$$
$$\mathcal{Q}_w(g,g) = \mathcal{M}^{-1}_{(1,\varepsilon w,1)}\mathcal{B}(\mathcal{M}_{(1,\varepsilon w,1)}g, \mathcal{M}_{(1,\varepsilon w,1)}g). \tag{158}$$

Since the collision integral is a quadratic operator, we get

$$\frac{1}{\varepsilon}\int \mathcal{M}_{(1,\varepsilon w,1)} A_w f_\varepsilon dv$$
$$= \int \mathcal{M}_{(1,\varepsilon w,1)}\tilde{A}_w \left(\frac{1}{2}\mathcal{Q}_w(f_\varepsilon, f_\varepsilon) - \frac{2}{\varepsilon^2}\mathcal{Q}_w\left(\sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}, \sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}\right)\right) dv . \tag{159}$$

Using (156) and (159), we obtain a control on the flux in terms of the modulated entropy and of the entropy dissipation whenever the velocities remain bounded, which is of course not always the case.

The last point consists then in gaining some integrability with respect to the v variable. In order to do that, we apply the strategy presented in the previous chapter

(Lemma 3.4). We introduce the projection P_w on the kernel of $\mathcal{L}_w$

$$f_\varepsilon = P_w f_\varepsilon + (f_\varepsilon - P_w f_\varepsilon), \tag{160}$$

where the first term is a polynomial of degree 2 in the v variable and thus is integrable against any polynomial in v, and the second term is expected to converge to 0.

Using (156), (159) and (160), together with the identity

$$\mathcal{Q}_w(\Pi_w f, \Pi_w f) = \frac{1}{2}\mathcal{L}_w((\Pi_w f)^2)\,,$$

(see chapter 4, Lemma 4.2), we eventually arrive at the following decomposition

$$\begin{aligned}
&\int \frac{1}{\varepsilon^2}(F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)})A_w dv \\
&= \frac{1}{2}\int \mathcal{M}_{(1,\varepsilon w,1)} A_w (\Pi_w f_\varepsilon)^2 dv \\
&+ \frac{1}{4}\int \mathcal{M}_{(1,\varepsilon w,1)} A_w (f_\varepsilon - \Pi_w f_\varepsilon)(f_\varepsilon + \Pi_w f_\varepsilon) dv \\
&+ \frac{1}{2}\int \mathcal{M}_{(1,\varepsilon w,1)} \tilde{A}_w \mathcal{Q}_w (f_\varepsilon - \Pi_w f_\varepsilon, f_\varepsilon + \Pi_w f_\varepsilon) dv \\
&- \frac{2}{\varepsilon^2}\int \mathcal{M}_{(1,\varepsilon w,1)} \tilde{A}_w \mathcal{Q}_w \left(\sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}, \sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}\right) dv \\
&= I_1 + I_{21} + I_{22} + I_3\,.
\end{aligned} \tag{161}$$

6.4.2 - Control by the entropy dissipation

The term I_3 measures in some sense the relaxation of F_ε to the manifold of local Maxwellians, and is therefore expected to be controlled by the entropy production.

By the elementary inequality

$$(x - y)\ln(x/y) \geq 4(\sqrt{x} - \sqrt{y})^2\,, \quad x, y > 0\,,$$

and the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left\| \mathcal{Q}_w \left(\sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}, \sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}\right)\right\|^2_{L^2(\nu_w^{-1}\mathcal{M}_{(1,\varepsilon w,1)} dv dx dt)} \\
&\qquad \leq C \int_0^{+\infty} \iint D(F_\varepsilon) dv dx dt \leq C C^{in} \varepsilon^{q+3}\,,
\end{aligned} \tag{162}$$

where

$$\nu_w = \iint |(v - v_*)\cdot \omega| \mathcal{M}_{(1,\varepsilon w,1)}(v_*) dv_* d\omega\,.$$

Hence

$$\|I_3\|_{L^2(dtdx)} \leq 2\sqrt{CC^{in}}\varepsilon^{(q-1)/2}\|\tilde{A}\|_{L^2(\upsilon M dv)}\,, \tag{163}$$

from which we conclude that I_3 converges strongly to 0 in $L^2(dtdx)$.

By the weighted coercivity estimate stated in chapter 1, Theorem 6.3 and the transformation (20), we also expect the entropy production will control the distance between f_ε and its hydrodynamic projection $\Pi_w f_\varepsilon$ on $\ker(\mathcal{L}_w)$.

From the identity

$$\mathcal{L}_w f_\varepsilon = \frac{\varepsilon}{2}\mathcal{Q}_w(f_\varepsilon, f_\varepsilon) - \frac{2}{\varepsilon}\mathcal{Q}_w\left(\sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}, \sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}\right)$$

the L^2 bound (162) and the continuity of $\mathcal{Q}_w$ obtained by translation invariance (20) from Lemma 3.3 of chapter 5

$$\|\mathcal{Q}_w(\phi,\phi)\|_{L^2(\upsilon_w^{-1}\mathcal{M}_{(1,\varepsilon w,1)}dv)} \leq C\|\phi\|_{L^2(\upsilon_w \mathcal{M}_{(1,\varepsilon w,1)}dv)}\|\phi\|_{L^2(\mathcal{M}_{(1,\varepsilon w,1)}dv)}$$

we get

$$\begin{aligned}
&\iint \chi_\varepsilon f_\varepsilon \mathcal{L}_w f_\varepsilon \mathcal{M}_{(1,\varepsilon w,1)} dvdx \\
&\leq \frac{1}{2}C\|\varepsilon\chi_\varepsilon f_\varepsilon\|_{L^\infty(\boldsymbol{R}^3, L^2(\mathcal{M}_{(1,\varepsilon w,1)}dv))}\|\chi_\varepsilon f_\varepsilon\|^2_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dvdx)} \\
&+ \|\chi_\varepsilon f_\varepsilon\|_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dvdx)}\left\|\frac{2}{\varepsilon}\mathcal{Q}_w\left(\sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}, \sqrt{\frac{F_\varepsilon}{\mathcal{M}_{(1,\varepsilon w,1)}}}\right)\right\|_{L^2(\upsilon_w^{-1}\mathcal{M}_{(1,\varepsilon w,1)}dvdx} \\
&\leq C'\sqrt{\eta}\|\chi_\varepsilon f_\varepsilon\|^2_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dvdx)} \\
&+ O(\varepsilon^{(q+1)/2})_{L^2(dt)}\|\chi_\varepsilon f_\varepsilon\|_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dvdx)} \\
&\leq C''\sqrt{\eta}\left(\|\chi_\varepsilon(f_\varepsilon - \Pi_w f_\varepsilon)\|^2_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dvdx)} + \frac{1}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon w,1)})\right) \\
&+ \frac{1}{\sqrt{\eta}}O(\varepsilon^{q+1})_{L^1(dt)}
\end{aligned}$$

using Cauchy-Schwarz inequality. Then, by the coercivity estimate

$$\|\chi_\varepsilon(f_\varepsilon - \Pi_w f_\varepsilon)\|^2_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dv)} \leq C\iint \chi_\varepsilon f_\varepsilon \mathcal{L}_w f_\varepsilon \mathcal{M}_{(1,\varepsilon w,1)} dvdx$$

we see that, for η small enough,

$$\begin{aligned}
\|\chi_\varepsilon(f_\varepsilon - \Pi_w f_\varepsilon)\|^2_{L^2(\upsilon_w\mathcal{M}_{(1,\varepsilon w,1)}dvdx)} &\leq \frac{C_\eta}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon w,1)}) \\
&+ \frac{1}{\sqrt{\eta}}O(\varepsilon^{q+1})_{L^1(dt)}\,,
\end{aligned} \tag{164}$$

where we recall that

$$\upsilon_w(v) \geq \frac{1}{C}(1 + |v - \varepsilon w|).$$

Truncating large velocities $|v - \varepsilon w|^2 \geq K$ in the integrand of I_{21} for some fixed $K > 0$ gives

$$\begin{aligned} &\|\chi_\varepsilon I_{21}^{<}\|_{L^1(dx)} \leq C\sqrt{K}\left(\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1}) + O(\varepsilon^{q+1})_{L^1(dt)}\right), \\ &\|\chi_\varepsilon I_{22}\|_{L^1(dx)} \leq C\left(\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{1,\varepsilon w,1}) + O(\varepsilon^{q+1})_{L^1(dt)}\right). \end{aligned} \tag{165}$$

6.4.3 - Control by the relative entropy

Finally we control large velocities in the integrand of I_{21}. By Young's inequality

$$\begin{aligned} &\frac{1}{\varepsilon}|F_\varepsilon - \mathcal{M}_{(1,\varepsilon w,1)}|\,|v - \varepsilon w|^2 \mathbf{1}_{|v-\varepsilon w|^2 \geq K} \\ &\qquad \leq \frac{4}{\varepsilon^2}\mathcal{M}_{(1,\varepsilon w,1)}\left(h(\varepsilon f_\varepsilon) + h^*\left(\frac{1}{4}\varepsilon|v - \varepsilon w|^2\right)\mathbf{1}_{|v-\varepsilon w|^2 \geq K}\right). \end{aligned}$$

Estimating then the tail of the Maxwellian implies that

$$\|I_{21}^{>}\|_{L^1(dx)} \leq \frac{4}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon w,1)}) + C\sqrt{K}\exp\left(-\frac{1}{4}K\right). \tag{166}$$

Finally we estimate the nonlinear term I_1. We start with the following explicit computation:

$$\frac{1}{2}\int \mathcal{M}_{(1,\varepsilon w,1)}A_w(\Pi_w f_\varepsilon)^2 dv = \left(z \otimes z - \frac{1}{3}|z|^2 I\right),$$

where

$$z = \int \mathcal{M}_{(1,\varepsilon w,1)}\Pi_w f_\varepsilon(v - \varepsilon w)dv\,.$$

By the Cauchy-Schwarz inequality

$$\|I_1\|_{L^1(dx)} \leq C\big\|\mathcal{M}_{(1,\varepsilon w,1)}(\Pi_w f_\varepsilon)^2\big\|_{L^1(dxdv)} \leq C\big\|\mathcal{M}_{(1,\varepsilon w,1)}f_\varepsilon^2\big\|_{L^1(dxdv)}\,.$$

As the L^2-norm of f_ε is controlled by the modulated entropy, we finally obtain

$$\|I_1\|_{L^1(dx)} \leq \frac{C}{\varepsilon^2}H(F_\varepsilon|\mathcal{M}_{(1,\varepsilon w,1)}). \tag{167}$$

Combining (161) with (163), (165), (166) and (167), we have completed the proof of Lemma 3.1.

7 - From Boltzmann to Navier-Stokes

We shall conclude this survey with a description of the main ideas in the proof of the incompressible Navier-Stokes limit of the Boltzmann equation.

7.1 - *Statement of the problem and main result*

We start from the dimensionless Boltzmann equation in the incompressible Navier-Stokes scaling, i.e. Kn = St = Ma = ε:

$$\begin{aligned} &\varepsilon\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon)\,,\\ &F_\varepsilon\big|_{t=0} = F_\varepsilon^{in}\,. \end{aligned} \tag{168}$$

We consider this problem in the infinite, 3-dimensional space, i.e. $x \in \boldsymbol{R}^3$, $v \in \boldsymbol{R}^3$ and $t \geq 0$; the unknown in the above Boltzmann equation is the number density $F_\varepsilon \equiv F_\varepsilon(t, x, v) \geq 0$, and $\mathcal{B}$ is the Boltzmann collision integral for a hard sphere gas.

The initial number density is assumed to satisfy the relative entropy bound

$$H(F_\varepsilon^{in}|M) \leq C^{in}\varepsilon^2\,, \tag{169}$$

where as before

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}}e^{-\frac{1}{2}|v|^2}\,.$$

As in the last two chapters, it is more convenient to consider the relative number density and the relative number density fluctuation

$$\begin{aligned} G_\varepsilon &= \frac{F_\varepsilon}{M}\,, &\text{and } g_\varepsilon &= \frac{F_\varepsilon - M}{\varepsilon M}\,,\\ G_\varepsilon^{in} &= \frac{F_\varepsilon^{in}}{M}\,, &\text{and } g_\varepsilon^{in} &= \frac{F_\varepsilon^{in} - M}{\varepsilon M}\,, \end{aligned} \tag{170}$$

instead of the number density F_ε.

The following notation for moments will be especially convenient:

$$\langle\phi\rangle = \int \phi(v)M(v)dv$$

for each $\phi \in L^1(Mdv)$.

Likewise, when dealing with quantities related to the entropy production, we shall use the notation

$$d\mu = |(v - v_*)\cdot\omega|MM_*dvdv_*d\omega$$

and we agree that the surface element $d\omega$ on the unit sphere $\boldsymbol{S}^2$ is normalized so that

$$\iiint d\mu = 1\,.$$

Then, integrals with respect to the measure $d\mu$ will be denoted

$$\langle\!\langle \psi \rangle\!\rangle = \iiint \psi(v, v_*, \omega) d\mu(v, v_*, \omega)\,,$$

for each $\psi \in L^1(d\mu)$.

Problem. Given F_ε^{in} satisfying the relative entropy bound (169) — let F_ε be, for each $\varepsilon > 0$, a renormalized solution of the scaled Boltzmann equation (168) with initial data F_ε^{in}.

Prove that the bulk velocity field fluctuation

$$u_\varepsilon(t,x) = \frac{1}{\varepsilon} \int_{\boldsymbol{R}^3} vF_\varepsilon(t,x,v)dv \rightharpoonup u(t,x)$$

in some weak topology, possibly modulo extraction of a subsequence, where u is a Leray solution of the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta_x u\,, \quad \operatorname{div}_x u = 0\,, \tag{171}$$

with initial data to be explicited and with kinematic viscosity μ (implicitly) determined in terms of Boltzmann's collision integral $\mathcal{B}$.

This problem is one part of the program initiated in [**5**]; it has been solved only recently in [**28**], [**29**]. This last reference [**29**] is the one described below; although its scope is more general than that of [**28**], it involves a new idea for handling unbounded collision kernels that actually simplifies the discussion in [**28**].

Theorem 1.1 (Golse-Saint-Raymond [**28**], [**29**]). *Let $F_\varepsilon^{in} \geq 0$ a.e. be a family of measurable functions on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ satisfying the relative entropy bound* (169) *and such that*

$$\frac{1}{\varepsilon} \int_{\boldsymbol{R}^3} vF_\varepsilon^{in} dv \rightharpoonup u^{in} \text{ in } L^1_{loc}(\boldsymbol{R}^3),$$

for some arbitrary $u^{in} \in L^2(\boldsymbol{R}^3)$. For each $\varepsilon > 0$, let F_ε be a renormalized solution of the scaled Boltzmann equation (168) *with initial data F_ε^{in}. Then, as $\varepsilon \to 0$, the family of fluctuations of bulk velocity field*

$$\frac{1}{\varepsilon} \int_{\boldsymbol{R}^3} vF_\varepsilon dv$$

is relatively compact in $L^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R}^3)$ *and each of its limit points is a weak solution of*

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta_x u\,, \quad \operatorname{div}_x u = 0\,, \qquad u\big|_{t=0} = Pu^{in}\,, \tag{172}$$

where P *denotes as previously the Leray projection onto divergence-free vector fields, and the kinematic viscosity is given by the formula*

$$\mu = \frac{1}{10}\int_{\boldsymbol{R}^3} \tilde{A} : AMdv\ .$$

Furtermore, this weak solution u *satisfies the Leray type energy inequality*

$$\frac{1}{2}\int_{\boldsymbol{R}^3} |u(t)|^2 dx + \int_0^{+}\int_{\boldsymbol{R}^3} \mu|\nabla_x u|^2 dxds \le \varliminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}H(F^{in}_\varepsilon|M)$$

for each $t \ge 0$.

We indeed recall from chapter 2 that the tensor field $A(v) = v^{\otimes 2} - \frac{1}{3}|v|^2 I$ and the vector field $B(v) = \frac{1}{2}(|v|^2 - 5)v$ both belong to the range of the linearization $\mathcal{L}_M$ at M of the collision integral, and denote

$$\tilde{A} = \mathcal{L}_M^{-1}A \in (\ker\mathcal{L}_M)^{\perp}\,, \quad \tilde{B} = \mathcal{L}_M^{-1}B \in (\ker\mathcal{L}_M)^{\perp}\,.$$

Because the linearized collision integral $\mathcal{L}_M$ is rotation-invariant, we recall that $\tilde{A}$ and $\tilde{B}$ are of the form

$$\tilde{A}(v) = a(|v|)A(v)\,, \quad \tilde{B}(v) = \beta(|v|)B(v)$$

where a and β are radial, scalar functions.

Notice that the above statement does not involve a heat equation. This, however, is by no means a limitation of the method in [**28**], [**29**], by which one can also derive a system consisting of the incompressible Navier-Stokes equation coupled to an energy equation under the assumption of Boussinesq balance.

7.2 - *Method of proof*

In the remaining part of this paper, we outline the main ideas in the proof of the Navier-Stokes limit theorem above (Theorem 1.1). Of course, this proof more or less follows the formal argument presented above. However, several key properties of the solutions of the scaled Boltzmann equation used in this formal argument — such as, for instance, the local conservation laws of momentum and energy — are not known to be satisfied by renormalized solutions. Hence the proof sketched below differs noticeably from the formal argument in several places, yet the general idea remains essentially the same.

7.2.1 - A priori bounds

By Theorem 7.2 of chapter 3, renormalized solutions of the scaled Boltzmann equation (168) relatively to M satisfy the entropy inequality for each $t \geq 0$:

$$H(F_\varepsilon(t)|M) + \frac{1}{4\varepsilon^2}\int_0^t \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(G_\varepsilon) d\mu dx ds \leq H(F_\varepsilon^{in}|M)\,,$$

where $d(G_\varepsilon)$ is the entropy production integrand defined by (29). Because of the initial entropy bound (169), one has, for each $t \geq 0$, the relative entropy bound

$$H(F_\varepsilon(t)|M) \leq C^{in}\varepsilon^2\,, \tag{173}$$

as well as the entropy production estimate

$$\int_0^{+\infty} \int_{\boldsymbol{R}^3} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} d(G_\varepsilon) d\mu dx ds \leq C^{in}\varepsilon^4\,. \tag{174}$$

We recall from Lemma 3.2 in chapter 5 that the relative entropy bound (173) implies that

$$\int \left\langle \left(\sqrt{G_\varepsilon} - 1\right)^2 \right\rangle dx \leq C^{in}\varepsilon^2\,; \tag{175}$$

likewise, the entropy production bound (176) implies that

$$\int_0^{+\infty} \int \left\langle\!\!\left\langle \left(\sqrt{G_\varepsilon' G_{\varepsilon *}'} - \sqrt{G_\varepsilon G_{\varepsilon *}}\right)^2 \right\rangle\!\!\right\rangle dx dt \leq C^{in}\varepsilon^4\,. \tag{176}$$

7.2.2 - Normalizing functions

As explained in our description of the DiPerna-Lions existence theorem, the Boltzmann equation can be equivalently renormalized with any admissible nonlinearity whose derivative saturates the quadratic growth of the collision integral.

Throughout the proof of the Navier-Stokes limit theorem, we shall essentially use two kinds of normalizing nonlinearities:

- compactly supported nonlinearities that coincide with the identity near the reference Maxwellian state; and
- variants of the maximal, i.e. square-root renormalization.

Nonlinearities of the first kind are used to define the renormalized form of the Boltzmann equation in which one passes to the limit as $\varepsilon \to 0$, while the square-root normalization is used to establish compactness properties of the family of solutions to the scaled Boltzmann equation.

The first kind of normalizing nonlinearities is defined through the class of bump functions $\gamma \in C^\infty(\boldsymbol{R}_+)$ such that

$$\gamma\big|_{[0,3/2]} \equiv 1\,, \quad \gamma\big|_{[2,+\infty)} \equiv 0\,, \quad \gamma \text{ is nonincreasing on } \boldsymbol{R}_+\,. \tag{177}$$

The Boltzmann equation is then renormalized with the nonlinearity

$$\Gamma(Z) = (Z-1)\gamma(Z)\,;$$

later on, we denote

$$\hat{\gamma}(Z) = \frac{d}{dZ}((Z-1)\gamma(Z)) = \Gamma'(Z)\,. \tag{178}$$

The scaled Boltzmann equation renormalized with Γ is put in the form

$$\partial_t(g_\varepsilon\gamma_\varepsilon) + \frac{1}{\varepsilon} v \cdot \nabla_x(g_\varepsilon\gamma_\varepsilon) = \frac{1}{\varepsilon^3}\hat{\gamma}_\varepsilon \mathcal{Q}(G_\varepsilon, G_\varepsilon)\,, \tag{179}$$

where we have denoted

$$\gamma_\varepsilon = \gamma(G_\varepsilon)\,, \qquad \hat{\gamma}_\varepsilon = \hat{\gamma}(G_\varepsilon)\,,$$

and where $\mathcal{Q}$ designates as previously the Boltzmann collision integral intertwined with the multiplication by M:

$$\mathcal{Q}(G,G) = M^{-1}\mathcal{B}(MG, MG)\,. \tag{180}$$

Later on, we shall pass to the limit in the momentum equation deduced from (179).

The second class of normalizing nonlinearities that we shall use to establish compactness properties of the number density fluctuations G_ε is defined as

$$\Gamma_\zeta(Z) = \sqrt{\zeta + Z}\,, \quad \zeta > 0$$

where the parameter ζ will be adapted to ε.

We saw in chapter 3 that renormalized solutions of the Boltzmann equation satisfy the local conservation of mass (i.e. the continuity equation); in terms of the number density fluctuation g_ε, this local conservation law is expressed as

$$\varepsilon\partial_t\langle g_\varepsilon\rangle + \operatorname{div}_x\langle v g_\varepsilon\rangle = 0\,. \tag{181}$$

Now, the entropy bound (173) implies that

(182) $\qquad (1+|v|^2)g_\varepsilon$ is relatively compact in weak $L^1_{loc}(dtdx; L^1(Mdv))$

—see chapter 5 , especially Lemma 3.1 in section 5.3.

Let us explain how we use (182). Modulo extraction of a subsequence, one has

$$g_\varepsilon \rightharpoonup g \text{ in } L^1_{loc}(dtdx; L^1((1+|v|^2)Mdv))$$

and hence

$$\langle g_\varepsilon\rangle \rightharpoonup \langle g\rangle \text{ and } \langle v g_\varepsilon\rangle \to \langle v g\rangle \text{ in } L^1_{loc}(dtdx)\,.$$

Passing to the limit as $\varepsilon \to 0$ in (181) leads to

$$\operatorname{div}_x\langle v g\rangle = 0\,,$$

so that, denoting

$$u = \langle v g\rangle$$

the relation above is the incompressibility condition in the Navier-Stokes equations

$$\operatorname{div}_x u = 0\,.$$

Let us now explain how the motion equation in the Navier-Stokes system is derived from the Boltzmann equation. This is of course the main part in the proof, and it involves several technicalities.

In particular, we shall need truncations in the velocity variable at a level that is tied to ε. For each function $\xi \equiv \xi(v)$, and each $K > 6$, we define

$$\xi_{K_\varepsilon}(v) = \xi(v)\mathbf{1}_{|v|^2\le K|\ln\varepsilon|}\,. \tag{183}$$

Multiplying each side of the scaled, renormalized Boltzmann equation (179) by each component of v_{K_ε} and averaging in v leads to

$$\partial_t\langle v_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\rangle + \operatorname{div}_x \boldsymbol{F}_\varepsilon(A) + \nabla_x \frac{1}{\varepsilon}\left\langle \frac{1}{3}|v|^2_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\right\rangle = \boldsymbol{D}_\varepsilon(v) \tag{184}$$

where $\boldsymbol{F}_\varepsilon(A)$ is the truncated, renormalized traceless part of the momentum flux

$$\boldsymbol{F}_\varepsilon(A) = \frac{1}{\varepsilon}\langle A_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\rangle \tag{185}$$

while $\boldsymbol{D}_\varepsilon(v)$ is the momentum conservation defect

$$\boldsymbol{D}_\varepsilon(v) = \frac{1}{\varepsilon^3}\left\langle\!\left\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon(G'_\varepsilon G'_{\varepsilon *} - G_\varepsilon G_{\varepsilon *})\right\rangle\!\right\rangle\,. \tag{186}$$

Notice that truncating large velocities in the number density, or large values thereof (which is what the renormalization procedure does) break the symmetries in the collision integral leading to the local conservation of momentum (see chapter 1): this accounts for the defect $\boldsymbol{D}_\varepsilon(v)$ on the right hand side of (184). As $\varepsilon \to 0$, $v_{K_\varepsilon} \to v$ while $G_\varepsilon \to 1$ so that $\hat{\gamma}_\varepsilon \to 1$: hence, the missing symmetries are restored in the integrand defining $\boldsymbol{D}_\varepsilon(v)$. Hence, one can hope that $\boldsymbol{D}_\varepsilon(v) \to 0$ as $\varepsilon \to 0$.

In fact, the strategy for establishing the Navier-Stokes limit theorem consists of the following three steps

- Step 1: prove that, modulo extraction of a subsequence

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to \langle vg \rangle = u \text{ in } w{-}L^1_{loc}(dtdx)\,,$$

while

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u \text{ in } C(\boldsymbol{R}_+; \mathcal{D}'(\boldsymbol{R}^3))\,,$$

where P denotes the Leray projection, i.e. the orthogonal projection on divergence-free vector fields in $L^2(\boldsymbol{R}^3)$;

- Step 2: likewise, prove that

$$\boldsymbol{D}_\varepsilon(v) \to 0 \text{ in } L^1_{loc}(dtdx)\,;$$

- Step 3: and finally prove that

$$P(\operatorname{div}_x \boldsymbol{F}_\varepsilon(A)) \to P\operatorname{div}_x\big(u^{\otimes 2}\big) - \nu\Delta_x u \text{ in } \mathcal{D}'(\boldsymbol{R}^*_+ \times \boldsymbol{R}^3)\,.$$

Once these three steps are completed, one applies P to both sides of (184), which gives

$$\partial_t P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle + P(\operatorname{div}_x \boldsymbol{F}_\varepsilon(A)) = P\boldsymbol{D}_\varepsilon(v)\,. \tag{187}$$

Taking limits in each term as $\varepsilon \to 0$ shows that u satisfies the Navier-Stokes motion equation. As for the initial condition, observe that it is guaranteed by the uniform convergence in t that is the second statement in step 1 above.

7.3 - *Vanishing of the momentum conservation defect*

We start with step 2, i.e. we explain how to prove

Proposition 3.1. *Under the same assumptions as in Theorem* 1.1

$$\boldsymbol{D}_\varepsilon(v) \to 0 \quad \text{in } L^1_{loc}(dtdx).$$

First, we start from the elementary formula

$$\begin{aligned} G'_\varepsilon G'_{\varepsilon*} - G_\varepsilon G_{\varepsilon*} &= \Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} + \sqrt{G_\varepsilon G_{\varepsilon*}}\Big) \\ &= \Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big)^2 + 2\sqrt{G_\varepsilon G_{\varepsilon*}}\Big(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\Big) \end{aligned}$$

and split the momentum conservation defect as

$$\boldsymbol{D}_\varepsilon(v) = \boldsymbol{D}^1_\varepsilon(v) + \boldsymbol{D}^2_\varepsilon(v)$$

with

$$\boldsymbol{D}^1_\varepsilon(v) = \frac{1}{\varepsilon^3}\left\langle\!\!\left\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon\Big(\sqrt{G'_\varepsilon G'_{\varepsilon *}} - \sqrt{G_\varepsilon G_{\varepsilon *}}\Big)^2\right\rangle\!\!\right\rangle$$

and

$$\boldsymbol{D}^2_\varepsilon(v) = \frac{2}{\varepsilon^3}\left\langle\!\!\left\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon\sqrt{G_\varepsilon G_{\varepsilon *}}\Big(\sqrt{G'_\varepsilon G'_{\varepsilon *}} - \sqrt{G_\varepsilon G_{\varepsilon *}}\Big)\right\rangle\!\!\right\rangle.$$

That $\boldsymbol{D}^1_\varepsilon(v) \to 0$ in $L^1_{loc}(dtdx)$ follows from the entropy production estimate (176). Setting

$$\Xi_\varepsilon = \frac{1}{\varepsilon^2}\sqrt{G_\varepsilon G_{\varepsilon *}}\Big(\sqrt{G'_\varepsilon G'_{\varepsilon *}} - \sqrt{G_\varepsilon G_{\varepsilon *}}\Big)$$

we further split $\boldsymbol{D}^2_\varepsilon(v)$ as

$$\begin{aligned}\boldsymbol{D}^2_\varepsilon(v) = &-\frac{2}{\varepsilon}\left\langle\!\!\left\langle v\mathbf{1}_{|v|^2>K_\varepsilon}\hat{\gamma}_\varepsilon\Xi_\varepsilon\right\rangle\!\!\right\rangle + \frac{2}{\varepsilon}\left\langle\!\!\left\langle v\hat{\gamma}_\varepsilon(1-\hat{\gamma}_{\varepsilon *}\hat{\gamma}'_\varepsilon\hat{\gamma}'_{\varepsilon *})\Xi_\varepsilon\right\rangle\!\!\right\rangle \\ &+\frac{1}{\varepsilon}\left\langle\!\!\left\langle (v+v_*)\hat{\gamma}_\varepsilon\hat{\gamma}_{\varepsilon *}\hat{\gamma}'_\varepsilon\hat{\gamma}'_{\varepsilon *}\Xi_\varepsilon\right\rangle\!\!\right\rangle.\end{aligned}$$

The first term is easily mastered by the entropy production estimate (176) and the following classical estimate on the tail of Gaussian integrals

$$\int_{\boldsymbol{R}^N} e^{-|v|^2/2}|v|^a\mathbf{1}_{|v|^2>R}\,dv = O\Big(R^{\frac{a+N}{2}-1}e^{-R/2}\Big) \text{ as } R \to +\infty\,.$$

Observe that the integrand in the third term has the same symmetries as the original collision integrand (before truncation in $|v|$ and renormalization). The third term is also mastered by a combination of the entropy production estimate (176) with the Gaussian tail estimate above.

The most difficult part in the analysis of the momentum conservation defect is by far the second term in the decomposition of $\boldsymbol{D}^2_\varepsilon(v)$ above. That it vanishes in $L^1_{loc}(dtdx)$ as $\varepsilon \to 0$ ultimately relies upon the following

Nonlinear compactness estimate

(188) $$(1+|v|)\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2 \text{ is uniformly integrable on } [0,T]\times K \times \boldsymbol{R}^3$$

for the measure $dtdxMdv$, for each $T > 0$ and each compact $K \subset \boldsymbol{R}^3$.

We shall not give any further details on the proof that $\boldsymbol{D}^2_\varepsilon(v) \to 0$ in $L^1_{loc}(dtdx)$, which is based on the above nonlinear compactness estimate together with the usual entropy production bound (176).

Let us however say a few words on the nonlinear compactness estimate itself. The relative entropy bound (173) is essentially as good as an $L^\infty(dt; L^2(Mdvdx))$ bound on g_ε *on the set of* (t, x, v)*'s such that* $G_\varepsilon(t, x, v) = O(1)$. Elsewhere, it essentially reduces to an $O(\varepsilon)$ bound in $L^\infty(dt; L^1(Mdvdx))$, which is quite not enough for the Navier-Stokes limit. This is why the first works on this limit *assumed* some variant of this nonlinear compactness estimate. For instance, in either [**5**] or [**43**], it was assumed that

$$(189) \qquad (1+|v|^2)\frac{g_\varepsilon^2}{1+G_\varepsilon} \text{ is uniformly integrable on } [0,T]\times K\times \boldsymbol{R}^3$$

whereas all that was known on this quantity was the estimate

$$(1+|v|^2)\frac{g_\varepsilon^2}{1+G_\varepsilon} = O(|\ln\varepsilon|) \text{ in } L^1_{loc}(dtdx; L^1(Mdv))\,.$$

This led to a decomposition of the number density fluctuation as

$$g_\varepsilon = g_\varepsilon^\flat + \varepsilon g_\varepsilon^\sharp$$

where the «good» part of the fluctuation is

$$g_\varepsilon^\flat = \frac{g_\varepsilon}{\frac{1}{2}+\frac{1}{2}G_\varepsilon} = O(1) \text{ in } L^\infty(dt; L^2(Mdvdx))\,,$$

while the «bad» part is

$$g_\varepsilon^\sharp = \frac{g_\varepsilon^2}{1+G_\varepsilon} = O(1) \text{ in } L^\infty(dt; L^1(Mdvdx))\,.$$

In later works — for instance in [**50**], [**52**] and [**28**] — this decomposition was slightly modified, as follows. Pick a bump function $\gamma \in C^\infty_c(\boldsymbol{R}^*_+)$ such that

$$\gamma\big|_{[\frac34,\frac54]} \equiv 1\,, \quad \operatorname{supp}(\gamma)\subset\left[\frac12,\frac32\right] \text{ and } 0\le\gamma\le 1$$

and define

$$g_\varepsilon^\flat = g_\varepsilon\gamma(G_\varepsilon)\,, \qquad g_\varepsilon^\sharp = \frac{1-\gamma(G_\varepsilon)}{\varepsilon}g_\varepsilon\,.$$

It was proved in [**28**] that

$$(190) \qquad |g_\varepsilon^\flat|^2 \text{ is uniformly integrable on } [0,T]\times K\times\boldsymbol{R}^3$$

for the measure $dtdxMdv$, while

$$(1+|v|^2)g_\varepsilon^\sharp = O\left(\frac{1}{\ln|\ln\varepsilon|}\right) \text{ in } L^1_{loc}(dtdx;L^1(Mdv))\,. \tag{191}$$

Observe the difference between these last two controls and (189): with the new definition of $g_\varepsilon^\flat$ and $g_\varepsilon^\sharp$, it is no longer true that $|g_\varepsilon^\flat|^2 \le Cg_\varepsilon^\sharp$, while $\left(\frac{g_\varepsilon}{1+G_\varepsilon}\right)^2 \le \frac{g_\varepsilon^2}{1+G_\varepsilon}$, so that (189) actually entailed that the square of the good part in the old flat-sharp decomposition is uniformly integrable — even with a quadratic weight in v.

In fact, the techniques in [**28**] did not allow adding a quadratic weight in v as in (189), so that this compactness assumption remained unproved; fortunately, it was possible to complete the proof of the Navier-Stokes limit for cut-off Maxwell molecules with only the bounds (190)-(191), and the weighted estimate

$$(1+|v|)^s(1-\gamma(G_\varepsilon)) = O\left(\frac{\varepsilon^2}{\sqrt{\ln|\ln\varepsilon|}}\right) \text{ in } L^1_{loc}(dtdx;L^1(Mdv))\,. \tag{192}$$

This control shows that the set where the bad part of the number density fluctuation dominates is small in weighted v-space. There is a definite lack of symmetry between the controls (191) — bearing on large values of g_ε — and (192) — bearing on large $|v|$'s. This lack of symmetry is remedied in the most recent variant (188) of these nonlinear compactness estimates (see [**29**]): we shall return to this when sketching the proof of (188).

7.4 - *The asymptotic momentum flux*

With the vanishing of conservation defects (Step 2 in the proof of the Navier-Stokes limit) settled in the previous section, we turn our attention to Step 3, i.e. passing to the limit in the divergence of the momentum flux modulo gradients. This is by far the most difficult part of our analysis, and does require several preparations. In the present section, we reduce the momentum flux to some asymptotic normal form, to which we eventually apply some compactness results to be described later.

Lemma 4.1. *Let Π be the $L^2(Mdv)$-orthogonal projection on* $\ker\mathcal{L}$*; then, under the same assumptions as in Theorem* 1.1

$$\boldsymbol{F}_\varepsilon(A) = 2\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle - 2\left\langle \tilde{A}\frac{1}{\varepsilon^2}\mathcal{Q}(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon})\right\rangle + o(1)_{L^1_{loc}(dtdx)}\,,$$

where we recall that the tensor field $\tilde{A}$ is defined by

$$\tilde{A} \perp \ker \mathcal{L} \text{ and } \mathcal{L}(\tilde{A}) = A = v \otimes v - \frac{1}{3}|v|^2 I\,.$$

The proof of this lemma is based upon splitting the momentum flux as

$$\begin{aligned}
\boldsymbol{F}_\varepsilon(A) &= \frac{1}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon \frac{G_\varepsilon - 1}{\varepsilon}\right\rangle \\
&= \left\langle A_{K_\varepsilon}\gamma_\varepsilon \left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle + \frac{2}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\rangle \\
&= \boldsymbol{F}^1_\varepsilon(A) + \boldsymbol{F}^2_\varepsilon(A)\,,
\end{aligned}$$

as a consequence of the elementary identity

$$\begin{aligned}
\frac{1}{\varepsilon}(G_\varepsilon - 1) &= \frac{1}{\varepsilon}(\sqrt{G_\varepsilon}-1)(\sqrt{G_\varepsilon}+1) \\
&= \frac{1}{\varepsilon}(\sqrt{G_\varepsilon}-1)^2 + \frac{2}{\varepsilon}(\sqrt{G_\varepsilon}-1)\,.
\end{aligned}$$

Then, one applies the following corollary of the nonlinear compactness estimate (188).

Corollary 4.2. *Under the same assumptions as in Theorem* 1.1

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} - \Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \to 0 \text{ in } L^2(dtdx; L^2((1+|v|)Mdv)$$

as $\varepsilon \to 0$.

With the corollary above, one can show that the term $\boldsymbol{F}^1_\varepsilon(A)$ in the decomposition of the momentum flux is asymptotically close to

$$\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle$$

(notice that the high velocity truncation is disposed of since $\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$ has at most polynomial growth in v as $|v| \to +\infty$). In order to deal with the second term $\boldsymbol{F}^2_\varepsilon(A)$, we introduce the following decomposition (already used in the previous chapter)

$$\begin{aligned}
\frac{2}{\varepsilon}\left\langle A\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\rangle &= \frac{2}{\varepsilon}\left\langle \tilde{A}\mathcal{L}\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\rangle \\
&= 2\left\langle \tilde{A}\mathcal{Q}\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}, \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)\right\rangle - \frac{2}{\varepsilon^2}\mathcal{Q}(\sqrt{G_\varepsilon}, \sqrt{G_\varepsilon}),
\end{aligned}$$

from which we deduce with the corollary above and Lemma 4.2 that $\boldsymbol{F}^2_\varepsilon(A)$ is close to

$$\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle - 2\left\langle \tilde{A}\,\frac{1}{\varepsilon^2}\mathcal{Q}(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon})\right\rangle .$$

Lemma 4.3. *For each $\phi \in \ker\mathcal{L}$, one has*

$$\mathcal{Q}(\phi,\phi) = \frac{1}{2}\mathcal{L}(\phi^2)\,.$$

Next, we explain how Lemma 4.1 is used in the proof of the Navier-Stokes limit. To begin with, since

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \simeq \frac{1}{2}g_\varepsilon\gamma_\varepsilon$$

one has

$$\left\langle A_{K_\varepsilon}\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle \simeq \langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle \otimes \langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle - \frac{1}{3}|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle|^2 I\ .$$

On the other hand, the entropy production estimate (176) implies that, modulo extraction of a subsequence, one has

$$\frac{1}{\varepsilon^2}\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right) \rightharpoonup q \text{ in } L^2(dtdxd\mu).$$

Passing to the limit in the scaled, renormalized Boltzmann equation (179) entails the relation

$$\begin{aligned}\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2} qb(v-v_*,\omega)Mdv_*d\omega &= v\cdot\nabla_x g\\ &= \frac{1}{2}A:\nabla_x u + \text{ terms that are odd in } v\ .\end{aligned}$$

Eventually we arrive at the following asymptotic form of the momentum flux:

Proposition 4.4. *Under the same assumptions as in Theorem* 1.1, *one has*

$$\begin{aligned}\boldsymbol{F}_\varepsilon(A) = {}& \langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle \otimes \langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle - \frac{1}{3}|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle|^2 I\\ &- \nu\left(\nabla_x u + (\nabla_x u)^T\right) + o(1)_{L^1_{loc}(dtdx)}\end{aligned}$$

where

$$u = \langle vg\rangle \text{ and } g = \lim_{\varepsilon\to 0} g_\varepsilon \text{ in } w\text{--}L^1_{loc}(dtdx;L^1(Mdv))\,.$$

7.5 - *Strong compactness of* $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$

In order to pass to the limit in the quadratic term $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ and to conclude that

$$P\mathrm{div}_x\left(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle - \frac{1}{3} |\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle|^2 I \right) \to P\mathrm{div}_x(u \otimes u - \frac{1}{3}|u|^2 I)$$

in the sense of distributions on $\boldsymbol{R}_+^* \times \boldsymbol{R}^3$ as $\varepsilon \to 0$, the weak convergence properties of g_ε established so far are clearly insufficient. One needs instead some *strong* compactness properties on the family $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$.

a) Strong compactness in the x-variable.

Velocity averaging is the natural way to obtain compactness in the space variable x for kinetic equations in the parabolic scaling (168). We refer to chapter 2 for an exposition of the main ideas and results in this direction.

For the purpose of studying the compactness of $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ in the x-variable, we use the nonlinear compactness estimate (188) coupled with the following variant of the L^2 case of the Velocity Averaging theorem.

Lemma 5.1. *Let* ϕ_ε *be a bounded family in* $L^2_{loc}(dtdx; L^2(Mdv))$ *such that* $|\phi_\varepsilon|^2$ *is locally uniformly integrable on* $\boldsymbol{R}_+^* \times \boldsymbol{R}^3 \times \boldsymbol{R}^3$ *for the Lebesgue measure. Assume that*

$$(\varepsilon\partial_t + v \cdot \nabla_x)\phi_\varepsilon \textit{ is bounded in } L^1_{loc}(dtdxdv).$$

Then, for each $\psi \in L^2(Mdv)$, *the family* $\langle \phi_\varepsilon \psi \rangle$ *is relatively compact in* $L^2_{loc}(dtdx)$ *with respect to the x-variable, meaning that, for each* $T > 0$ *and each compact* $K \subset \boldsymbol{R}^3$, *one has*

$$\iint_{[0,T]\times K} |\langle \phi_\varepsilon \psi \rangle(t, x+y) - \langle \phi_\varepsilon \psi \rangle(t,x)|^2 dtdx \to 0$$

as $y \to 0$ *uniformly in* ε.

See [28] for the proof — which mixes features of the proofs of Theorem 2.1 and Theorem 3.3 in chapter 2.

Now, we apply the lemma above to

$$\phi_\varepsilon = \frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}$$

since

$$(\varepsilon\partial_t + v \cdot \nabla_x)\phi_\varepsilon = \frac{1}{\varepsilon^2} \frac{\mathcal{Q}(G_\varepsilon, G_\varepsilon)}{2\sqrt{\varepsilon^c + G_\varepsilon}} = O(1)_{L^1_{loc}(dtdxdv)}$$

for $c \in (1, 2)$, by the entropy production estimate (176). Since

$$\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon} \simeq \frac{1}{2} g_\varepsilon ,$$

applying the Velocity Averaging lemma above leads to the following compactness («in the x-variable») result

Proposition 5.2. *Under the same assumptions as in Theorem* 1.1, *for each* $T > 0$ *and* $K \subset \boldsymbol{R}^3$ *compact, one has*

$$\iint_{[0,T]\times K} |\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle(t, x+y) - \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle(t, x)|^2 dt dx \to 0$$

uniformly in ε as $y \to 0$.

<u>b) Strong compactness in the t-variable</u>.

It remains to obtain compactness in the time variable. As we shall see, the solenoidal part of $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ is strongly compact in the t-variable, but its orthogonal complement — which is a gradient field — is not because of high frequency oscillations in t.

Proposition 5.3. *Under the assumptions of Theorem* 1.1, *modulo extraction of a subsequence, one has*

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u$$

in $C(\boldsymbol{R}_+; w{-}L^2_x)$ and in $L^2_{loc}(dtdx)$ as $\varepsilon \to 0$.

Proof. Indeed, Proposition 5.2 together with the translation invariance of the Leray projection P and the fact that P is a pseudo-differential operator of order 0 implies that

$$\iint_{[0,T]\times K} |P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle(t, x+y) - P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle(t, x)|^2 dt dx \to 0 \tag{193}$$

uniformly in ε as $y \to 0$. On the other hand, the conservation law (187) implies that

$$\partial_t \int_{\boldsymbol{R}^3} P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \cdot \xi dx = O(1) \text{ in } L^1_{loc}(dt) \tag{194}$$

for each compactly supported, solenoidal vector field $\xi \in H^3(\boldsymbol{R}^3)$, since we know from Lemma 4.1 and the bounds (175) and (176), that $\boldsymbol{F}_\varepsilon(A)$ is bounded in $L^1_{loc}(dtdx)$. Also,

$$g_\varepsilon \gamma_\varepsilon \leq (1 + \sqrt{2}) \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}$$

so that (175) implies that

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle = O(1) \text{ in } B(\boldsymbol{R}_+; L^2(\boldsymbol{R}^3))\,, \tag{195}$$

(where $B(X, Y)$ denotes the class of bounded maps from X to Y).

Since the class of H^3, compactly supported solenoidal vector fields is dense in that of all H^3 solenoidal vector fields (see Appendix A of [**42**]), (195) and (194) imply that

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \text{ is relatively compact in } C(\boldsymbol{R}_+; w{-}L^2(\boldsymbol{R}^3))\,. \tag{196}$$

As for the $L^2_{loc}(dtdx)$ compactness, observe that (196) implies that

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \text{ is relatively compact in } L^2_{loc}(dtdx)$$

where χ_δ designates any mollifying sequence. Hence

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \cdot P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \rightharpoonup Pu \cdot Pu \star \chi_\delta \text{ in } L^1_{loc}(dtdx)$$

as $\varepsilon \to 0$. By (193),

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \to P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$$

in $L^2_{loc}(dtdx)$ uniformly in ε as $\delta \to 0$. With this, we conclude that

$$|P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle|^2 \rightharpoonup |Pu|^2 \text{ in } L^1_{loc}(dtdx)$$

which implies that $P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to Pu$ strongly in $L^2_{loc}(dtdx)$. □

Next, consider

$$\nabla_x \pi_\varepsilon = \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle - P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle\,.$$

Since

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \rightharpoonup u \text{ in } L^2_{loc}(dtdx) \text{ and } \mathrm{div}_x u = 0$$

one has

$$\nabla_x \pi_\varepsilon \rightharpoonup 0 \text{ in } L^2_{loc}(dtdx) \tag{197}$$

as $\varepsilon \to 0$. Decompose then

$$\begin{aligned} P\mathrm{div}_x(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle) &= P\,\mathrm{div}_x(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle) \\ &\quad + P\,\mathrm{div}_x(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \nabla_x \pi_\varepsilon) \\ &\quad + P\,\mathrm{div}_x(\nabla_x \pi_\varepsilon \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle) \\ &\quad + P\,\mathrm{div}_x(\nabla_x \pi_\varepsilon \otimes \nabla_x \pi_\varepsilon). \end{aligned}$$

By Proposition 5.3, the first term converges to $P\mathrm{div}_x(u \otimes u)$ in the sense of distributions, while the second and third terms converge to 0 in the sense of distributions because of (197).

As for the last term, for any mollifying sequence ξ_δ, denote

$$\pi_\varepsilon^\delta = \xi_\delta \star_x \xi_\delta \star_x \pi_\varepsilon\,, \quad \lambda_\varepsilon^\delta = \xi_\delta \star_x \xi_\delta \star_x \left\langle \frac{1}{3}|v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \right\rangle .$$

Then one has

$$\begin{aligned} \varepsilon\partial_t \nabla_x \pi_\varepsilon^\delta + \nabla_x \lambda_\varepsilon^\delta &\to 0 \text{ in } L^1_{loc}(\boldsymbol{R}_+; H^s_{loc}(\boldsymbol{R}^3)) \\ \varepsilon\partial_t \lambda_\varepsilon^\delta + \frac{5}{3}\Delta_x \lambda_\varepsilon^\delta &\to 0 \text{ in } L^1_{loc}(\boldsymbol{R}_+; H^s_{loc}(\boldsymbol{R}^3)) \end{aligned}$$

as a result of (184), the vanishing of momentum and energy conservation defects (see Proposition 3.1 for the momentum, and proceed analogously for the energy) and the fact that $\boldsymbol{F}_\varepsilon(A)$ is bounded in $L^1_{loc}(dtdx)$ (see Lemma 7.4, and the bounds (175) and (176)). From the above system, P.-L. Lions and N. Masmoudi observed in [**43**] that

$$\operatorname{div}_x\left(\nabla_x \pi_\varepsilon^\delta \otimes \nabla_x \pi_\varepsilon^\delta\right) = \frac{1}{2}\nabla_x \left(|\nabla_x \pi_\varepsilon^\delta|^2 - \frac{5}{3}|\lambda_\varepsilon^\delta|^2 \right) + o(1)_{L^1_{loc}(dtdx)} .$$

By the strong compactness in the x-variable (see Proposition 5.2)

$$P\operatorname{div}_x(\nabla_x \pi_\varepsilon \otimes \nabla_x \pi_\varepsilon) \to 0$$

in the sense of distributions. Collecting the observations above, we have just proved that

Proposition 5.4. *Under the assumptions of Theorem* 1.1, *modulo extraction of a subsequence, one has*

$$P\operatorname{div}_x(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon\rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon\rangle) \to P\operatorname{div}_x(u \otimes u)$$

in the sense of distributions on $\boldsymbol{R}_+^* \times \boldsymbol{R}^3$ *as* $\varepsilon \to 0$.

With this Proposition, all three steps in the proof of the Navier-Stokes limit (Theorem 1.1) are completed.

7.6 - *The nonlinear compactness estimate*

It only remains to prove the nonlinear compactness estimate (188), on which the two most important steps in the proof of the Navier-Stokes limit — i.e. the vanishing of conservation defects and the limiting form of the momentum flux — are based.

The nonlinear compactness estimate (188) will of course be obtained from statement 1) in Theorem 3.8 of the second chapter. In fact, we first observe that, for each $c \in (1,2)$

$$\phi_\varepsilon^\delta = \left(\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}\right)^2 \gamma\left(\varepsilon\delta\left(\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}\right)\right)$$

satisfies

$$\phi_\varepsilon^\delta = O(1) \text{ in } L_t^\infty(L^1(Mdvdx))$$

while

$$(\varepsilon\partial_t + v \cdot \nabla_x)\phi_\varepsilon^\delta = O(1) \text{ in } L^1_{loc}(dtdxMdv)\,.$$

We next let $\delta \to 0$ and remove the ε^c from under the square root (in that order) so that

$$\phi_\varepsilon^\delta \simeq \left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right)^2 .$$

In order to apply Theorem 3.8 to ϕ_ε^δ, it remains to prove that this family is uniformly integrable in the v-variable. Together with Theorem 3.8, this was one of the new key ideas in [28]. In fact, we prove that

Proposition 6.1. *Under the assumptions of Theorem* 1.1, *for each* $T > 0$ *and each compact* $K \subset \boldsymbol{R}^3$, *the family*

$$(1 + |v|)\left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right)^2$$

is uniformly integrable in the v-variable on $[0,T] \times K \times \boldsymbol{R}^3$ *for the measure* $dtdxMdv$.

This proposition improves upon the result in [28], that only applied to cutoff Maxwell molecules. Its proof is fairly technical, so that we shall only sketch the main new idea in it.

Start from the identity (already used to study the asymptotic momentum flux in section 4)

$$\mathcal{L}_M\left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right) = \varepsilon\mathcal{Q}_M\left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}, \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right) - \frac{1}{\varepsilon}\mathcal{Q}_M(\sqrt{G_\varepsilon}, \sqrt{G_\varepsilon})\,. \tag{198}$$

Next, we recall the Bardos-Caflisch-Nicolaenko spectral gap for $\mathcal{L}$ in weighted space (see chapter 1, Theorem 6.3)

$$\langle\phi\mathcal{L}_M\phi\rangle \geq C_0\langle(1 + |v|)\phi^2\rangle\,, \quad \phi \in (\ker\mathcal{L})^\perp$$

together with the Golse-Perthame-Sulem continuity estimate for $\mathcal{Q}$ (see chapter 5, Lemma 3.3)

$$\|\mathcal{Q}_M(\phi,\phi)\|_{L^2((1+|v|)^{-1}Mdv)} \leq C\|\phi\|_{L^2(Mdv)}\|\phi\|_{L^2((1+|v|)Mdv)}\,.$$

Using both estimates in the identity above leads to the following control

$$\left(1-O(\varepsilon)\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|_{L^2(Mdv)}\right)\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}-\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|_{L^2((1+|v|)Mdv)}$$
$$\leq O(\varepsilon)_{L^2_{t,x}}+O(\varepsilon)\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|^2_{L^2(Mdv)}.$$

This control suggests that

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\text{ is close to its hydrodynamic projection }\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$$

precisely in that weighted L^2 space that appears in the statement of Proposition 6.1. Since the hydrodynamic projection $\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$ is as regular in v as one can hope for (being a quadratic polynomial in v), this eventually entails uniform integrability in v once the difficulties related to the (t,x) dependence in the estimate above have been handled. As we already indicated above, the remaining part of the proof is too technical to be described here, and we refer the interested reader to [**29**] for a complete argument.

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Abstract

This article surveys recent mathematical results on the kinetic theory of gases. Specifically, the following topics are discussed in some detail

- the global existence theory of R. DiPerna and P.-L. Lions for the Boltzmann equation, and

-the derivation of the classical models of fluid mechanics (i.e. the Euler or Navier-Stokes equations) from the Boltzmann equation.

Among all existing results on these topics, we have chosen to discuss mostly those bearing on solutions that are global in time and for arbitrary initial data satisfying only a priori estimates with intrinsic physical meaning — typically, bounds on the total mass, energy or entropy.

Consequently, the mathematical methods presented here are well adapted to handling limits of sequences of functions with little or no uniform regularity. In particular, we study compactness arguments in L^p spaces implied by controls on derivatives of solutions coming from the partial differential equations satisfied by these solutions.

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