Riv. Mat. Univ. Parma (7) **7** (2007), 159-171

T. Tshikuna-Matamba (*)

A note on nearly $\alpha-$Kenmotsu submersions (**)

1 - Introduction

In [TM1], many classes of almost contact metric manifolds, furnished with the Kenmotsu metric, have been characterized. Among them, nearly Kenmotsu manifolds seem to be interesting according to the result in [TM3]; there, it is shown that a Riemannian submersion with nearly Kenmotsu structure as total space, has minimal fibres and preserves the sectional holomorphic curvature tensor on horizontal distribution. Also, semi-invariant submanifolds of nearly Kenmotsu manifolds are studied by M.M. Tripathi and S.S. Shukla in [T-S].

Following D. Janssens and L. Vanhecke [J-V], who defined $\alpha-$Kenmotsu manifolds, we define a nearly $\alpha-$Kenmotsu structure. This class includes the classes of nearly Kenmotsu, nearly cosymplectic, nearly-K-cosymplectic and closely cosymplectic structures.

This paper is organized in the following way.

In Section 2, we recall some background notions on almost Hermitian and almost contact metric manifolds which will be needed in the sequel.

Section 3 is devoted to almost contact metric submersions. Here we will review fundamental properties and describe the structure of the base space or that of the fibre submanifolds.

In Section 4, we examine the geometry of the fibres; the properties of the O'Neill's tensors are used to prove the minimality and the superminimality of the

(*) Département de Mathématiques, Institut Supérieur Pédagogique, B.P. 282-Kananga, République Démocratique du Congo, e-mail: tshikmat@yahoo.fr

(**) Received 18th December 2006. AMS classification 53C15, 53C25.

fibres. It is shown that *the fibres of a nearly a−Kenmotsu submersion are minimal.* If they are superminimal, then the horizontal distribution is integrable.

Section 5 is concerned with the curvature property. We show that *a nearly a−Kenmotsu submersion preserves the holomorphic sectional curvature tensor on horizontal vector fields.*

I am grateful to Professor M. Falcitelli for her judicious comments.

2 - Preliminaries on manifolds

An almost Hermitian manifold is a Riemannian manifold (M, g) equipped with a tensor field , J, of type $(1, 1)$ satisfying the following conditions:

(i) $J^2D = -D$, and
(ii) $g(JD, JE) = g(D, E)$, for all D, $E \in \chi(M)$.

Any almost Hermitian manifold, (M, g, J), admits a differentail $2-$form, Ω, defined by

$$\Omega(D, E) = g(D, JE),$$

and called the fundamental form or the Kähler form. Almost Hermitian manifolds are of even dimension say, $2m$. From the classification of almost Hermitian structures, obtained by A. Gray and L.M. Hervella in [G-H], we shall be intersted with the nearly Kähler structure. This is defined by

$$(\nabla_D \Omega)(D, E) = 0.$$

An almost contact structure on a differentiable manifold, M, is a triple (φ, ξ, η) where:

(i) ξ is a distinguished vector field,
(ii) η is a differential $1-$form such that $\eta(\xi) = 1$, and
(iii) φ is a tensor field of type $(1, 1)$ satisfying

$$\varphi^2 D = -D + \eta(D)\xi,$$

for all $D \in \chi(M)$.

If, in addition, M admits a Riemannian metric g such that

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E),$$

then g is called a compatible metric. In this case, $(M, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. Almost contact metric manifolds are of odd dimension noted, $2m+1$.

As in the case of almost Hermitian manifolds, the fundamental $2-$form, ϕ, of an almost contact metric manifold is defined by

$$\phi(D,E) = g(D,\varphi E).$$

Let us recall the defining relations of those structures which will be used in this study.

An almost contact metric manifold is said to be:

(1) *cosymplectic* if $\nabla\varphi = 0$;
(2) *closely cosymplectic* if $(\nabla_D\varphi)D = 0$, and $d\eta = 0$;
(3) *nearly cosymplectic* if $(\nabla_D\varphi)D = 0$;
(4) *nearly-K-cosymplectic* if $(\nabla_D\varphi)E + (\nabla_E\varphi)D = 0 = \nabla_D\xi$;
(5) *nearly Kenmotsu* if $(\nabla_D\varphi)E + (\nabla_E\varphi)D = -\eta(E)\varphi D - \eta(D)\varphi E$;
(6) *Kenmotsu* if $(\nabla_D\varphi)E = g(\varphi D,E)\xi - \eta(E)\varphi D$.

Let a be a real number, in [J-V], Janssens and Vanhecke have defined an $a-$Kenmotsu manifold by setting

$$(\nabla_D\varphi)E = a\cdot\{g(\varphi D,E)\xi - \eta(E)\varphi D\}.$$

Following this, a nearly $a-$Kenmotsu manifold is one defined by

$$(\nabla_D\varphi)E + (\nabla_E\varphi)D = a\cdot\{-\eta(E)\varphi D - \eta(D)\varphi E\}.$$

Taking $E = D$, the above relation gives rise to

$$(\nabla_D\varphi)D = a\cdot\{-\eta(D)\varphi D\},$$

which defines a nearly $a-$Kenmotsu as pointed in [TM4]. Setting $a = 0$, the last relation reduces to the nearly cosymplectic structure.

3 - Almost contact metric submersions

Let us recall, from [O'N], that a Riemannian submersion is a surjective mapping

$$\pi : M \longrightarrow B,$$

between Riemannian manifolds such that

(i) π is of maximal rank;
(ii) $\pi_*/(Ker\,\pi_*)^\perp$ is a linear isometry.

The tangent bundle $T(M)$, of the total space M, admits an orthogonal decomposition

$$T(M) = H(M) \oplus V(M).$$

We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and the horizontal projections respectively. A vector field X of the horizontal distribution, $H(M)$, is called a basic vector field if it is $\pi-$related to a vector field X_* of the base space B. Such a vector field means that $X_* = \pi_* X$. On the base space, tensors and other objects will be denoted by a prime $'$ while those on the fibres will be specified by a carret $\hat{}$.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'+1}, g', \varphi', \xi', \eta')$ be almost contact metric manifolds. By an almost contact metric submersion of type I, one understands a Riemannian submersion

$$\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$$

satisfying:

(i) $\pi_* \varphi = \varphi' \pi_*$,
(ii) $\pi_* \xi = \xi'$.

When the base space is an almost Hermitian manifold, $(B^{2m'}, g', J')$, then the Riemannian submersion

$$\pi : M^{2m+1} \longrightarrow B^{2m'},$$

is called an almost contact metric submersion of type II if

$$\pi_* \varphi = J' \pi_*.$$

If a property holds for both type I and type II submersions, we will denote

$$\pi : M \longrightarrow B$$

without refering to the dimension.

Now, we overview some of the fundamental properties of such types of submersions.

Proposition 3.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Then*

(a) $\pi^* \phi' = \phi$;
(b) $\pi^* \eta' = \eta$;
(c) *the horizontal and vertical distributions are $\varphi-$invariant*;
(d) $\eta(U) = 0$ *if* $U \in V(M)$;
(e) $\mathcal{H}(\nabla_X \varphi)Y$ *is basic associated to* $(\nabla'_{X_*} \varphi')Y_*$ *if* X *and* Y *are basic.*

Proof. See Watson [W1]. □

Proposition 3.2. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. Then*

(a) $\pi^* \Omega' = \phi$;
(b) *the horizontal and vertical distributions are $\varphi-$invariant;*
(c) $\eta(X) = 0$ if $X \in H(M)$;
(d) *$\mathcal{H}(\nabla_X \varphi)Y$ is basic associated to $(\nabla'_{X_*} J')Y_*$ if X and Y are basic.*

Proof. See again Watson [$W1$]. □

Now, from a given structure of the total space, we want to determine the corresponding structure on the base space and the fibres.

Proposition 3.3. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is a nearly $\alpha-$Kenmotsu manifold, then the base space inherits the structure of the total space while the fibres are nearly Kähler manifolds.*

Proof. Let X and Y be basic vector fields. According to the defining relation of a nearly $\alpha-$Kenmotsu manifold, we have

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = \alpha \cdot \{-\eta(Y)\varphi X - \eta(X)\varphi Y\}.$$

Since $\mathcal{H}(\nabla_X \varphi)Y$ is basic associated to $(\nabla'_{X_*}\varphi')Y_*$, $\mathcal{H}(\nabla_Y \varphi)X$ is basic associated to $(\nabla'_{Y_*}\varphi')X_*$ and $\pi^*\eta' = \eta$, we get

$$(\nabla'_{X_*}\varphi')Y_* + (\nabla'_{Y_*}\varphi')X_* = \alpha \cdot \{-\eta'(Y_*)\varphi' X_* - \eta'(X_*)\varphi' Y_*\}.$$

The base space is then a nearly $\alpha-$Kenmotsu manifold.

Concerning the structure of the fibres, let us consider U and V tangent to the fibres. The defining relation of the total space gives rise to

$$(\nabla_U \varphi)V + (\nabla_V \varphi)U = 0,$$

because $\eta(U) = 0 = \eta(V)$ from Proposition 3.1(d). It is known that, the fibres of an almost contact metric submersion of type I are almost Hermitian manifolds. So, on the fibres, one has $\hat{\varphi}U = JU$ and $\hat{\varphi}V = JV$. Therefore, the relation

$$(\hat{\nabla}_U \hat{\varphi})V + (\hat{\nabla}_V \hat{\varphi})U = 0$$

defines a nearly Kähler structure. □

Proposition 3.4. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ *be an almost contact metric submersion of type II. If the total space is a nearly* $\alpha-$*Kenmotsu manifold, then the base space is a nearly Kähler manifold.*

Proof. It is not hard to show that a nearly $\alpha-$Kenmotsu manifold verifies the relation

$$(\nabla_D \phi)(D, E) = \alpha \cdot \eta(D)\phi(E, D).$$

Considering the basic vector fields X and Y, one gets $(\nabla_X \phi)(X, Y) = 0$ because of the vanishing of η on horizontal distribution from Proposition 3.2(c).

Since $\pi^* \Omega' = \phi$ by Proposition 3.2(a), one obtains

$$(\nabla'_{X_*} \Omega')(X_*, Y_*) = 0,$$

which defines a nearly Kähler structure on the base space. □

4 - The geometry of the fibres

The O'Neill configuration tensors, T and A, on the total space of a Riemannian submersion are defined in [O'N] by setting

$$T_D E = \mathcal{H}\nabla_{\mathcal{V}D}\mathcal{V}E + \mathcal{V}\nabla_{\mathcal{V}D}\mathcal{H}E,$$

$$A_D E = \mathcal{V}\nabla_{\mathcal{H}D}\mathcal{H}E + \mathcal{H}\nabla_{\mathcal{H}D}\mathcal{V}E.$$

Among the fundamental properties of these tensors, we recall

(4.1) $$T_U V = T_V U;$$

(4.2) $$T_E = T_{\mathcal{V}E};$$

(4.3) $$\mathcal{H}\nabla_U V = T_U V;$$

(4.4) $$A_X Y = -A_Y X;$$

(4.5) $$A_E = A_{\mathcal{H}E}\,.$$

If X is basic, then

(4.6) $$\mathcal{H}\nabla_U X = A_X U,$$

and $[U, X]$ is vertical.

It is known that T is used in the geometry of the fibres and A is the integrability tensor of the horizontal distribution.

Proposition 4.1. *Let $\pi : M \longrightarrow B$ be an almost contact metric submersion of type I or type II such that the total space is a nearly Kenmotsu manifold. Then*

(a) $T_U \varphi U = \varphi T_U U$;
(b) $T_U \xi = 0$;
(c) $A_X \varphi X = 0$;
(d) $A_\xi \xi = 0$.

Proof. The defining relation of a nearly Kenmotsu manifold reduces to

$$(\nabla_D \varphi)D = -\eta(D)\varphi D.$$

Consider the case of a type I submersion. Since $\eta(U) = 0$, the above relation becomes $(\nabla_U \varphi)U = 0$ from which

$$T_U \varphi U - \varphi T_U U = 0$$

via the horizontal projection.

In the case of a type II submersion, since φU is vertical in the light of Proposition 3.2(b), the horizontal projection leads to

$$\mathcal{H}(\nabla_U \varphi)U = -\eta(U)\mathcal{H}(\varphi U) = 0$$

because $\mathcal{H}(\varphi U) = 0$.

As $\mathcal{H}(\nabla_U \varphi)U = 0$, one deduces $T_U \varphi U = \varphi T_U U$ which establishes (a) whose (b) is a consequence.

Let us examine (c). On a basic vector field X, we have

$$(\nabla_X \varphi)X = -\eta(X)\varphi X.$$

Consider a type I submersion. It is known, from Proposition 3.1(c), that φX is horizontal. Therefore, the vertical projection

$$\mathcal{V}(\nabla_X \varphi)X = -\eta(X)\mathcal{V}(\varphi X) = 0$$

because $\mathcal{V}(\varphi X) = 0$. Since $\mathcal{V}(\nabla_X \varphi)X = 0$, we deduce $A_X \varphi X = \varphi A_X X$.

In the case of a type II submersion, one gets $(\nabla_X \varphi)X = 0$ because $\eta(X) = 0$; thus, $A_X \varphi X - \varphi A_X X = 0$. On the other hand, by virtue of (4.4), $A_X X = 0$ so that $A_X \varphi X = 0$ is true. This is the proof of (c) from which (d) follows. □

Theorem 4.1. *Let $\pi : M \longrightarrow B$ be an almost contact metric submersion of type I or type II. If the total space is a nearly a–Kenmotsu manifold, then*

(a) $T_U \varphi U = \varphi T_U U$;
(b) $T_\xi \xi = 0$;

(c) $A_X \varphi X = 0$;
(d) $A_\xi \xi = 0$.

Proof. Recall that a nearly $a-$Kenmotsu manifold is defined by

$$(\nabla_D \varphi)E + (\nabla_E \varphi)D = a \cdot \{-\eta(E)\varphi D - \eta(D)\varphi E\}.$$

Setting $D = E = U$ in the above relation, one gets

$$(\nabla_U \varphi)U = -a \cdot \eta(U)\varphi U,$$

which is the relation in Proposition 4.1. □

As a consequence of the above Theorem 4.1, one has

Corollary 4.1. *Let $\pi : M \longrightarrow B$ be an almost contact metric submersion of type I or type II. If the total space is a nearly $a-$Kenmotsu manifold, then the fibres are minimal submanifolds.*

Proof. In [TM2], it is shown that if the O'Neill tensor T verifies

$$T_U \varphi V = \varphi T_U V,$$

then the fibres are minimal. Thus, considering Proposition 4.1 and Theorem 4.1(a) and (b), the proof follows. □

Now, let us turn our attention to another subject. The superminimality of the fibres. We will follow Watson, [W2], who studied this for an almost Hermitian submersion. Note that, in [F-P], a result has been obtained concerning the superminimality of the fibres of an almost Kähler submersion.

Definition 4.1. *Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold and $\bar{M}$ a $\varphi-$invariant submanifold of M. If, $(\nabla_V \varphi) = 0$ for all V tangent to $\bar{M}$, then $\bar{M}$ is said to be superminimal.*

Note that, a superminimal $\varphi-$invariant almost contact metric submanifold of an almost contact metric manifold is minimal. But, this may not be true for a superminimal $\varphi-$invariant almost Hermitian submanifold of an almost contact metric manifold.

Remark 4.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. In order to verify superminimality of the almost Hermitian fibres, $(\hat{M}, \hat{J}, \hat{g})$, there are four components of $g((\nabla_V \varphi)E, F)$ to be considered on the* *total space. We find*

(SM-1) $g((\nabla_V\varphi)U,W)=g(\hat{\nabla}_V\hat{J}U-\hat{J}\hat{\nabla}_VU,W)$,
(SM-2) $g((\nabla_V\varphi)U,X)=g(\varphi T_VU-T_V\varphi U,X)$,
(SM-3) $g((\nabla_V\varphi)X,U)=-g((\nabla_V\varphi)U,X)$,
(SM-4) $g((\nabla_V\varphi)X,Y)=-g(A_{\varphi X}Y+A_X\varphi Y,V)$.

In the case of an almost contact metric submersion of type II, it is known that the fibres are almost contact metric manifolds. To verify superminimality of the fibres, we consider

(SM-5) $g((\nabla_V\varphi)U,W)=g(\hat{\nabla}_V\hat{\varphi}U-\hat{\varphi}\hat{\nabla}_VU,W)$,
(SM-6) $g((\nabla_V\varphi)U,X)=g(\varphi T_VU-T_V\varphi U,X)$,
(SM-7) $g((\nabla_V\varphi)X,U)=-g((\nabla_V\varphi)U,X)$,
(SM-8) $g((\nabla_V\varphi)X,Y)=-g(A_{\varphi X}Y+A_X\varphi Y,V)$.

Proposition 4.2. *Let $\pi:M^{2m+1}\longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the fibres are superminimal, then they are Kähler.*

Proof. The vanishing of (SM-1) calculation is equivalent to the assertion that the fibres are Kähler. □

Proposition 4.3. *Let $\pi:M\longrightarrow B$ be an almost contact metric submersion of type I or type II. If the total space is cosymplectic, then the fibres are superminimal.*

Proof. Since the total space is cosymplectic, we have $(\nabla_E\varphi)F=0$. Let us consider a vector field, V, tangent to the fibres. Then $(\nabla_V\varphi)F=0$ which shows that the fibres are superminimal. □

Theorem 4.2. *Let $\pi:M\longrightarrow B$ be an almost contact metric submersion of type I or type II such that the total space is nearly Kenmotsu. If the fibres are superminimal, then the horizontal distribution is completeley integrable.*

Proof. It is known that the defining relation of a nearly Kenmotsu manifold can reduce to

$$(\nabla_D\varphi)D+\eta(D)\varphi D=0.$$

We then have

$$0=g(U,(\nabla_X\varphi)X)+\eta(X)g(\varphi X,U).$$

Since φX is horizontal, $g(\varphi X, U) = 0$, so, it remains

$$0 = g(U, (\nabla_X \varphi)X)$$

$$0 = g(U, A_X \varphi X - \varphi A_X X)$$

$$0 = g(U, A_X \varphi X).$$

Therefore, $A_X \varphi X = 0$. From the superminimality of the fibres, we conclude that $A \equiv 0$.

Consider the case of a type II submersion, the vanishing of η on horizontal vector fields yields $(\nabla_X \varphi)X = 0$. From this, we can obtain $A_X \varphi Y = A_{\varphi X} Y$. Combining this with calculation (SM-8) gives rise to $A \equiv 0$. □

Theorem 4.3. *Let $\pi : M \longrightarrow B$ be an almost contact metric submersion of type I or type II with the total space a nearly $a-$Kenmotsu manifold. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. Since, by Theorem 4.1(c), $A_X \varphi X = 0$, we deduce $A \equiv 0$ as in the proof of Theorem 4.2. □

5 - Curvature

Let us recall that the Riemannian curvature tensor R of a Kähler manifold satisfies the K_1-identity (the Kähler identity) defined by

$$R(D, E, F, G) = R(D, E, JF, JG).$$

Other K_i-identities (i = 1, 2, 3) have been studied by A. Gray in [G], but their interrelations with the theory of Riemannian submersions can be found in [W-V].

Let (M^{2m}, g, J) be an almost Hermitian manifold. The K_i-curvature properties are defined in the following way.

(1) K_1 : if $R(D, E, F, G) = R(D, E, JF, JG)$;
(2) K_2 : if $R(D, E, F, G) = R(JD, E, JF, G) + R(JD, JE, F, G) + R(JD, E, F, JG)$;
(3) K_3 : if $R(D, E, F, G) = R(JD, JE, JF, JG)$.

The notation NK_1 means that the manifold is nearly Kähler and possesses the K_1-curvature property. It is known that $NK_1 = K_1$, thus, we will consider K_1-curvature property.

In their study of curvature tensors of almost contact metric manifolds, D. Janssens and L. Vanhecke, [J-V], have defined the *cosymplectic curvature* *property* by setting

$$R(D,E,F,G) = R(D,E,\varphi F,\varphi G).$$

Kenmotsu and Sasakian curvature properties are also defined in [J-V] and recalled in [TM5].

We shall be interested by the cosymplectic curvature property which will intertwine with the Kähler one.

First we have.

Theorem 5.1 [J-V]. *Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. If it satisfies the condition*

$$(\nabla_D \varphi)E = 0,$$

then it verifies the cosymplectic curvature property.

Proof. For an almost contact metric manifold, the Ricci identity is given by

$$R(D,E)\varphi - \varphi R(D,E) = [\nabla_D, \nabla_E]\varphi - \nabla_{[D,E]}\varphi. \tag{5.1}$$

The condition on M being equivalent to $\nabla\varphi = 0$, the right hand side of (5.1) vanishes. We get $R(D,E)\varphi F - \varphi R(D,E)F = 0$ which gives

$$g(R(D,E)\varphi F, \varphi G) = g(\varphi R(D,E)F, \varphi G) = -g(R(D,E)F, \varphi^2 G)$$

from which we get

$$g(R(D,E)\varphi F, \varphi G) = -g(R(D,E)F, -G) - g(R(D,E)F, \eta(G)\xi) \tag{5.2}$$

It remains to show that $g(R(D,E)F, \eta(G)\xi) = 0$. Indeed,

$$g(R(D,E)F, \eta(G)\xi) = g(R(D,E)F, \xi)\eta(G);$$

but

$$g(R(D,E)F, \xi) = R(D,E,F,\xi) = -R(D,E,\xi,F) = -g(R(D,E)\xi, F).$$

Since, in such a situation, $\nabla_D \xi = 0$, we get $R(D,E)\xi = 0$ from which we deduce $g(R(D,E)F, \xi) = 0$ so that (5.2) becomes

$$g(R(D,E)\varphi F, \varphi G) = g(R(D,E)F, G);$$

hence $R(D,E,\varphi F,\varphi G) = R(D,E,F,G)$ follows immediately. □

Theorem 5.2. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space verifies the condition*

$$(\nabla_D \phi)(D,E) = \alpha \cdot \eta(D)\phi(E,D),$$

then the fibres possess the Kähler identity.

Proof. If $a = 0$, then the above condition reduces to $(\nabla_D \phi)(D, E) = 0$ which leads to the case that the fibres are nearly Kähler. According to Proposition 3.3, they possess the NK_1-identity which is the Kähler one.

Suppose $a \neq 0$; the vanishing of η on vertical vector fields gives rise to $(\nabla_U \phi)(U, V) = 0$ from which the proof follows as in the case where $a = 0$. □

Considering the fact that, a nearly $a-$Kenmotsu is also defined by the relation

$$(\nabla_D \phi)(D, E) = a \cdot \eta(D)\phi(E, D),$$

the above Theorem 5.2 can be replaced by the following

Theorem 5.3. *The fibres of a nearly $a-$Kenmotsu submersion of type I verify the Kähler identity.*

Recall that for an almost contact metric manifold $(M^{2m+1}g, \varphi, \xi, \eta)$, the $\varphi-$holomorphic sectional curvature tensor is defined by

$$H_\varphi(E) = \|E\|^{-4} g(R(E, \varphi E)E, \varphi E),$$

where $g(E, \xi) = 0$.

Theorem 5.4. *Let $\pi : M \longrightarrow B$ be a nearly $a-$Kenmotsu submersion of type I or type II. Then the $\varphi-$holomorphic sectional curvature tensor is preserved on the horizontal distribution.*

Proof. Watson, [W1], has shown that

$$H_\varphi(X) = H'_\varphi(X_*) - 3\|X\|^{-4}\|A_X \varphi X\|^2.$$

Thus, it suffices to show that $A_X \varphi X = 0$. We can refer to Theorem 4.1(c). □

References

[F-P] M. FALCITELLI and A. M. PASTORE, *A note on almost Kähler and nearly Kähler submersions,* J. Geom. **69** (2000), 79-87.

[G] A. GRAY, *Curvature identities for Hermitian and almost Hermitian manifolds,* Tôhoku Math. J. **28** (1976), 601-612.

[G-H] A. GRAY and L. M. HERVELLA, *The sixteen classes of almost Hermitian manifolds and their linear invariants,* Ann. Mat. Pura Appl. **123** (1980), 35-58.

[J-V] D. JANSSENS, and L. VANHECKE, *Almost contact structures and curvature tensors,* Kodai Math. J. **4** (1981), 1-27.

[O'N] B. O'NEILL, *The fundamental equations of a submersion,* Michigan Math. J. **13** (1966), 459-469.

[T-S] M. M. TRIPATHI and S. S. SHUKLA, *Semi-invariant submanifolds of nearly-Kenmotsu manifolds,* Bull. Calcutta Math. Soc. **95** (2003), 17-30.

[TM1] T. TSHIKUNA-MATAMBA, *Nouvelles classes de variétés de Kenmotsu,* An. Sti. Univ. "Al.I.Cuza" Iasi, Mat. (N.S.) **38** (1992), 167-175.

[TM2] T. TSHIKUNA-MATAMBA, *On the φ-linearity of the configuration tensor of an almost contact metric submersion,* Bull. Math. Soc. Sci. Math. Roumanie **37** (1993), 165-174.

[TM3] T. TSHIKUNA-MATAMBA, *Submersions métriques presque de contact sur les nouvelles variétés de Kenmotsu,* An. Sti. Univ. "Al.I.Cuza" Iasi, Mat. (N.S.) **40** (1994), 117-126.

[TM4] T. TSHIKUNA-MATAMBA, *On the structure of the base space and the fibres of an almost contact metric submersion,* Houston J. Math. **23** (1997), 291-305.

[TM5] T. TSHIKUNA-MATAMBA, *Almost contact metric submersions and curvature tensors,* Extracta Math. **20** (2005), 291-306.

[W1] B. WATSON, *The differential geometry of two types of almost contact metric submersions,* in "*The mathematical heritage*" *of C. F. Gauss*, World Sci. Publishing, River Edge, NJ 1991, 827-861.

[W2] B. WATSON, *Superminimal fibres in an almost Hermitian submersion,* Boll. Unione Mat. Ital. 8**(1-B)** (2000), 159-172.

[W-V] B. WATSON and L. VANHECKE, *K_i−curvatures and almost Hermitian submersions,* Rend. Sem. Mat. Univ. Politec. Torino **36** (1977/78), 205-224.

Abstract

The purpose of this paper is to discuss some geometric properties of Riemannian submersions whose total space is a nearly $a-$Kenmotsu manifold.

* * *