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Existence for some vectorial elliptic problems with measure data (**)

1 - Introduction

The study of elliptic boundary value problems with measure data

$$
(1.1)\qquad \begin{cases} -\operatorname{div}\,(a(x,u(x),Du(x)))=\mu & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}
$$

received great attention starting from [**2**]. In such a paper the scalar case $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ has been considered and the model operator is the p-laplacian $a_i(x,u,\xi)=|\xi|^{p-2}\xi_i$. Some years later [**3**], the anisotropic p_i-laplacian has been dealt with: $a_i(x,u,\xi)=|\xi_i|^{p_i-2}\xi_i$, that is, each component of the gradient D_iu may have a different exponent p_i; this seems useful when dealing with some reinforced materials, [**11**]; see also [**10**], example 1.7.1, page 169. Let us point out that both [**2**] and [**3**] deal with the scalar case $u:\Omega\to\mathbb{R}$; as far as the vectorial case $u:\Omega\to\mathbb{R}^N$ is concerned, some contributions appeared in [**7**] and [**4**], where the vectorial p-laplacian is considered. In this paper we deal with the anisotropic

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vector valued case:

$$\begin{cases} -\sum_{i=1}^{n} D_i(|D_i u|^{p_i-2} D_i u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ with $n \geq 3$, $N \geq 1$, Ω is a bounded open set and $\mu = (\mu^1, \dots, \mu^N)$ is a finite Radon measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$. In this paper we prove existence of weak solutions for the Dirichlet problem (1.2): in Section 2 we give the precise result, whose proof appears in Section 3. We thank the referee for useful remarks and suggestions.

2 - Notations and statements

We study existence of weak solutions for the Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{n} D_i(|D_i u|^{p_i-2} D_i u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ with $n \geq 3$, $N \geq 1$, Ω is a bounded open set and $\mu = (\mu^1, \dots, \mu^N)$ is a finite Radon measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$. We need some restrictions on the exponents $p_i \geq 2$; we introduce the harmonic mean

$$\overline{p} = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} \right)^{-1}$$

and we assume that

$$\overline{p} < n \text{ and } \frac{\overline{p}(n-1)}{(\overline{p}-1)n} < p_i < \frac{\overline{p}(n-1)}{n-\overline{p}} \quad \forall\, i \in \{1, \dots, n\}. \tag{2.2}$$

The right hand side of (2.2) can be written as

$$p_i \left[1 - \frac{n(\overline{p}-1)}{\overline{p}(n-1)} \right] = p_i \frac{n-\overline{p}}{\overline{p}(n-1)} < 1$$

thus

$$1 \leq p_i - 1 < \frac{n(\overline{p}-1)}{\overline{p}(n-1)} p_i$$

so we can take exponents $q_1, \dots, q_n$ such that

$$p_i - 1 < q_i < \frac{n(\overline{p}-1)}{\overline{p}(n-1)} p_i \qquad \forall\, i \in \{1, \dots, n\},$$

and we set

$$\underline{q} = \min_i q_i, \qquad \overline{q} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}\right)^{-1}, \qquad \frac{1}{\overline{q}^*} = \frac{1}{\overline{q}} - \frac{1}{n}.$$

In this paper we will prove the following

Theorem 2.1. *Under the previous assumptions there exists* $u \in W_0^{1,\underline{q}}(\Omega, \mathbb{R}^N) \cap L^{\overline{q}^*}(\Omega, \mathbb{R}^N)$ *with* $D_i u \in L^{q_i}(\Omega, \mathbb{R}^N)\ \forall\, i \in \{1, \dots, n\}$ *such that*

$$\int_\Omega \sum_{i=1}^{n}\sum_{\beta=1}^{N} |D_i u|^{p_i-2} D_i u^\beta D_i \varphi^\beta d\mathcal{L}^n = \sum_{\beta=1}^{N}\int_\Omega \varphi^\beta d\mu^\beta \qquad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{2.3}$$

In this theorem we may take $n = 3$, $p_1 = p_2 = 2$ and $p_3 \in (2, +\infty)$; this example shows that our theorem cannot be derived from [**5**], [**13**].

The previous theorem improves on [**9**]: we remove the restriction on the support of the measure μ and we allow a wider range for p_i.

3 - Proof of the Theorem

Let $\{\varsigma_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero. We mollify our measure μ and we obtain smooth functions

$$f_k = \mu * \rho_{\varsigma_k} \rightharpoonup^* \mu$$

with

$$\sup_k \|f_k\|_{L^1(\Omega)} \leq |\mu|(\mathbb{R}^n) < +\infty. \tag{3.1}$$

We set $\underline{p} = \min_i p_i$ and we use direct methods in the calculus of variations in order to find

$$u_k \in \mathcal{V} = \left\{ u \in W_0^{1,\underline{p}}(\Omega, \mathbb{R}^N) : D_i u \in L^{p_i}(\Omega, \mathbb{R}^N)\ \forall\, i \in \{1, \dots, n\} \right\},$$

such that

$$\int_\Omega \sum_{i=1}^{n}\sum_{\beta=1}^{N} |D_i u_k|^{p_i-2} D_i u_k^\beta D_i \varphi^\beta d\mathcal{L}^n = \sum_{\beta=1}^{N}\int_\Omega \varphi^\beta f_k^\beta d\mathcal{L}^n \qquad \forall \varphi \in \mathcal{V}. \tag{3.2}$$

We bound u_k: we use a componentwise truncation argument [**7**], [8] and estimates on level sets [**2**]; we set

$$M(t) = \begin{cases} 0 & 0 \le t \le L \\ t - L & L < t < L + 1 \\ 1 & t \ge L + 1 \\ -M(-t) & t < 0 \end{cases}$$

and we use the following test function φ in (3.2): we fix $\alpha \in \{1, \dots, N\}$ and we take $\varphi = (\varphi^1, \dots, \varphi^N)$ with $\varphi^\beta = 0$ for $\beta \neq \alpha$ and $\varphi^\alpha = M(u_k^\alpha)$, where u_k^α is the α-th component of $u_k = (u_k^1, \dots, u_k^N)$. We define

$$B_{L,k} = \{x \in \Omega : L \le |u_k^\alpha(x)| < L + 1\},$$

and we insert the previous φ into (3.2): we get

$$\int_{B_{L,k}} \sum_{i=1}^{n} |D_i u_k|^{p_i - 2} |D_i u_k^\alpha|^2 dx = \int_{\Omega} f_k^\alpha M(u_k^\alpha)\, dx. \tag{3.3}$$

Since $|M(t)| \le 1$, we can use (3.1) so that the right hand side of (3.3) is bounded by $|\mu|(\mathbb{R}^n)$. Let us look at the left hand side: since $p_i \ge 2$, we have

$$|D_i u_k^\alpha|^{p_i - 2} \le |D_i u_k|^{p_i - 2}$$

thus we get

$$\int_{B_{L,k}} \sum_{i=1}^{n} |D_i u_k^\alpha|^{p_i} dx \le c_1 \qquad \forall k, L \in \mathbb{N}. \tag{3.4}$$

Since

$$q_i < \frac{n(\overline{p} - 1)}{\overline{p}(n - 1)} p_i,$$

we can select

$$\theta \in \left(0, \frac{n(\overline{p} - 1)}{\overline{p}(n - 1)}\right) \subset (0, 1)$$

such that $q_i \le \theta p_i$ for every i. Now, without loss of generality, we can assume that

$$q_i = \theta p_i$$

for every i and

$$\theta \text{ is close to } \frac{n(\overline{p} - 1)}{\overline{p}(n - 1)}.$$

Let us remark that $1 < q_i < p_i$ for every $i \in \{1, \dots, n\}$ and $\overline{q} < \overline{p} < n$. Now we follow [**3**]: we set

$$\lambda = \frac{(1-\theta)\overline{q}^*}{\theta}$$

so that $\lambda > 1$; we use Holder inequality:

$$\begin{aligned}
\int_\Omega |D_i u_k^a|^{q_i} dx &= \int_\Omega |D_i u_k^a|^{q_i} (1 + |u_k^a|)^{-\lambda q_i/p_i} (1 + |u_k^a|)^{\lambda q_i/p_i} dx \\
&\le \left[\int_\Omega |D_i u_k^a|^{p_i} (1 + |u_k^a|)^{-\lambda} dx \right]^{q_i/p_i} \left[\int_\Omega (1 + |u_k^a|)^{\lambda q_i/(p_i - q_i)} dx \right]^{1 - q_i/p_i} \\
&= \left[\sum_{L=0}^{+\infty} \int_{B_{L,k}} |D_i u_k^a|^{p_i} (1 + |u_k^a|)^{-\lambda} dx \right]^{q_i/p_i} \left[\int_\Omega (1 + |u_k^a|)^{\lambda q_i/(p_i - q_i)} dx \right]^{1 - q_i/p_i} \\
&\le \left[\sum_{L=0}^{+\infty} (1 + L)^{-\lambda} \int_{B_{L,k}} |D_i u_k^a|^{p_i} dx \right]^{q_i/p_i} \left[\int_\Omega (1 + |u_k^a|)^{\overline{q}^*} dx \right]^{1 - q_i/p_i}.
\end{aligned} \tag{3.5}$$

We keep in mind (3.4) and $\lambda > 1$: we get

$$\int_\Omega |D_i u_k^a|^{q_i} dx \le c_2 \left[\int_\Omega (1 + |u_k^a|)^{\overline{q}^*} dx \right]^{1 - q_i/p_i}. \tag{3.6}$$

We use the anisotropic embedding theorem [**12**] and we get

$$\begin{aligned}
\|u_k^a\|_{L^{\overline{q}^*}(\Omega)} &\le c_3 \prod_{i=1}^n \|D_i u_k^a\|_{L^{q_i}(\Omega)}^{1/n} \le c_4 \prod_{i=1}^n \left[\int_\Omega (1 + |u_k^a|)^{\overline{q}^*} dx \right]^{[(1/q_i) - (1/p_i)]/n} \\
&= c_4 \left[\int_\Omega (1 + |u_k^a|)^{\overline{q}^*} dx \right]^{(1-\theta)/(\theta \overline{p})}.
\end{aligned} \tag{3.7}$$

We recall that $\overline{p} < n$, so

$$(1-\theta)/(\theta\overline{p}) < 1/\overline{q}^*$$

thus (3.7) gives

$$\|u_k^a\|_{L^{\overline{q}^*}(\Omega)} \le c_5 \quad \forall k \in \mathbb{N}, \quad \forall a = 1, \dots, N. \tag{3.8}$$

We insert this bound into (3.6) and we get

$$\left\| D_i u_k^a \right\|_{L^{q_i}(\Omega)} \le c_6 \quad \forall k \in \mathbb{N}, \quad \forall i = 1, \dots n, \quad \forall a = 1, \dots, N. \tag{3.9}$$

Because of weak compactness, there exists $u \in W_0^{1,\underline{q}}(\Omega, \mathbb{R}^N) \cap L^{\overline{q}^*}(\Omega, \mathbb{R}^N)$, with $D_i u \in L^{q_i}(\Omega, \mathbb{R}^N)$ $\forall\, i \in \{1, \dots, n\}$, $\overline{\sigma}_i \in L^{\frac{q_i}{p_i-1}}(\Omega, \mathbb{R}^N)$ and $h_i \in L^{q_i}(\Omega, \mathbb{R})$ for every $i \in \{1, \dots, n\}$ such that, up to a subsequence,

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,\underline{q}}(\Omega, \mathbb{R}^N), \tag{3.10}$$

$$u_k \to u \quad \text{a.e. in } \Omega, \tag{3.11}$$

$$D_i u_k \rightharpoonup D_i u \quad \text{in } L^{q_i}(\Omega, \mathbb{R}^N) \quad \forall\, i \in \{1, \dots, n\}, \tag{3.12}$$

$$|D_i u_k|^{p_i - 2} D_i u_k \rightharpoonup \overline{\sigma}_i \quad \text{in } L^{\frac{q_i}{p_i-1}}(\Omega, \mathbb{R}^N) \;\; \forall\, i \in \{1, \dots, n\}, \tag{3.13}$$

$$|D_i u_k - D_i u| \rightharpoonup h_i \ge 0 \quad \text{in } L^{q_i}(\Omega, \mathbb{R}) \;\; \forall\, i \in \{1, \dots, n\}. \tag{3.14}$$

We claim that

$$h_i = 0 \quad \text{in } \Omega \quad \forall\, i \in \{1, \dots, n\} \tag{3.15}$$

thus

$$D_i u_k \to D_i u \quad \text{in } L^1(\Omega, \mathbb{R}^N) \quad \forall\, i \in \{1, \dots, n\}. \tag{3.16}$$

We prove claim (3.15) following [4]. We fix a smooth $v : \mathbb{R}^n \to \mathbb{R}^N$; later, v will be chosen as a linear Taylor polynomial of u. We take cut off functions $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\eta : \mathbb{R}^N \to \mathbb{R}$ such that

$$\phi \in C_c^\infty(\Omega, \mathbb{R}),$$

$$\eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R}),$$

with $0 \le \phi \le 1$ and $0 \le \eta \le 1$. We take $\psi : \mathbb{R}^N \to \mathbb{R}^N$ as follows

$$\psi(z) = \gamma(|z|) \frac{z}{|z|}$$

where $\gamma \in C^1((0, +\infty), \mathbb{R})$ with γ and γ' non-negative and bounded; moreover, we require that $\psi_{| \text{ supp } (\eta)} = \text{ Id }$. Note that

$$\frac{\partial \psi^a}{\partial z^s}(z) = \gamma'(|z|) \frac{z^s z^a}{|z|^2} + \frac{\gamma(|z|)}{|z|} \left(\delta_{sa} - \frac{z^s z^a}{|z|^2} \right)$$

thus

$$\sum_{a=1}^N \xi^a \sum_{s=1}^N \frac{\partial \psi^a}{\partial z^s}(z) \xi^s \ge 0 \quad \forall z, \xi \in \mathbb{R}^N. \tag{3.17}$$

For every $j \in \{1, \dots, n\}$ it results that

$$
\begin{aligned}
0 &\le \int_\Omega |D_j u_k - D_j u| \eta(u_k - v)\phi dx \\
&\le \int_\Omega |D_j u_k - D_j v| \eta(u_k - v)\phi dx + \int_\Omega |D_j v - D_j u| \eta(u_k - v)\phi dx \\
&\le \left(\int_\Omega |D_j u_k - D_j v|^{p_j} \eta(u_k - v)\phi dx\right)^{1/p_j} \left(\int_\Omega \eta(u_k - v)\phi dx\right)^{1-1/p_j} \\
&+ \int_\Omega |D_j v - D_j u| \eta(u_k - v)\phi dx.
\end{aligned} \tag{3.18}
$$

Let $\tilde{p} = \max_i p_i$. Then, using the monotonicity of $z \to |z|^{p-2}z$ and the equality $\psi =$Id on supp(η), we get

$$
\begin{aligned}
0 &\le T_k^j := \int_\Omega |D_j u_k - D_j v|^{p_j} \eta(u_k - v)\phi dx \\
&\le \sum_{i=1}^n \int_\Omega |D_i u_k - D_i v|^{p_i} \eta(u_k - v)\phi dx \\
&\le 8^{\tilde{p}-1} \sum_{i=1}^n \int_\Omega \langle |D_i u_k|^{p_i-2} D_i u_k - |D_i v|^{p_i-2} D_i v, D_i u_k - D_i v\rangle \eta(u_k - v)\phi dx \\
&= 8^{\tilde{p}-1} \sum_{i=1}^n \int_\Omega \langle |D_i u_k|^{p_i-2} D_i u_k, D_i(\psi(u_k - v))\rangle \eta(u_k - v)\phi dx \\
&- 8^{\tilde{p}-1} \sum_{i=1}^n \int_\Omega \langle |D_i v|^{p_i-2} D_i v, D_i u_k - D_i v\rangle \eta(u_k - v)\phi dx \\
&= 8^{\tilde{p}-1} \Bigg\{ \sum_{i=1}^n \int_\Omega \langle |D_i u_k|^{p_i-2} D_i u_k, D_i(\psi(u_k - v))\rangle \phi dx \\
&- \sum_{i=1}^n \int_\Omega \langle |D_i v|^{p_i-2} D_i v, D_i u_k - D_i v\rangle \eta(u_k - v)\phi dx \\
&- \sum_{i=1}^n \int_\Omega \langle |D_i u_k|^{p_i-2} D_i u_k, D_i(\psi(u_k - v))\rangle [1 - \eta(u_k - v)]\phi dx \Bigg\}.
\end{aligned} \tag{3.19}
$$

We recall (3.2) and we get

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Omega}\langle|D_iu_k|^{p_i-2}D_iu_k, D_i(\psi(u_k-v))\rangle\phi dx\\
&=\sum_{i=1}^{n}\int_{\Omega}\langle|D_iu_k|^{p_i-2}D_iu_k, D_i(\psi(u_k-v)\phi)\rangle dx\\
&-\sum_{i=1}^{n}\int_{\Omega}\langle|D_iu_k|^{p_i-2}D_iu_k, \psi(u_k-v)D_i\phi\rangle dx\\
&=\int_{\Omega}\langle\mu*\rho_{\varsigma_k}, \psi(u_k-v)\phi\rangle dx\\
&-\sum_{i=1}^{n}\int_{\Omega}\langle|D_iu_k|^{p_i-2}D_iu_k, \psi(u_k-v)D_i\phi\rangle dx.
\end{aligned}\tag{3.20}$$

Moreover, (3.17) implies

$$\begin{aligned}
&-\sum_{i=1}^{n}\int_{\Omega}\langle|D_iu_k|^{p_i-2}D_iu_k, D_i(\psi(u_k-v))\rangle[1-\eta(u_k-v)]\phi dx\\
&\leq\sum_{i=1}^{n}\int_{\Omega}\sum_{a=1}^{N}|D_iu_k|^{p_i-2}D_iu_k^a\sum_{s=1}^{N}\frac{\partial\psi^a}{\partial z^s}(u_k-v)D_iv^s[1-\eta(u_k-v)]\phi dx.
\end{aligned}\tag{3.21}$$

We insert (3.20) and (3.21) into (3.19): we get

$$\begin{aligned}
&0\leq T_k^j:=\int_{\Omega}|D_ju_k-D_jv|^{p_j}\eta(u_k-v)\phi dx\\
&\leq 8^{\tilde{p}-1}\Bigg\{\int_{\Omega}\langle\mu*\rho_{\varsigma_k}, \psi(u_k-v)\phi\rangle dx-\sum_{i=1}^{n}\int_{\Omega}\langle|D_iu_k|^{p_i-2}D_iu_k, \psi(u_k-v)D_i\phi\rangle dx\\
&+\sum_{i=1}^{n}\int_{\Omega}\sum_{a=1}^{N}|D_iu_k|^{p_i-2}D_iu_k^a\sum_{s=1}^{N}\frac{\partial\psi^a}{\partial z^s}(u_k-v)D_iv^s[1-\eta(u_k-v)]\phi dx\\
&-\sum_{i=1}^{n}\int_{\Omega}\langle|D_iv|^{p_i-2}D_iv, D_iu_k-D_iv\rangle\eta(u_k-v)\phi dx\Bigg\}.
\end{aligned}\tag{3.22}$$

We now use (3.11), (3.12), (3.13), (3.14) and we keep in mind that $\eta, \psi, D\psi$ are continuous and bounded: for every $j = 1, \dots, n$ we get

$$
\begin{aligned}
0 \le T^j_\infty &:= \limsup_{k\to+\infty} T^j_k \\
&\le 8^{\tilde{p}-1}\Bigg\{ \sup_{z\in\mathbb{R}^N} |\psi(z)| \int_\Omega \phi d|\mu| - \sum_{i=1}^n \int_\Omega \langle \overline{\sigma}_i, \psi(u-v) D_i\phi\rangle dx \\
&+ \sum_{i=1}^n \int_\Omega \sum_{a=1}^N \overline{\sigma}_i^a \sum_{s=1}^N \frac{\partial \psi^a}{\partial z^s}(u-v) D_i v^s [1-\eta(u-v)]\phi dx \\
&- \sum_{i=1}^n \int_\Omega \langle |D_i v|^{p_i-2} D_i v, D_i u - D_i v\rangle \eta(u-v)\phi dx \Bigg\},
\end{aligned} \tag{3.23}
$$

$$
0 \le \lim_{k\to+\infty} \int_\Omega |D_j u_k - D_j u| \eta(u_k - v)\phi dx = \int_\Omega h_j \eta(u-v)\phi dx, \tag{3.24}
$$

$$
\lim_{k\to+\infty} \int_\Omega |D_j v - D_j u| \eta(u_k - v)\phi dx = \int_\Omega |D_j v - D_j u| \eta(u-v)\phi dx, \tag{3.25}
$$

$$
\lim_{k\to+\infty} \int_\Omega \eta(u_k - v)\phi dx = \int_\Omega \eta(u-v)\phi dx. \tag{3.26}
$$

Take the limsup in (3.18). We get, for every $j = 1, \dots, n$,

$$
\begin{aligned}
&0 \le \int_\Omega h_j \eta(u-v)\phi dx \le \\
&\left(T^j_\infty\right)^{1/p_j} \left(\int_\Omega \eta(u-v)\phi dx \right)^{1-1/p_j} + \int_\Omega |D_j v - D_j u| \eta(u-v)\phi dx
\end{aligned} \tag{3.27}
$$

where T^j_∞ has been estimated in (3.23). For a. e. $a \in \Omega$ and every $j = 1, \dots, n$, we have (see [**6**] and [**1**])

$$
\lim_{R\to 0^+} \fint_{B(a,R)} \frac{|u(x) - [u(a) + Du(a)(x-a)]|}{R} dx = 0, \tag{3.28}
$$

$$
\lim_{R\to 0^+} \fint_{B(a,R)} |Du(x) - Du(a)| dx = 0, \tag{3.29}
$$

$$
\lim_{R\to 0^+} \fint_{B(a,R)} |\overline{\sigma}_j(x) - \overline{\sigma}_j(a)| dx = 0, \tag{3.30}
$$

$$\lim_{R\to 0^+} \fint_{B(a,R)} |h_j(x) - h_j(a)|dx = 0, \tag{3.31}$$

$$\limsup_{R\to 0^+} \frac{|\mu|(B(a,R))}{\mathcal{L}^n(B(a,R))} < +\infty \tag{3.32}$$

where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$ and

$$\fint_A f(x)dx = \frac{1}{\mathcal{L}^n(A)} \int_A f(x)dx.$$

We choose v to be the linear Taylor polynomial of u in a:

$$v(x) = u(a) + Du(a)(x - a).$$

Since we are still free to select ϕ, η and ψ, let

$$\phi_R(x) = \frac{1}{R^n}\tilde{\phi}\Big(\frac{x-a}{R}\Big), \quad \tilde{\phi} \in C_c^\infty(B(0,1),\mathbb{R}), \quad 0 \le \tilde{\phi} \le 1, \quad \int_{\mathbb{R}^n} \tilde{\phi}(x)dx = 1$$

$$\eta_R(y) = \tilde{\eta}\Big(\frac{y}{R}\Big), \quad \tilde{\eta} \in C_c^\infty(B(0,1),\mathbb{R}), \quad 0 \le \tilde{\eta} \le 1, \quad \tilde{\eta}_{|B(0,\frac{1}{2})} = 1,$$

$$\psi_R(y) = R\tilde{\psi}\Big(\frac{y}{R}\Big),$$

with $\tilde{\psi}$ as in the above discussion. Then (3.27) can be written as follows:

$$\begin{aligned} 0 &\le \int_{B(a,R)} h_j \eta_R(u-v)\phi_R dx \\ &\le \Big(T^j_{\infty,R}\Big)^{1/p_j} \left(\int_{B(a,R)} \eta_R(u-v)\phi_R dx \right)^{1-1/p_j} + \int_{B(a,R)} |D_j v - D_j u| \eta_R(u-v)\phi_R dx \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} 0 \le T^j_{\infty,R} &:= \limsup_{k\to+\infty} \int_{B(a,R)} |D_j u_k - D_j v|^{p_j} \eta_R(u-v)\phi_R dx \\ &\le 8^{\tilde{p}-1} \Bigg\{ \sup_{z\in\mathbb{R}^N} |\psi_R(z)| \int_{B(a,R)} \phi_R d|\mu| - \sum_{i=1}^n \int_{B(a,R)} \langle \overline{\sigma}_i, \psi_R(u-v) D_i\phi_R\rangle dx \\ &+ \sum_{i=1}^n \int_{B(a,R)} \sum_{a=1}^N \overline{\sigma}_i^a \sum_{s=1}^N \frac{\partial \psi_R^a}{\partial z^s}(u-v) D_i v^s [1 - \eta_R(u-v)]\phi_R dx \\ &- \sum_{i=1}^n \int_{B(a,R)} \langle |D_i v|^{p_i - 2} D_i v, D_i u - D_i v\rangle \eta_R(u-v)\phi_R dx \Bigg\}. \end{aligned} \tag{3.34}$$

Recalling (3.28), ... , (3.32) and the definitions of η_R, ϕ_R, we get

$$\lim_{R\to 0^+} \int_{B(a,R)} h_j \eta_R(u-v)\phi_R dx = h_j(a), \tag{3.35}$$

$$\lim_{R\to 0^+} T^j_{\infty,R} = 0, \tag{3.36}$$

$$0 \le \int_{B(a,R)} \eta_R(u-v)\phi_R dx \le 1, \tag{3.37}$$

$$\lim_{R\to 0^+} \int_{B(a,R)} |D_j v - D_j u| \eta_R(u-v)\phi_R dx = 0. \tag{3.38}$$

As far as (3.36) is concerned, we give some details in the Appendix. Let us take $R \to 0^+$ in (3.33): we get, for every $j = 1, \dots, n$,

$$h_j(a) = 0 \qquad \text{for a.e. } a \in \Omega \tag{3.39}$$

thus our claim (3.15) is proved. Now we have (3.16), thus, up to a subsequence,

$$Du_k \to Du \text{ a.e. in } \Omega. \tag{3.40}$$

The previous pointwise convergence and the estimate

$$\left\| |D_i u_k|^{p_i-2} D_i u_k \right\|_{L^{\frac{q_i}{p_i-1}}(\Omega)} = \left\| D_i u_k \right\|^{p_i-1}_{L^{q_i}(\Omega)} \le c_7, \qquad \forall k \in \mathbb{N}, \qquad \forall i \in \{1, \dots, n\}$$

allow us to get

$$|D_i u_k|^{p_i-2} D_i u_k \rightharpoonup |D_i u|^{p_i-2} D_i u \ \text{ in } \ L^{\frac{q_i}{p_i-1}}(\Omega, \mathbb{R}^N) \qquad \forall i \in \{1, \dots, n\}. \tag{3.41}$$

Thus we can pass to the limit, as $k \to +\infty$, in (3.2) and it turns out that u solves (2.3). This ends the proof. □

4 - Appendix

For the convenience of the reader, we give some details about the proof of (3.36). We look at inequality (3.34): it suffices to prove that all the four integrals go to zero when $R \to 0^+$. We keep in mind that

$$|\psi_R(z)| \le c_8 R \tag{4.1}$$

$$0 \le \phi_R(x) \le \frac{1}{R^n} \tag{4.2}$$

then

$$0 \leq \sup_{z \in \mathbb{R}^N} |\psi_R(z)| \int_{B(a,R)} \phi_R d|\mu| \leq c_8 R \frac{|\mu|(B(a,R))}{R^n}$$

and the right hand side goes to zero because of (3.32). Now we deal with the second integral in the right hand side of (3.34):

$$\begin{aligned} &\left| \int_{B(a,R)} \langle \overline{\sigma}_i(x), \psi_R(u(x) - v(x)) D_i \phi_R(x) \rangle dx \right| \\ &\leq \left| \int_{B(a,R)} \langle \overline{\sigma}_i(a), \psi_R(u(x) - v(x)) D_i \phi_R(x) \rangle dx \right| \\ &+ \left| \int_{B(a,R)} \langle \overline{\sigma}_i(x) - \overline{\sigma}_i(a), \psi_R(u(x) - v(x)) D_i \phi_R(x) \rangle dx \right| = I_R + II_R. \end{aligned}$$

Note that

$$D_i \phi_R(x) = \frac{1}{R^n} \left[D_i \tilde{\phi}\Big(\frac{x-a}{R}\Big) \right] \frac{1}{R}.$$

We recall that

$$\tilde{\psi}(z) = z \text{ for } z \in B\left(0, \frac{1}{2}\right) \subset \text{ supp } (\tilde{\eta})$$

then

$$\psi_R(u - v) = u - v \quad \text{ on } \left\{ \frac{|u - v|}{R} \leq \frac{1}{2} \right\}$$

thus it is convenient to let

$$A_R = \left\{ x \in B(a,R) : \frac{|u(x) - v(x)|}{R} > \frac{1}{2} \right\}.$$

We define ω_n to be the n dimensional Lebesgue measure of the unit ball in $\mathbb{R}^n$; then $\mathcal{L}^n(B(a,R)) = \omega_n R^n$ and

$$0 \leq \frac{\mathcal{L}^n(A_R)}{R^n} \leq 2\omega_n \fint_{B(a,R)} \frac{|u(x) - v(x)|}{R} dx.$$

We use (3.28) and we get

$$\lim_{R\to 0^+} \frac{\mathcal{L}^n(A_R)}{R^n} = 0. \tag{4.3}$$

Now we are able to deal with I_R as follows:

$$\begin{aligned} 0 \le I_R = &\Bigg| \frac{1}{R^n} \int_{A_R} \langle \overline{\sigma}_i(a), \psi_R(u(x)-v(x)) \Big[D_i\tilde{\phi}\Big(\frac{x-a}{R}\Big)\Big] \frac{1}{R} \rangle dx \\ &+ \frac{1}{R^n} \int_{B(a,R)\setminus A_R} \langle \overline{\sigma}_i(a), \psi_R(u(x)-v(x)) \Big[D_i\tilde{\phi}\Big(\frac{x-a}{R}\Big)\Big] \frac{1}{R} \rangle dx \Bigg| \\ \le &\frac{\mathcal{L}^n(A_R)}{R^n} |\overline{\sigma}_i(a)| c_8 \big\|D_i\tilde{\phi}\big\|_{L^\infty(\mathbb{R}^n)} + \omega_n |\overline{\sigma}_i(a)| \big\|D_i\tilde{\phi}\big\|_{L^\infty(\mathbb{R}^n)} \fint_{B(a,R)} \frac{|u(x)-v(x)|}{R} dx. \end{aligned} \tag{4.4}$$

Using (4.3) and (3.28) we get

$$\lim_{R\to 0^+} I_R = 0. \tag{4.5}$$

As far as II_R is concerned, we argue as follows:

$$0 \le II_R \le c_8 \omega_n \big\|D_i\tilde{\phi}\big\|_{L^\infty(\mathbb{R}^n)} \fint_{B(a,R)} |\overline{\sigma}_i(x) - \overline{\sigma}_i(a)| dx$$

thus we can use (3.30) and we get

$$\lim_{R\to 0^+} II_R = 0. \tag{4.6}$$

Limits (4.5) and (4.6) guarantee that the second integral in (3.34) goes to zero when $R \to 0^+$. In a similar manner we can show that the third integral goes to zero. The fourth integral is easier to be dealt with. Thus we have established (3.36).

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Abstract

We study the Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{n} D_i(|D_i u|^{p_i-2} D_i u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ *with* $n \geq 3$, $N \geq 1$, Ω *is a bounded open set and* $\mu = (\mu^1, \ldots, \mu^N)$ *is a finite Radon measure on* $\mathbb{R}^n$ *with values into* $\mathbb{R}^N$. *Under some restrictions on the exponents* $p_i \geq 2$, *we prove existence of a weak solution* u.

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