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Periodic solutions for semilinear parabolic problems with nonlinear dynamical boundary condition (**)

1 - Introduction

The subject of the present paper is motivated by a semilinear parabolic problem with time derivate in the boundary condition of the form

$$\partial_t u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + q(x)u = f(x,t,u) \text{ in } Q := \Omega \times P, \tag{1.1}$$

$$\partial_t u + \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_j} u \cos(\upsilon, x_i) + c(x,t,u) = 0 \text{ on } \Sigma := \partial\Omega \times P, \tag{1.2}$$

$$u(x,t+\omega) = u(x,t) \text{ in } Q,\ \omega > 0, \tag{1.3}$$

of which we want to give a mathematical proof of existence of weak periodic solutions. The symmetric density matrix $\{a_{i,j}(x)\}_{n\times n}$ in the diffusion term, has as elements continuous functions defined in $\overline{\Omega}$, a bounded regular set of R^n, $n \geq 1$ with boundary $\partial\Omega$.

We represent with $P := R/\omega Z$ the period interval $[0,\omega]$ and with υ the outward normal vector on $\partial\Omega$. Consequently, for functions definited in P, we are automatically imposing the time periodicity. The physical meaning of model is the following: A perfect solid heat conductor Ω, is placed in contact with a fluid and the

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temperature within the solid at position x and at the time t is designed by $u(x,t)$. If the boundary $\partial\Omega$ is permeable to heat, then the rate at which heat flows through $\partial\Omega$ is described by $\sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_j} u \cos(\upsilon, x_i)$ and $c(x,t,u)$ denotes a t-periodic nonlinear source term. The rate of accumulation of heat in the fluid is given by condition (1.2).

We emphasize that the main feature of problem (1.1)-(1.3) arises by setting a not standard boundary condition since it also involves derivates with respect to time of the unknown which is very natural in many mathematical model as: heat transfer in a solid in contact with a moving fluid; Stefan problem; diffusion in porous media; chemical engineering; problems arising in fluid dynamics. Problems under dynamical boundary condition have received a great deal of attention in literature.

The well-posedness in some Bessel potential spaces of parabolic semilinear reaction-diffusion problem and parabolic systems is investigate in [**6**], [**7**] using the framework of semigroups. Again the theory of semigroups in Banach spaces is applied in [**8**] to prove an existence and the uniqueness result for parabolic and hyperbolic equations. Instead, the question of the blow-up of solutions is studied in [**15**] and [**6**]. Uniform estimates in Hölder spaces for the solutions of parabolic systems are given in [**3**]. The solubility of parabolic equations with dynamical boundary condition in Hölder weighted spaces is treated in [**13**]. Finally, we recall [**5**], [**9**], [**14**] and [**2**] where the interested reader can find an exhaustive list of references on related papers.

However, in all previous papers the periodic case is not studied. It is worth to mentioning that the starting point to approach the periodicity of solutions, relies on the next theorem 0 for maximal monotone operators, Faedo-Galerkin approximations and a suitable fixed point argument.

Theorem 0 ([**1**], [**4**], [**11**]). *Let L be a linear closed, densely defined operator from the reflexive Banach space V to V^*, L maximal monotone and let A be a bounded hemicontinuous monotone mapping from V to V^*, then $L + A$ is maximal monotone in $V \times V^*$. Moreover, if $L + A$ is coercive, then Range$(L + A) = V^*$.*

The outline of this work is the following. Section 2, is mainly devoted to detail the assumptions and some definitions used along the paper; this leads to the definition of some functional spaces. In order to establish the existence of periodic solutions, the equation under consideration is transformed into an abstract problem to make maximal monotone theory applicable. Therefore, in the third section, we define two mappings L , A and show that satisfy the assumptions of theorem 0. Finally, the last section is concerned by a fixed point argument and one shows, by means of the Schauder fixed point theorem, the existence of weak periodic solutions for approximating problems, obtained by means of the Faedo-Galerkin procedure. We derive an uniform energy estimate for these approximating solutions u_k. The same Galerkin method is useful to show estimates on u_{kt} in the norm of the spaces $L^2(\Omega \times P)$ and $L^2(\partial\Omega \times P)$. Uniform estimates, constitute the core of our development, in that permit to establish the existence of periodic solutions for our problem, passing to the limit on the approximating solutions u_k.

2 - Assumptions and definitions

Concerning the periodicity of weak solutions to (1.1)-(1.3), we shall use a static formulation which consists to look for them in some appropriate t-periodic functional space.

Indeed, let

$$V := L^2(P; W^{1,2}(\Omega))$$

denote the space of periodic functions endowed with the equivalent norm

$$\|v\|_V := \Big(\int_Q |\, v(x,t)\,|^2 \, dxdt + \int_Q |\, \nabla v(x,t)\,|^2 \, dxdt + \int_\Sigma |\, \widetilde{u}(x,t)\,|^2 \, dsdt \Big)^{1/2} \tag{2.1}$$

where $\widetilde{u}$ stands for the trace of $u \in V$. Recall that for regular domains Ω, every $v \in W^{1,2}(\Omega)$ has a trace in $W^{1/2}(\partial\Omega)$ and in view of the trace Sobolev theorem, $W^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ compactly.

The topological dual of V is the space

$$V^* := L^2(P; W^{-1,2}(\Omega)).$$

endowed with the $\|.\|_*$ norm. The pairing of duality between V and V^* shall be denoted by $< .,. >$.

The structural assumptions on data are accounted below:

H$_1$) $a_{ij} \in C(\overline{\Omega})$ and there exists a positive constant a such that

$$a \mid \xi \mid^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \ \text{ for all } \xi \in R^n$$

H$_2$) q is a continuous function with

$$0 < \delta := \min_{\overline{\Omega}} q(x);$$

H$_3$) $f \in L^2(\Omega \times P \times R)$ with the growth condition

$$| f(x,t,\xi) | \leq C(1 + | \xi |);$$

H$_4$) $c : \partial\Omega \times P \times R \to R$ is a Caratheodory function, with $c(x, t, \xi)$ nondecreasing in $\xi \in R$ for a.e. $(x, t) \in \partial\Omega \times P$ and

$$| c(x,t,\xi) | \leq C(d(x,t) + | \xi |)$$

$$\xi c(x,t,\xi) \geq C_3 | \xi |^2 - c_4(x,t),\ C_3 > 0$$

with $d \in L^2(\partial\Omega \times P)$, $c_4 \in L^1(\partial\Omega \times P)$.

The notion of weak solution for our problem may be introduced in the following way

Definition 2.1. A function u is a weak periodic solution to (1.1)-(1.3) if hold

(a) $u \in V$, $u_t \in L^2(\Omega \times P)$;

(b) By the trace theorem, there exists the spatial trace of u on Σ and has a distributional time derivate $\partial_t\widetilde{u}$ belonging to $L^2(\partial\Omega \times P)$

and

$$\int_Q \partial_t u\sigma dxdt + \sum_{i,j=1}^{n} \int_Q a_{ij}(x)\partial_{x_j}u\partial_{x_i}\sigma dxdt + \int_Q q(x)u\sigma dxdt \tag{2.2}$$
$$+ \int_\Sigma \partial_t\widetilde{u}\ \widetilde{\sigma} dsdt + \int_\Sigma c(x,t,\widetilde{u})\widetilde{\sigma} dsdt = \int_Q f(x,t,u)\sigma dxdt, \forall\sigma \in V.$$

Remark. If the weak periodic solution u possesses the additional regularity $u \in C^1(P; C^2(\overline{\Omega}))$, then owing to the divergence theorem u is a classic solution.

Fixed $w \in L^2(Q)$, consider the problem

$$\partial_t u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) + q(x)u = f(x,t,w) \text{ in } Q, \tag{2.3}$$

$$\partial_t\widetilde{u} + \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_j}\widetilde{u}\cos(\upsilon, x_i) + c(x,t,\widetilde{u}) = 0 \text{ on } \Sigma\ , \tag{2.4}$$

$$u(x, t + \omega) = u(x,t) \text{ in } Q,\ \omega > 0. \tag{2.5}$$

In agreement with definition 2.1, a function u defined in Q is called a weak periodic solution of (2.3)-(2.4), if

$$u \in V,\ u_t \in L^2(\Omega \times P) \text{ with } \partial_t\widetilde{u} \in L^2(\partial\Omega \times P)$$

and satisfies

$$\int_Q \partial_t u\sigma dxdt + \sum_{i,j=1}^{n} \int_Q a_{ij}(x)\partial_{x_j}u\partial_{x_i}\sigma dxdt + \int_Q q(x)u\sigma dxdt$$

$$+\int_{\Sigma}\partial_t\widetilde{u}\,\widetilde{\sigma}dsdt+\int_{\Sigma}c(x,t,\widetilde{u})\widetilde{\sigma}dsdt=\int_{Q}f(x,t,w)\sigma dxdt,\forall\sigma\in V \tag{2.6}$$

3 - Existence of periodic solutions

In this section, our main purpose is to prove the existence of weak periodic solutions. This shall be done introducing two appropriate mappings, in order to formulate (2.3)-(2.5) as an abstract problem to which apply theorem 0.

Let $L : D \to V^*$ be a closed skew-adjoint i.e. $L = -L^*$ (integrating by parts and using the periodicity) linear operator, defined by

$$< L(u),\sigma >:=\int_{Q}\partial_t u\sigma\, dxdt+\int_{\Sigma}\partial_t\widetilde{u}\,\widetilde{\sigma}\, dsdt,\ \forall\sigma\in V$$

on the dense set

$$D := \{u \in V : u_t \in L^2(\Omega\times P) \text{ and } \partial_t\widetilde{u}\in L^2(\partial\Omega\times P)\}$$

because $C^\infty(\overline{Q}) \subset D$ is dense in V.

As a consequence of a result in [**11**], Lemma 1.1, p. 313, the operator L is maximal monotone from V to V^*.

Let $A : V \to V^*$ be the mapping defined by setting

$$< A(u),\sigma >:=\sum_{i,j=1}^{n}\int_{Q}a_{ij}(x)\partial_{x_j}u\partial_{x_i}\sigma\, dxdt+\int_{Q}q(x)u\sigma\, dxdt$$

$$+\int_{\Sigma}c(x,t,\widetilde{u})\widetilde{\sigma}\, dsdt,\quad \forall\sigma\in V.$$

The next proposition concerns with the properties of the latter operator.

Proposition 3.1. *If $H_1)$–$H_4)$ are fulfilled, then A is:*
(i) *hemicontinuous;*
(ii) *monotone;*
(iii) *coercive.*

Proof. (i) Applying the Hölder inequality one has
$|< A(u),\sigma >|\le \beta(\sum_{i,j=1}^{n}\int_Q \mid \partial_{xj}u \mid\mid \partial_{x_i}\sigma \mid dxdt) + M\int_Q \mid u \mid\mid \sigma \mid dxdt)$
$+C(\int_\Sigma \mid d(x,t) \mid\mid \widetilde{\sigma} \mid dsdt + \int_\Sigma \mid \widetilde{u} \mid \widetilde{\sigma} dsdt)$
$\le (\beta\|\nabla u\|_{L^2(Q)} + M\|u\|_{L^2(Q)} + C\|d\|_{L^2(\Sigma)} + C\|\widetilde{u}\|_{L^2(\Sigma)})\|\sigma\|_V$

$\leq \widetilde{C}(\|d\|_{L^2(\Sigma)} + \|u\|_V)\|\sigma\|_V$

($M := \max_{\overline{\Omega}} q(x)$ and $\beta := \max_{\overline{\Omega}} \mid a_{ij}(x) \mid$),

from which one obtains

$$\|A(u)\|_* \leq \widetilde{C}(\|d\|_{L^2(\Sigma)} + \|u\|_V).$$

At this point, the hemicontinuity assertion is ensured by [**10**], Theorems 2.1 and 2.3.

(ii) $< A(u_1) - A(u_2), u_1 - u_2 >$

$= \sum_{i,j=1}^n \int\limits_Q a_{ij}(x)\partial_{x_j}(u_1 - u_2)\partial_{x_i}(u_1 - u_2)\, dxdt$

$+ \int\limits_Q q(x)(u_1 - u_2)^2\, dxdt$

$+ \int\limits_\Sigma (c(x,t,\widetilde{u}_1) - c(x,t,\widetilde{u}_2))(\widetilde{u}_1 - \widetilde{u}_2)dsdt \geq 0.$

(iii) $< A(u), u >= \sum_{i,j=1}^n \int\limits_Q a_{ij}(x)\partial_{x_j}u\partial_{x_i}u\, dxdt$

$+ \int\limits_Q q(x)u^2\, dxdt + \int\limits_\Sigma c(x,t,\widetilde{u})\widetilde{u}dsdt$

$\geq a\|\nabla u\|^2_{L^2(Q)} + \delta\|u\|^2_{L^2(Q)} + C_3\|\widetilde{u}\|^2_{L^2(\Sigma)} - \int\limits_\Sigma c_4(x,t)dsdt$

$\geq C'\|u\|^2_V - \int\limits_\Sigma c_4(x,t)dsdt,$

hence

$$\frac{< A(u), u >}{\|u\|_V} \geq C'\|u\|_V - \frac{\int\limits_\Sigma c_4(x,t)dsdt}{\|u\|^2_V} \to +\infty, \text{ as } \|u\|_V \to +\infty.$$

The proof is complete. ■

Besides, let $G \in V^*$ denote a linear functional defined by

$$< G, \sigma >:= \int\limits_Q f(x,t,w)\sigma dxdt,\ \forall \sigma \in V$$

then, we can rewrite problem (2.6) in an abstract form like

$$Lu + Au = G. \tag{3.1}$$

A such setting makes possible to apply the methods of the functional analysis.

Theorem 3.2. *If* $H_1)$–$H_4)$ *hold, problem* (3.1) *has a unique weak periodic solution.*

Proof. The existence of weak periodic solutions is a consequence of theorem 0, while the uniqueness descends from the strict monotonicity. ■

Next, we shall establish some energy estimates through the Faedo-Galerkin method. Let $\{s_h\}_h$ be an orthonormal base of the space $W^{1,2}(\Omega)$, we turn our interest to the finite-dimensional problem

$$\begin{aligned}&\int_\Omega \partial_t u_k s_h \; dx + \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\partial_{x_j} u_k \partial_{x_i} s_h \; dx + \int_\Omega q(x) u_k s_h \; dx \\ &+ \int_{\partial\Omega} \partial_t \widetilde{u}_k \widetilde{s}_h \; ds + \int_{\partial\Omega} c(x,t,\widetilde{u}_k)\widetilde{s}_h \; ds = \int_\Omega f(x,t,w) s_h \; dx,\end{aligned} \tag{3.2}$$

whose solution $u_k(x,t) := \sum_{h=1}^k y_h(t) s_h(x)$ is called a periodic Faedo-Galerkin approximation solution of $u(x,t)$ with the time periodic functions $y_h(t)$ satisfying

$$\begin{aligned}&2y_h'(t) + y_h(t)\Big(\sum_{i,j=1}^n \int_\Omega a_{ij}(x)\partial_{x_j} s_h \partial_{x_i} s_h \; dx + \int_\Omega q(x) \; dx\Big) \\ &= \int_\Omega f(x,t,w) s_h \; dx - \int_{\partial\Omega} c\Big(x,t,\sum_{h=1}^k y_h(t)\widetilde{s}_h(x)\Big)\widetilde{s}_h \; ds, \; h = 1,2,...k.\end{aligned}$$

Multiplying (3.2) by $y_h(t)$, integrating the resulting relation over $(0\,,\omega)$ and summing for $h = 1,2,...,k$, we get

$$\begin{aligned}&\int_Q u_k \partial_t u_k \; dxdt + \sum_{i,j=1}^n \int_Q a_{ij}(x)\partial_{x_j} u_k \partial_{x_i} u_k \; dxdt + \int_Q q(x) u_k^2 \; dxdt \\ &+ \int_\Sigma c(x,t,\widetilde{u}_k)\widetilde{u}_k \; dsdt = \int_Q f(x,t,w) u_k \; dxdt.\end{aligned} \tag{3.3}$$

By the periodicity of u_k, the first term of the left side in (3.3)

$$\int_Q u_k \partial_t u_k \; dxdt = 0,$$

by which

$$\begin{aligned}&\sum_{i,j=1}^n \int_Q a_{ij}(x)\partial_{x_j} u_k \partial_{x_i} u_k \; dxdt + \int_Q q(x) u_k^2 \; dxdt \\ &+ \int_\Sigma c(x,t,\widetilde{u}_k)\widetilde{u}_k \; dsdt = \int_Q f(x,t,w) u_k \; dxdt.\end{aligned}$$

Hence,

$$a\int_Q |\nabla u_k|^2\,dxdt + \delta\int_Q |u_k|^2\,dxdt$$

$$+C_3\int_\Sigma |\widetilde{u}_k|^2\,dsdt - \int_\Sigma c_4(x,t)dsdt \le \frac{1}{2}\int_Q |f(x,t,w)|^2\,dxdt + \frac{1}{2}\int_Q |u_k|^2\,dxdt,$$

thus,

$$\min\Big(\Big(a-\frac{1}{2}\Big),\,\delta,\,C_3\Big)\Big(\int_Q |\nabla u_k|^2\,dxdt + \int_Q |u_k|^2\,dxdt + \int_\Sigma |\widetilde{u}_k|^2\,dsdt\Big)$$

$$\le \frac{1}{2}\int_Q |f(x,t,w)|^2\,dxdt + \int_\Sigma |c_4(x,t)|\,dsdt.$$

This last inequality provides the energy estimate

$$\int_Q |\nabla u_k|^2\,dxdt + \int_Q |u_k|^2\,dxdt + \int_\Sigma |\widetilde{u}_k|^2\,dsdt \le L \tag{3.4}$$

where with $L > 0$, we denote various constants independent of k.

Now, we seek estimates of u_{kt}, $\widetilde{u}_{kt}$ in $L^2(\Omega\times P)$, respectively in $L^2(\partial\Omega\times P)$. To this aim, we consider the problem

$$\begin{aligned}\int_\Omega \partial_t u_k s_h\,dx + \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\partial_{x_j}u_k\partial_{x_i}s_h\,dx + \int_\Omega q(x)u_k s_h\,dx\\ + \int_{\partial\Omega} \partial_t\widetilde{u}_k\widetilde{s}_h\,ds + \int_{\partial\Omega} c(x,t,\widetilde{u}_k)\widetilde{s}_h\,ds = \int_\Omega f(x,t,w)s_h\,dx.\end{aligned} \tag{3.5}$$

Multiplying (3.5) by $y_h'(t)$ and summing for h one obtains

$$\begin{aligned}\int_\Omega |\partial_t u_k|^2\,dx + \sum_{i,j=1}^n \int_\Omega a_{ij}(x)(\partial_{x_j}u_k)\partial_t(\partial_{x_i}u_k)\,dx + \int_\Omega q(x)u_k\partial_t u_k\,dx\\ + \int_{\partial\Omega} |\partial_t\widetilde{u}_k|^2\,ds + \int_{\partial\Omega} c(x,t,\widetilde{u}_k)\partial_t\widetilde{u}_k\,ds = \int_\Omega f(x,t,w)\partial u_k\,dx.\end{aligned} \tag{3.6}$$

Integrating over $(0,\omega)$, using H_4), the periodicity and the Young inequality, we infer that

$$\int_Q \mid \partial_t u_k \mid^2 dxdt + \int_\Sigma \mid \partial_t \widetilde{u}_k \mid^2 dsdt \tag{3.7}$$

$$\leq \frac{C^2}{2\varepsilon} \int_\Sigma \mid d(x,t) \mid^2 dsdt + \varepsilon \int_\Sigma \mid \partial_t \widetilde{u}_k \mid^2 dsdt$$

$$+ \frac{C^2}{2\varepsilon} \int_\Sigma |\widetilde{u}_k \mid^2 dsdt + \varepsilon \int_Q \mid \partial_t u_k \mid^2 dxdt + \frac{C^2 \mid Q \mid}{2\varepsilon} + \frac{C^2}{2\varepsilon} \int_Q \mid w \mid^2 dxdt.$$

Concluding, one has

$$\int_Q \mid \partial_t u_k \mid^2 dxdt + \int_\Sigma \mid \partial_t \widetilde{u}_k \mid^2 dsdt \leq L. \tag{3.8}$$

Owing to (3.4) and (3.8), we deduce that u_k is bounded in D, namely

$$\|u_k\|_D \leq C, \; \forall k \in N.$$

Passing to subsequence, if necessary still denoted by u_k, we obtain

$$u_k \rightharpoonup u \text{ in } D$$

$$\nabla u_k \rightharpoonup \nabla u \text{ in } L^2(Q)$$

$$\partial_t u_k \rightharpoonup \partial_t u \text{ in } L^2(\Omega \times P)$$

$$\partial_t \widetilde{u}_k \rightharpoonup \partial_t \widetilde{u} \text{ in } L^2(\partial\Omega \times P).$$

A result of [**11**], Theorem 5.1, guarantees that the sequence u_k is precompact in $L^2(Q)$ so that

$$u_k \to u \text{ in } L^2(Q) \text{ and a.e. in } Q.$$

According to a trace theorem (see [**12**], Theorem 3.4.1) we also have

$$u_k \to u \text{ in } L^2(P; L^2(\partial\Omega)).$$

4 - Fixed Points

In order to find the solutions to (1.1)-(1.3), we employ a fixed point argument based on the Schauder fixed point theorem. To this end, we define the mapping

$$\Phi : L^2(Q) \to L^2(Q)$$

by setting

$$\Phi(w) = u$$

where u is the unique solution corrisponding to w, of (2.3)-(2.5). The mapping Φ is well-defined and one has

Proposition 4.1. *The mapping Φ is continuous in $L^2(Q)$.*

Proof. Let w_k be a sequence of $L^2(Q)$ such that $w_k \to w$ strongly in $L^2(Q)$ proceeding as in section 3, we infer that Φ is continuous since $u_k = \Phi(w_k)$ converges strongly to $u = \Phi(w)$ in $L^2(Q)$. ■

Besides,

Lemma 4.2. *There exists a constant $R > 0$ such that*

$$\|\Phi(w)\|_{L^2(Q)} \leq R, \ \forall w \in L^2(Q).$$

Proof. The result follows from (3.5) passing to the limit on n. ■

Since $\Phi(L^2(Q)) \subset D$, the compact embedding of D in $L^2(Q)$ gives us the compacteness of Φ from $L^2(Q)$ into itself.

We close the section, stating the main result of the paper

Theorem 4.3. *If* $H_1)$–$H_4)$ *are fulfilled, problem* (1.1)-(1.3) *admits weak periodic solutions.*

Proof. The above arguments show that the mapping Φ is both continuous and compact. By Schauder's fixed point theorem, Φ has a fixed point and we thus have a weak periodic solution for our problem with a dynamic boundary condition. This completes the proof of the theorem. ■

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Abstract

We study the existence of weak periodic solutions for semilinear parabolic problems with a semilinear dynamical boundary condition. Methods to investigate the periodicity utilize the functional frame of some results on maximal monotone operators and the Schauder fixed point theorem.

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