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On sets of unique best approximants (**)

For a non-empty subset M of a metric space (X, d) and $x \in X$, let $P_M(x)$ denote the set of all best approximants to x in M i.e. $P_M(x) = \{m_0 \in M : d(x, m_0) = \text{dist}(x, M)\}$. The concept and properties, defined in terms of best approximants or derived from it, are called approximative. For discussing approximative properties, S. B. Steckin [**5**] introduced and discussed some sets in Banach spaces. In this paper, we also consider the sets introduced by Steckin and extend some of the results proved in [**5**] to metric spaces.

Let M be a subset of a metric space (X, d) and $x \in X$. An element $m_0 \in M$ is said to be a best approximation to x if $d(x, m_0) \leq d(x, m)$ for all $m \in M$ i.e. $d(x, m_0) = d(x, M) \equiv \inf\{d(x, m) : m \in M\}$. The set of all such $m_0 \in M$ is denoted by $P_M(x)$. The set M is said to be

(i) proximinal if every element of X has a best approximation in M,

(ii) semi-Chebyshev if each element of X has at most one best approximation in M,

(iii) Chebyshev if each element of X has exactly one best approximation in M,

(iv) antiproximinal if $P_M(x) = \emptyset$ for each $x \in X \backslash M$, and

(v) approximatively compact if for every $x \in X$ and every minimizing sequence $\langle m_n \rangle$ in M i.e. satisfying $\lim_{n\to\infty} d(x, m_n) = d(x, M)$ has a subsequence $\langle m_{n_i} \rangle$ converging to an element of M.

For a given non-empty set M of a metric space (X, d), let

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$$E_M = \{x \in X : P_M(x) \neq \emptyset\},$$

$$U_M = \{x \in X : P_M(x) \quad \text{is empty or singleton}\},$$

$$T_M = \{x \in X : P_M(x) \quad \text{is singleton}\}.$$

The set M is proximinal (respectively semi-Chebyshev, respectively Chebyshev) if $E_M = X$ (respectively $U_M = X$, respectively $T_M = X$) and antiproximinal if $E_M \subset M$.

In order to determine T_M, we consider another set T'_M as under. Let

$$P_\delta(x) = P_{\delta,M}(x) = M \cap B_{d(x,M)+\delta}(x), \delta > 0,$$

$$D_M(x) = \lim_{\delta \to 0^+} \text{diam}\,(P_{\delta,M}(x)).$$

We define

$$T'_M = \{x \in X : D_M(x) = 0\}.$$

Here, diam (M) is the diameter of the set M and $P_\delta(x)$ is called the set of δ-nearest points to x.

Before we discuss approximative properties, we recall a few definitions and some elementary facts.

A subset A of a metric space (X, d) is said to be residual in X if A is a countable intersection of dense open subsets of X. Equivalently, if complement of A is of first category in X. A property is called generic if it is true for all elements of a residual set.

For a metric space (X, d) and a closed interval $I = [0, 1]$, a continuous mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X, \lambda \in I$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a convex metric space [**6**]. A convex metric space (X, d) is said to be strongly convex or an M-space [**2**] if for each pair $x, y \in X$ and every $\lambda \in I$, there exists exactly one point $z \in X$ such that $z = W(x, y, \lambda)$.

Every normed linear space is strongly convex but not conversely. If (X, d) is a convex metric space then for each two distinct points $x, y \in X$ and for every λ, $0 < \lambda < 1$, there exists at least one point $z \in X$ such that $d(x, z) = (1 - \lambda)d(x, y)$ and $d(z, y) = \lambda d(x, y)$. For strongly convex metric spaces such a z is always unique.

Let $G[x, y]$ denote the line segment joining x and y i.e. $G[x, y] = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$; $G(x, y, -)$ denote the ray starting from x and passing through y.

A metric space (X, d) is called externally convex [**2**] if for all distinct points x, y such that $d(x, y) = \lambda$ and $k > \lambda$ there exists a unique $z \in X$ such that $d(x, y) + d(y, z) = d(x, z) = k$ i.e. z lying on the ray $G(x, y, -)$.

Every normed linear space is externally convex.

A convex metric space (X, d) is said to be strictly convex [**3**] if for every $x, y \in X$ and $r > 0$, $d(u, x) \leq r$, $d(u, y) \leq r$ imply $d(u, W(x, y, \lambda)) < r$ unless $x = y$, where u is arbitrary but fixed point of X.

A relation between T_M and $T_M^{'}$ is given by:

Proposition 1. *If M is a closed subset of a complete metric space (X, d) then $T_M^{'} \subset T_M$.*

Proof. Let $x \in T_M^{'}$ i.e. $D_M(x) = 0$. This implies $\lim\limits_{\delta \to 0} \operatorname{diam}(P_{\delta,M}(x)) = 0$. Now $\operatorname{diam} P_M(x) \leq \operatorname{diam}(P_{\delta,M}(x))$ for all δ. This implies $\operatorname{diam} P_M(x) \leq 0 \Rightarrow P_M(x)$ is either empty or singleton. We claim that $P_M(x) \neq \emptyset$. Let $<u_n>$ be a minimizing sequence in M for x. Since $D_M(x) = 0$, given $\varepsilon > 0$ there exists some $\delta_1 > 0$ such that $\operatorname{diam} P_{\delta,M}(x) < \varepsilon$ whenever $0 < \delta < \delta_1$. Fix up such a $\delta > 0$, $u_n \in P_{\delta,M}(x)$ for all $n \geq N$ for a suitable N. Therefore, $d(u_n, u_m) < \varepsilon$ for all $n, m \geq N$. This shows that $\langle u_n \rangle$ is a Cauchy sequence in M. Since M is complete, $\langle u_n \rangle \to u_0 \in M$. Since $\lim\limits_{n \to \infty} d(x, u_n) = d(x, M), d(x, u_0) = d(x, M)$ and so $u_0 \in P_M(x)$ i.e. $P_M(x) \neq \emptyset$. Therefore $P_M(x)$ is a singleton and so $x \in T_M$. Hence $T_M^{'} \subset T_M$.

For approximatively compact sets M, we have

Proposition 2. *If M is an approximatively compact subset of a complete metric space (X, d) then $T_M^{'} = T_M = U_M$.*

Proof. Clearly, $T_M \subset U_M$. Since M is approximatively compact, $P_M(x) \neq \emptyset$ (sec e.g. [**4**], p.382) and so $U_M \subset T_M$. Hence $T_M = U_M$.

Since M, being approximatively compact, is proximinal and so closed, $T_M^{'} \subset T_M$ by Proposition l. Now we show that $T_M \subset T_M^{'}$.

Let $x \in T_M$ and suppose $P_M(x) = \{u_0\}$. Now $D_M(x) = \lim\limits_{\delta \to 0^+} \operatorname{diam} P_{\delta,M}(x)$. If $u \in P_{\delta,M}(x)$ then $d(x, u) \leq d(x, M) + \delta$. For $\delta > 0$, we can choose $n > \dfrac{1}{\delta} \equiv N_1$ such that $d(x, u) < d(x, M) + \dfrac{1}{n}$ i.e. $u \in P_{\frac{1}{n},M}(x)$ for all $n \geq N_1$. Since $P_{\delta,M}(x) \subset P_{\frac{1}{n},M}(x)$, $\operatorname{diam} P_{\delta,M}(x) < \operatorname{diam} P_{\frac{1}{n},M}(x)$ for all $n \geq N_1 \Rightarrow D_M(x) \leq \operatorname{diam} P_{\frac{1}{n},M}(x)$ for all $n \geq N_1$.

Thus we can find N_1 such that $\operatorname{diam} P_{\frac{1}{n},M}(x) > \frac{1}{2}D_M(x)$ for all $n \geq N_1$. We can pick minimizing sequences $\langle v_n\rangle$, $\langle v_n^{'}\rangle$ for x such that $\{v_n, v_n^{'}\} \subset P_{\frac{1}{n},M}(x)$ for all $n \geq N_2$ and $d(v_n, v_n^{'}) > \frac{1}{2}D_M(x)$. Let $N = \max\{N_1, N_2\}$. Then $\operatorname{diam} P_{\frac{1}{n},M}(x) > \frac{1}{2}D_M(x)$, $\{v_n, v_n^{'}\} \subset P_{\frac{1}{n},M}(x)$ and $d(v_n, v_n^{'}) > \frac{1}{2}D_M(x)$ for all $n \geq N$. Since M is approximatively compact, there are subsequences $\langle v_{n_k}\rangle$, $\langle v_{n_k}^{'}\rangle$ of $\langle v_n\rangle$, $\langle v_n^{'}\rangle$ respectively each converging to $u_0 = P_M(x)$. Since $d(v_{n_k}, v_{n_k}^{'}) \leq d(v_{n_k}, P_M(x)) + d(P_M(x), v_{n_k}^{'})$, $d(v_{n_k}, v_{n_k}^{'}) \to 0$ as $n_k \to \infty$.Thus $0 \geq \frac{1}{2}D_M(x) \Rightarrow D_M(x) = 0 \Rightarrow x \in T_M^{'}$ and so $T_M \subset T_M^{'}$. Hence $T_M^{'} = T_M = U_M$.

The following theorem generalizes and extends a result of Steckin [**5**] (see also Braess [**1**], p.29) proved for strictly convex Banach spaces.

Theorem. *If M is an approximatively compact subset of a complete strictly convex metric space (X, d) which is externally convex then T_M and $T_M^{'}$ are residual sets.*

To prove this result, we firstly establish few lemmas:

Lemma 1. *In a metric space (X, d), the set $G_a = \{x \in X : D_M(x) < a\}$, $a > 0$ is an open set.*

Proof. Let $x \in G_a$ i.e. $D_M(x) < a$. Then there is some $\delta_1 > 0$ such that $\operatorname{diam} P_{\delta,M}(x) < a$ for $0 < \delta < \delta_1$. If $y \in X$ is such that $d(y, x) < \frac{\delta}{3}$, we claim that $P_{\frac{\delta}{3},M}(y) \subset P_{\delta,M}(x)$.

Suppose $u \in P_{\frac{\delta}{3},M}(y)$ i.e. $d(y, u) < d(y, M) + \frac{\delta}{3}$. Consider

$$\begin{aligned} d(x, u) &\leq d(x, y) + d(y, u) \\ &< \frac{\delta}{3} + d(y, M) + \frac{\delta}{3} \\ &= d(y, M) + \frac{2\delta}{3} \\ &\leq d(y, x) + d(x, M) + \frac{2\delta}{3} \\ &< d(y, M) + \delta. \end{aligned}$$

This implies $u \in P_{\delta,M}(x)$ and so $P_{\frac{\delta}{3},M}(y) \subset P_{\delta,M}(x)$ and therefore $\operatorname{diam} P_{\frac{\delta}{3},M}(y) \leq \operatorname{diam} P_{\delta,M}(x) < a$ for $0 < \delta < \delta_1$. This gives $\lim_{\delta\to 0} P_{\frac{\delta}{3},M}(y) < a$ and so $D_M(y) < a$ i.e. $y \in G_a$. Hence $B_{\frac{\delta}{3}}(x) \subset G_a$ i.e. G_a is open.

Lemma 2. [**6**] *In a convex metric space* (X, d), $d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)$, $x, y \in X$, $0 \leq \lambda \leq 1$.

Proof. Consider

$$\begin{aligned} d(x,y) \leq & d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y) \\ & \leq \lambda d(x,x) + (1-\lambda)d(x,y) + \lambda d(x,y) + (1-\lambda)d(y,y) \\ & = d(x,y). \end{aligned}$$

The result follows.

Lemma 3. *Let M be a subset of a convex metric space (X, d) and $x \in X$. If $m_0 \in P_M(x)$ and $y = W(x, m_0, \lambda)$ then $m_0 \in P_M(y)$.*

Proof. $m_0 \in P_M(x) \Rightarrow d(x, m_0) = d(x, M)$ i.e. $d(x, m_0) \leq d(x, m)$ for all $m \in M$. By Lemma 2, $d(x, W(x, m_0, \lambda)) + d(W(x, m_0, \lambda), m_0) = d(x, m_0)$. Consider

$$\begin{aligned} d(W(x, m_0, \lambda), m_0) &= d(x, m_0) - d(x, W(x, m_0, \lambda)) \\ &\leq d(x, m) - d(x, W(x, m_0, \lambda)) \quad \text{for all } m \in M \\ &\leq d(x, W(x, m_0, \lambda)) + d(W(x, m_0, \lambda), m) \\ &\quad - d(x, W(x, m_0, \lambda) \quad \text{for all } m \in M \\ &= d(W(x, m_0, \lambda), m) \quad \text{for all } m \in M. \end{aligned}$$

This implies that $m_0 \in P_M(y)$.

Corollary. *If M is a subset of a convex metric space (X, d) then $W(x, Px, \lambda) \in T'_M$ for each $x \in T'_M$ and $\lambda \in [0, 1]$, where $Px \equiv P_M(x)$.*

Proof. Let $x \in T'_M$ and $z \in W(x, Px, \lambda)$. Then by the above lemma, $Px \in Pz$. Since

$$d(x, z) = (1-\lambda)d(x, Px) \quad \text{and } d(Px, z) = \lambda d(x, Px) \tag{1}$$

we have, $d(z, M) = d(z, Px) = \lambda d(x, Px) = \lambda d(x, M)$. Therefore (1) gives

$$d(x, z) = d(x, Px) - \lambda d(x, Px) = d(x, M) - d(z, M). \tag{2}$$

We claim that $B[z, d(z, M) + \delta] \subset B[x, d(x, M) + \delta]$. Let $u \in B[z, d(z, M) + \delta]$ i.e. $d(u, z) \leq d(z, M) + \delta$. Consider

$$\begin{aligned} d(u,x) &\le d(u,z) + d(z,x) \\ &\le d(z,M) + \delta + d(z,x) \\ &= d(z,M) + \delta + d(x,M) - d(z,M) \text{ by (2)} \\ &= d(x,M) + \delta \end{aligned}$$

i.e. $u \in B[x, d(x,M) + \delta]$ and so $B[z, d(z,M) + \delta] \subset B[x, d(x,M) + \delta]$. Consequently,

$$M \cap B[z, d(z,M) + \delta] \subset M \cap B[x, d(x,M) + \delta]$$

i.e. $P_{\delta,M}(z) \subset P_{\delta,M}(x)$ and so $\lim_{\delta \to 0} \operatorname{diam}(P_{\delta,M}(z)) \le \lim_{\delta \to 0} \operatorname{diam}(P_{\delta,M}(x))$ i.e $D_M(z) \le D_M(x) = 0$, which gives $D_M(z) = 0$ i.e. $z \in T_M^{'}$. Hence $W(x, Px, \lambda) \in T_M^{'}$ for all $x \in T_M^{'}$ and $\lambda \in [0,1]$.

Lemma 4. *If G is a subset of an externally convex M-space (X,d) and $x \in X$ then $P_G(W(x, v_0, \lambda))$ is at most singleton for each $v_0 \in P_G(x)$.*

Proof. Let $v_0 \in P_G(x)$. Then by Lemma 3, $v_0 \in P_G(W(x, v_0, \lambda))$. Suppose $v_1 \in P_G(W(x, v_0, \lambda))$. Two cases arise:

Case I $d(x, v_1) \ne d(x, W(x, v_0, \lambda)) + d(W(x, v_0, \lambda), v_1)$

$$\begin{aligned} \text{i.e. } d(x, v_1) &< d(x, W(x, v_0, \lambda)) + d(W(x, v_0, \lambda), v_1) \\ &= d(x, W(x, v_0, \lambda)) + d(W(x, v_0, \lambda), v_0) \\ &= d(x, v_0), \end{aligned}$$

which is not possible.

Case II $d(x, v_1) = d(x, W(x, v_0, \lambda)) + d(W(x, v_0, \lambda), v_1)$.

This implies $v_1 \in G(x, x_\lambda, -)$; where $x_\lambda \equiv W(x, v_0, \lambda)$. Thus $v_0, v_1 \in G(x, x_\lambda, -)$. Now

$$\begin{aligned} d(x, v_1) &= d(x, x_\lambda) + d(x_\lambda, v_1) \\ &= d(x, x_\lambda) + d(x_\lambda, v_0) \\ &= d(x, v_0) \end{aligned}$$

i.e. $d(x, x_\lambda) + d(x_\lambda, v_0) = d(x, v_0)$ and $d(x, x_\lambda) + d(x_\lambda, v_0) = d(x, v_1)$. Therefore by external convexity, we get $v_0 = v_1$.

Since every strictly convex metric space is an M-space [**2**], we have

Corollary. *If X is a strictly convex metric space with external convexity then $P_G(W(x, v_0, \lambda))$ is at most singleton for each $v_0 \in P_G(x)$.*

Lemma 5. *If (X, d) is a strictly convex metric space with external convexity then U_M is dense in X.*

Proof. Let $x \in X$ be arbitrary. If $P_M(x) = \emptyset$ then $x \in U_M \subset \overline{U_M}$. Suppose $P_M(x) \neq \emptyset$. Let $v_0 \in P_M(x)$. Define $x_n = W\left(x, v_0, 1 - \frac{1}{n}\right), n = 1, 2, \ldots$ i.e. x_n lies on the line segment joining x, v_0 and so by Lemma 3, $v_0 \in P_M(x_n)$ for all n. Since X is strictly convex, $P_M(x_n) = \{v_0\}$ for all n by Lemma 4. Thus $x_n \in U_M$ for all n. We claim that $\langle x_n \rangle \to x$. Consider

$$
\begin{aligned}
d(x_n, x) &= d\left(W\left(x, v_0, 1 - \frac{1}{n}\right), x\right) \\
&= \frac{1}{n} d(x, v_0) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore $\langle x_n \rangle \to x$ and so $x \in \overline{U_M}$. Hence $X = \overline{U_M}$ i.e. U_M is dense in X.

Since every normed linear space is externally convex and for proximinal sets M, $P_M(x) \neq \emptyset$, we have

Corollary. **[5]** *If M is a proximinal subset of a strictly convex normed linear space X then the set T_M is dense in X.*

Proof of Theorem. Since M is approximatively compact, $T_M' = T_M = U_M$ by Proposition 2. By Lemma 5, U_M is dense in the strictly convex space X. Since $T_M' \subset G_a$ for all $a > 0$, $U_M \subset G_a$ for all $a > 0$. This implies $\overline{U_M} \subset \overline{G_a} \Rightarrow X \subset \overline{G_a}$ for all $a > 0 \Rightarrow G_a$ is dense in X for all $a > 0$. Now

$$
\begin{aligned}
T_M' &= \{x \in X : D_M(x) = 0\} \\
&= \bigcap_{n \in N} \left\{x \in X : D_M(x) < \frac{1}{n}\right\} \\
&= \bigcap_{n \in N} G_{\frac{1}{n}}.
\end{aligned}
$$

Hence T_M' and T_M are residual sets in X.

Since every normed linear space is externally convex, we have

Corollary. [**5**] *If M is an approximatively compact subset of a strictly convex Banach space X then T_M and $T_M^{'}$ are residual sets.*

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Abstract

S. B. Steckin [**5**] *proved that for a proximinal subset M of a strictly convex Banach space X, the set* $\{x \in X : x$ has a unique best approximation in $M\}$ *is dense in X and is a residual set in X if M is approximatively compact subset of X. We extend these results to convex metric spaces.*

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