Riv. Mat. Univ. Parma (7) **7** (2007), 173-183

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On increasing sequences of topologies on a set (**)

1 - Introduction

A set equipped with two topologies is called a bitopological space. It was introduced by Weston[**14**]. Kelly[**4**] initiated the systematic study of bitopological spaces. Later on this notion was investigated by Lane [**8**], Patty [**10**], Fletcher, Hoyle III and Patty [**2**], Reilly ([**12**], [**13**]), Raghavan and Reilly [**11**] and others. Kovár ([**5**], [**6**], [**7**]) considered three topologies on a set. In this paper we consider an increasing sequence of topologies on a set and define (ω)topological spaces. We study different properties of (ω)topological spaces concerning compactness, local compactness, paracompactness and separation axioms.

2 - (ω)topological spaces

We denote the set of real numbers and the set of natural numbers by R and N respectively. k, l, m, n etc. denote the elements of N.

Definition 2.1. If $\{\mathcal{J}_n\}$ is a sequence of topologies on a set X with $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ for all n then the pair $(X, \{\mathcal{J}_n\})$ is called a *(ω)topological space.*

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(**) Received 26th June 2007. AMS classification 54A10.

A set G in X is said to be (ω)*open* if G is $(\mathcal{J}_n)$open for some n. F is said to be (ω) *closed* if $X - F$ is (ω)open. It is clear that the unions and intersections of a finite number of (ω)open sets are (ω)open. But we cannot say so for arbitrary unions (see Example 2.1) and intersections (since a topological space does not have this property). We call a set $(\sigma\omega)$*open* (resp. $(\delta\omega)$ *closed*) if it is the union (resp. intersection) of a countable number of (ω)open (resp. (ω)closed) sets. Since for any n, an arbitrary union (resp. intersection) of $(\mathcal{J}_n)$open (resp. $(\mathcal{J}_n)$closed) sets is a $(\mathcal{J}_n)$open (resp. $(\mathcal{J}_n)$closed) set, it follows that an uncountable union (resp. intersection) of (ω)open (resp. (ω) closed) sets can be expressed as a countable union (resp. intersection) of (ω)open (resp. (ω)closed) sets and hence is a $(\sigma\omega)$open (resp. $(\delta\omega)$closed) set. Also it is clear that the complement of a $(\sigma\omega)$open (resp. $(\delta\omega)$closed) set is a $(\delta\omega)$closed (resp. $(\sigma\omega)$open) set.

If for some topology $\mathcal{J}$ on X, $\mathcal{J}_n = \mathcal{J}$ for all n then $(X, \{\mathcal{J}_n\})$ is identified with the topological space $(X, \mathcal{J})$.

Throughout the paper, unless mentioned otherwise, X denotes the (ω)topological space $(X, \{\mathcal{J}_n\})$. For any set $A \subset X$, $(\mathcal{J}_n)cl\, A$ denotes the closure of A with respect to the topology $\mathcal{J}_n$, $\mathcal{J}_n|A$ denotes the subspace topology of $\mathcal{J}_n$ on A.

Definition 2.2. If $Y \subset X$ then $(Y, \{\mathcal{J}_n|Y\})$ is called a *subspace* of $(X, \{\mathcal{J}_n\})$.

Definition 2.3. For a set $A \subset X$, $(\omega)cl\, A$ is the intersection of all (ω)closed sets containing A. It follows that $(\omega)cl\, A$ is a $(\delta\omega)$closed set.

Definition 2.4. A set $A \subset X$ is said to be (ω)*dense* in X if for every nonempty (ω)open set G, $A \cap G \neq \emptyset$.

Definition 2.5. A filterbase $\mathcal{F}$ in X is said to be (ω)*convergent* to $x_0 \in X$ if for every (ω)open set U with $x_0 \in U$ there exists an $A \in \mathcal{F}$ such that $A \subset U$.

Example 2.1. Let $\mathcal{T}$ denote the indiscrete topology of the set of real numbers R and $\mathcal{T}_n$ denote the power set of the set $N_n = \{1, 2, 3, ..., n\}$. We write $\mathcal{J}_1 = \mathcal{T} \cup \mathcal{T}_1$, $\mathcal{J}_2 = \mathcal{T} \cup \mathcal{T}_2$. In general, we write $\mathcal{J}_n = \mathcal{T} \cup \mathcal{T}_n$. Then $(R, \{\mathcal{J}_n\})$ is a (ω)topological space. For each n, N_n is $(\mathcal{J}_n)$open. But $N = \cup_{n=1}^{\infty} N_n$ is not $(\mathcal{J}_n)$open for any n.

3 - (ω)compactness and (ω)separation axioms

Definition 3.1. X is said to be (ω)*compact* if every (ω)open cover of X has a finite subcover.

Remark 3.1. If X is (ω)compact then it is clear that the topological space $(X, \mathcal{J}_n)$ is compact for all n. But the converse is not true. This is shown in Example 3.1.

Definition 3.2. X is said to be (ω)*Hausdorff* if for any two distinct points x, y of X, there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Remark 3.2. If for some n, $(X, \mathcal{J}_n)$ is a Hausdorff topological space then X is (ω)Hausdorff. But the converse is not true as shown by the following example.

Example 3.1. Let $\mathcal{J}$ be the indiscrete topology of R and τ_n be the subspace topology $\mathcal{U}|I_n$ of the usual topology $\mathcal{U}$ of R on $I_n = (-n, n)$. If $\mathcal{J}_n = \mathcal{J} \cup \tau_n$ then $(R, \{\mathcal{J}_n\})$ is a (ω)topological space on R which is (ω)Hausdorff but the topological space $(R, \mathcal{J}_n)$ is not Hausdorff for any n. If $J_n = [-n, n]$, $\mathcal{D}_n = \mathcal{U}|J_n$ and $\mathcal{S}_n = \mathcal{J} \cup \mathcal{D}_n$ then $(R, \{\mathcal{S}_n\})$ is not (ω)compact but the topological space $(R, \mathcal{S}_n)$ is compact for all n.

Definition 3.3. X is said to be (ω)*regular* if given a (ω)closed set F and a point $x \in X$ with $x \notin F$, there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

Example 3.2. Let us consider the increasing sequence $\{\mathcal{T}_n\}$ of topologies on N defined by $\mathcal{T}_n = \{N\} \cup P\{1, 2, 3, ..., n\}$, where $P\{1, 2, 3, ..., n\}$ denotes the power set of the set $\{1, 2, 3, ..., n\}$. Then the (ω)topological space $(N, \{\mathcal{T}_n\})$ is (ω)Hausdorff but not (ω)regular.

Definition 3.4. X is said to be (ω)*normal* if given two (ω)closed sets A and B with $A \cap B = \emptyset$, there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Definition 3.5. X is said to be *completely* (ω)*normal* if for each pair A, B of subsets of X satisfying

$$(A \cap ((\mathcal{J}_m)cl\ B)) \cup (((\mathcal{J}_m)cl\ A) \cap B) = \emptyset$$

for some m, there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

From Definitions 3.4 and 3.5 it is clear that every completely (ω)normal space is (ω)normal. But the converse is not true as shown by the following example.

Example 3.3. Let us consider the increasing sequence $\{\mathcal{J}_n\}$ of topologies on N defined as follows

$$\mathcal{J}_n=\{\emptyset,\{1\},N\}\cup(\bigcup_{i=1}^{n}\{\{1,2,3,...i,i+1\},\{1,2,3,...i,i+2\},\{1,2,3,...i,i+1,i+2\}\}).$$

Then it is easy to see that $(N,\{\mathcal{J}_n\})$ forms a (ω)topological space on N which is (ω)normal but neither (ω)regular nor (ω)Hausdorff.

If $N_4=\{1,2,3,4\}$ then it can easily be verified that the subspace $(N_4,\{\mathcal{J}_n|N_4\})$ is not (ω)normal. Hence (Theorem 3.13) $(N,\{\mathcal{J}_n\})$ is not completely (ω)normal.

It is easy to see that (ω)Hausdorffness, (ω)regularity and complete (ω)normality are hereditary properties. But (ω)normality is not a hereditary property.

Theorem 3.1. *If X is (ω)compact and K is a (ω)closed subset of X then K is (ω)compact.*

The proof is omitted.

Theorem 3.2. *If for each n, $(X\ ,\ \mathcal{J}_n)$ is a Hausdorff topological space and $(X,\ \{\mathcal{J}_n\})$ is (ω) compact then $\mathcal{J}_n=\mathcal{J}_{n'}$ for all n,n'.*

Proof. Let $n<n'$. Then $\mathcal{J}_n\subset\mathcal{J}_{n'}$. If $G\in\mathcal{J}_{n'}$ then $F=X-G$ is $(\mathcal{J}_{n'})$closed and hence by Theorem 3.1, F is (ω)compact. Therefore F is $(\mathcal{J}_n)$compact. Since $(X, \mathcal{J}_n)$ is Hausdorff, F is $(\mathcal{J}_n)$closed and so G is $(\mathcal{J}_n)$open. Therefore $\mathcal{J}_{n'}\subset\mathcal{J}_n$. □

Theorem 3.3. *X is (ω)Hausdorff iff for each $x\in X$,*

$$\{x\}=\bigcap_{n\in N}\{(\mathcal{J}_n)cl\ U\mid U\in\mathcal{J}_n\ \textit{with}\ x\in U\}.$$

The proof is omitted.

Theorem 3.4. *X is (ω)Hausdorff iff each (ω)convergent filterbase in X (ω)converges to exactly one point.*

Proof. Firstly assume X is (ω)Hausdorff and $\mathcal{F}$ be a filterbase in X which is (ω)convergent to $x\in X$. If $y\in X$ is a point distinct from x then there exists an n such that for some $U,V\in\mathcal{J}_n$, we have $x\in U$, $y\in V$ and $U\cap V=\emptyset$. By hypothesis there exists some $A_1\in\mathcal{F}$ such that $A_1\subset U$. Since any two elements of $\mathcal{F}$ have nonempty intersection there can be no element $A_2\in\mathcal{F}$ such that $A_2\subset V$. Thus $\mathcal{F}$ cannot (ω)converge to y.

Conversely suppose each (ω)convergent filterbase in X is (ω)convergent to a unique point. If possible suppose there exist a pair of distinct points x, y such that for any n and any $U, V \in \mathcal{J}_n$ with $x \in U$, $y \in V$ we have $U \cap V \neq \emptyset$. Then the family $\mathcal{F} = \{U \cap V \mid U, V \in \mathcal{J}_n,\ x \in U,\ y \in V, n \in N\}$ is a filterbase in X which is (ω)convergent to both x and y. Thus we arrive at a contradiction. □

Theorem 3.5. *Let X be (ω)Hausdorff, $x \in X$ and K be a (ω)compact subset of X with $x \notin K$. Then there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $x \in U$, $K \subset V$ and $U \cap V = \emptyset$.*

Proof. For each $y \in K$, there exists an $n_y \in N$ such that for some $U_{n_y}, V_{n_y} \in \mathcal{J}_{n_y}$, we have $x \in U_{n_y}$, $y \in V_{n_y}$ and $U_{n_y} \cap V_{n_y} = \emptyset$. Then the family $\{V_{n_y} \mid y \in K\}$ is a (ω)open cover of K and hence there is a finite subcover $\{V_{n_{y_1}}, V_{n_{y_2}}, ..., V_{n_{y_k}}\}$. Let $U = \cap_{i=1}^{k} U_{n_{y_i}}$ and $V = \cup_{i=1}^{k} V_{n_{y_i}}$. Since $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ for each n, U and V are $(\mathcal{J}_m)$open sets where $m = \max\{n_{y_1}, n_{y_2}, ..., n_{y_k}\}$. Also we have $x \in U$, $K \subset V$ and $U \cap V = \emptyset$. □

In a Hausdorff topological space every compact subset is a closed set. Here we get the result as follows.

Theorem 3.6. *If X is (ω)Hausdorff and $K \subset X$ is (ω)compact then K is a $(\delta\omega)$closed set.*

Proof. Let $x \in X - K$. Then by Theorem 3.5 there exists an $n_x \in N$ such that for some $U_x, V_x \in \mathcal{J}_{n_x}$, $x \in U_x$, $K \subset V_x$ and $U_x \cap V_x = \emptyset$. Therefore $X - K \subset \cup\{U_x | x \in X - K\} \subset \cup\{X - V_x | \ x \in X - K\} \subset X - K$ and so $X - K = \cup\{U_x | x \in X - K\}$. Therefore $X - K$ is $(\sigma\omega)$open and hence K is $(\delta\omega)$closed. □

We now give an example of a (ω)compact set in a (ω)Hausdorff space which is not (ω)closed.

Example 3.4. The interval $[a, b] \subset R$ is (ω)compact in the (ω)Hausdorff space $(R, \{\mathcal{J}_n\})$ of Example 3.1. Its complement $A = (-\infty, a) \cup (b, \infty)$ is not (ω)open, since it is not $(\mathcal{J}_n)$open for any n. But $A = \cup_{k=1}^{\infty}\{(-k, a) \cup (b, k)\}$ and so it is $(\sigma\omega)$ open. Thus $[a, b]$ is not (ω)closed but $(\delta\omega)$closed.

Theorem 3.7. *X is (ω)regular iff for any point $x \in X$ and any (ω)open set G containing x, there exists an n such that for some $(\mathcal{J}_n)$open set U containing x, we have $(\mathcal{J}_n)$cl $U \subset G$.*

The proof is omitted.

Theorem 3.8. *If X is (ω)compact and (ω)Hausdorff then X is (ω)regular.*

Proof. Follows from Theorem 3.1. and 3.5. □

Theorem 3.9. *Let X be (ω)regular. If F be a (ω)closed subset of X and K is a (ω)compact subset of X with $F \cap K = \emptyset$ then there exists an n such that for some $U, V \in \mathcal{J}_n$, we have $F \subset U$, $K \subset V$ and $U \cap V = \emptyset$.*

Proof. Similar to Theorem 3.5. □

Theorem 3.10. *X is (ω)normal iff given a (ω)closed set F and a (ω)open set W with $F \subset W$, there exists an n such that for some $(\mathcal{J}_n)$open set U, $F \subset U \subset (\mathcal{J}_n)cl\ U \subset W$.*

The proof is omitted.

Theorem 3.11. *If X is (ω)compact and (ω)regular then X is (ω)normal.*

Proof. Follows from Theorem 3.1 and 3.9. □

Corollary 3.1. *If X is (ω)compact and (ω)Hausdorff then X is (ω)normal.*

Before we prove the next theorem(Urysohn's lemma [3]), we introduce the following definition.

Definition 3.6. A function $f : X \to [0, 1]$ is said to be $(\sigma\omega)$continuous if for every open subset G of $[0, 1]$, $f^{-1}(G)$ is $(\sigma\omega)$open.

Theorem 3.12. *If X is (ω)normal then for any two (ω)closed sets A and B with $A \cap B = \emptyset$, there exists a $(\sigma\omega)$continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Since $A \subset X - B$ and $X - B$ is (ω)open, by Theorem 3.10 there exists a positive integer $n\left(\dfrac{1}{2}\right)$ such that for some $U_{n(\frac{1}{2})} \in \mathcal{J}_{n(\frac{1}{2})}$, we have

$$A \subset U_{n(\frac{1}{2})} \subset (\mathcal{J}_{n(\frac{1}{2})})cl\ U_{n(\frac{1}{2})} \subset X - B.$$

By using similar process we get $U_{n(\frac{1}{4})} \in \mathcal{J}_{n(\frac{1}{4})}$ and $U_{n(\frac{3}{4})} \in \mathcal{J}_{n(\frac{3}{4})}$ such that

$$A \subset U_{n(\frac{1}{4})} \subset (\mathcal{J}_{n(\frac{1}{4})})cl\ U_{n(\frac{1}{4})} \subset U_{n(\frac{1}{2})} \subset (\mathcal{J}_{n(\frac{1}{2})})cl\ U_{n(\frac{1}{2})}$$
$$\subset U_{n(\frac{3}{4})} \subset (\mathcal{J}_{n(\frac{3}{4})})cl\ U_{n(\frac{3}{4})} \subset X - B.$$

By repeating the same process we get for $t \in D = \{\frac{l}{2^m} \mid 0 < l < 2^m, l, m \in N\}$, a set $U_{n(t)} \in \mathcal{J}_{n(t)}$ for some positive integer $n(t)$ such that for $s, t \in D$ with $s < t$ we have

$$(\mathcal{J}_{n(s)})cl\ U_{n(s)} \subset U_{n(t)}.$$

If we define $U_{n(0)} = \emptyset$ and $U_{n(1)} = X$ then the above relation is also true when s, t coincide with 0 or 1. For $t \neq 0, 1$, we have

$$A \subset U_{n(t)} \subset (\mathcal{J}_{n(t)})cl\ U_{n(t)} \subset X - B.$$

Now we define the function $f : X \to [0, 1]$ by

$$f(x) = \inf\{t \mid x \in U_{n(t)}\}.$$

Then $f(A) = 0$ and $f(B) = 1$ and for $a \in (0, 1)$,

$$\{x \in X \mid f(x) < a\} = \bigcup_{t<a} U_{n(t)},$$

$$\{x \in X \mid f(x) > a\} = \bigcup_{t>a} \left[(\mathcal{J}_{n(t)})cl\ U_{n(t)}\right]^c.$$

Since the sets on the right hand side of the above two equalities are $(\sigma\omega)$open, it follows that f is $(\sigma\omega)$continuous. □

Theorem 3.13. *X is completely (ω)normal iff every subspace of it is (ω)normal.*

Proof. The necessity follows from the fact that a complete (ω)normal space is (ω)normal and complete (ω)normality is a hereditary property.

To prove the sufficiency, let A and B be two subsets of X such that

$$(A \cap ((\mathcal{J}_m)cl\ B)) \cup (((\mathcal{J}_m)cl\ A) \cap B) = \emptyset \tag{3.1}$$

for some m. Let us write

$$D = (X - (\mathcal{J}_m)cl\ A) \cup (X - (\mathcal{J}_m)cl\ B).$$

Then

$$(D \cap (\mathcal{J}_m)cl\ A) \cap (D \cap (\mathcal{J}_m)cl\ B) = \emptyset.$$

Since the subspace $(D, \{\mathcal{J}_n|D\})$ is (ω)normal, there exists an l such that for some $U, V \in \mathcal{J}_l|D$, we have $D \cap (\mathcal{J}_m)cl\ A \subset U$, $D \cap (\mathcal{J}_m)cl\ B \subset V$ and $U \cap V = \emptyset$.

From (3.1) we get $A \cap (\mathcal{J}_m)cl\ B = \emptyset$ and so $A \subset X - (\mathcal{J}_m)cl\ B$. Similarly $B \subset X - (\mathcal{J}_m)cl\ A$. Therefore $A \subset D \cap (\mathcal{J}_m)cl\ A$ and $B \subset D \cap (\mathcal{J}_m)cl\ B$.

If $V_1 = V \cap (X - (\mathcal{J}_m)cl\ A)$ then $D \cap (\mathcal{J}_m)cl\ B \subset V_1$. Also since $V \in \mathcal{J}_l|D$ and $X - (\mathcal{J}_m)cl\ A \in \mathcal{J}_m$ it follows that $V_1 \in \mathcal{J}_n$ where $n = \max\{l, m\}$. Since $U \in \mathcal{J}_l|D$ there exists $U_1 \in \mathcal{J}_l$ such that $U_1 \cap D = U$. Then $U_1, V_1 \in \mathcal{J}_n$, $A \subset U_1$, $B \subset V_1$ and $U_1 \cap V_1 \subset U_1 \cap V = (U_1 \cap D) \cap V$ (since $V \subset D$) $= U \cap V = \emptyset$. □

4 - Local (ω)compactness and (ω)paracompactness

Definition 4.1. X is said to be *locally* (ω)*compact* if for each point x of X, there exists an n such that for some ($\mathcal{J}_n$)open neighbourhood U of x, ($\mathcal{J}_n$)*cl* U is (ω)compact.

Definition 4.2. A collection $\mathcal{U}$ of subsets of X is said to be *locally finite* if each $x \in X$ has a ($\mathcal{J}_n$)open neighbourhood meeting a finitely many $U \in \mathcal{U}$.

It is clear that (ω)compactness implies local (ω)compactness.

Definition 4.3. A (ω)Hausdorff space X is said to be (ω)*paracompact* if each (ω)open cover of X has a locally finite ($\mathcal{J}_n$)open refinement for some n.

It follows from the definitions that a (ω)compact (ω)Hausdorff space is (ω)paracompact.

The (ω)topological space $(R, \{\mathcal{J}_n\})$ of Example 3.1, is locally (ω)compact and (ω)paracompact but not (ω)compact.

Theorem 4.1. *Let X be (ω)Hausdorff. Then X is locally (ω)compact iff for each point x and (ω)open set G containing x, there exists an n such that for some ($\mathcal{J}_n$)open set U containing x, we have ($\mathcal{J}_n$)clU $\subset G$ and ($\mathcal{J}_n$)clU is (ω)compact.*

Proof. Suppose X is locally (ω)compact and G is a (ω)open set containing x. Then there is an l such that for some ($\mathcal{J}_l$)open set V with $x \in V$ and $A = (\mathcal{J}_l)cl\, V$ is (ω)compact. The subspace $(A, \{\mathcal{J}_n|A\})$ is then (ω)compact and (ω)Hausdorff and hence, by Theorem 3.8 it is (ω)regular. Therefore, by Theorem 3.7 there is an m such that for some ($\mathcal{J}_m|A$)open set W containing x, we have $(\mathcal{J}_m|A)cl\, W \subset G \cap A$. Let $W = H \cap A$ where $H \in \mathcal{J}_m$. If $U = H \cap V$ then $U \in \mathcal{J}_n$ where $n = \max\{l, m\}$, $x \in U$ and

$$(\mathcal{J}_n)cl\ U = ((\mathcal{J}_n)cl\ U) \cap A \text{ (since } A \text{ is } (\mathcal{J}_n)\text{closed)}$$

$$= (\mathcal{J}_n|A)cl\ U.$$

Therefore, by Theorem 3.1 ($\mathcal{J}_n$)*cl* U is (ω)compact. Also

$$(\mathcal{J}_n)cl\ U \subset ((\mathcal{J}_n|A))cl\ W \subset G.$$

The converse is obviously true. □

From the above theorem we get the following theorem which is an improvement of Theorem 3.8.

Theorem 4.2. *If X is (ω)Hausdorff and locally (ω)compact then X is (ω)regular.*

It is easy to see that a (ω)closed subspace of a locally (ω)compact space is locally (ω)compact. The next theorem gives another source of locally (ω)compact spaces.

Theorem 4.3. *If X is (ω)Hausdorff and locally (ω)compact and $G \subset X$ is a (ω)open set then the subspace $(G, \{\mathcal{J}_n|G\})$ is locally (ω)compact.*

The proof is omitted.

The next theorem is a sort of converse of the above theorem.

Theorem 4.4. *Let X be (ω)Hausdorff and Y be (ω)dense subset of X. If $(Y, \{\mathcal{J}_n|Y\})$ is locally (ω)compact then Y is $(\sigma\omega)$open set in X.*

Proof. For $y \in Y$, we choose an $n_y \in N$ such that for some $\big(\mathcal{J}_{n_y}|Y\big)$open set U_y with $y \in U_y$ and $\big(\mathcal{J}_{n_y}|Y\big)cl\ U_y$ is (ω)compact. For some $G_y \in \mathcal{J}_{n_y}$, we have $U_y = G_y \cap Y$. Let $a \in G_y$ and H be any (ω)open set containing a. Then $G_y \cap H \neq \emptyset$ and $G_y \cap H$ is (ω)open in X. Since Y is (ω)dense in X, $\big(G_y \cap Y\big) \cap H = \big(G_y \cap H\big) \cap Y \neq \emptyset$. It thus follows that $a \in (\omega)cl\ \big(G_y \cap Y\big)$ and hence

$$G_y \subset (\omega)cl\big(G_y \cap Y\big) \tag{3.2}$$

Since $\big(\mathcal{J}_{n_y}|Y\big)cl\ U_y$ is (ω)compact, by Theorem 3.6 it is a $(\delta\omega)$closed subset of X. Now $G_y \cap Y \subset \big(\mathcal{J}_{n_y}|Y\big)cl\ U_y$ and so $(\omega)cl\big(G_y \cap Y\big) \subset \big(\mathcal{J}_{n_y}|Y\big)clU_y \subset Y$. Therefore by (3.2), $G_y \subset Y$ which implies that $Y = \cup\big\{G_y \mid y \in Y\big\}$. Hence Y is $(\sigma\omega)$open. □

Theorem 4.5. *If X is (ω)paracompact then every (ω)closed subset of X is (ω)paracompact.*

The proof is omitted.

The following theorem is also an improvement of Theorem 3.8.

Theorem 4.6. *If X is (ω)paracompact then X is (ω)regular.*

Proof. Suppose A is a (ω)closed set with $x \notin A$. For every $y \in A$ there exists an $n_y \in N$ such that for some $(\mathcal{J}_{n_y})$open sets U_y and V_y, we have $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. Then the family $\big\{V_y \mid y \in A\big\} \cup \{X - A\}$ forms a (ω)open cover of X. Since X is (ω)paracompact, for some n, there exists a locally finite $(\mathcal{J}_n)$open refinement $\mathcal{C}$ of this (ω)open cover. Let $V = \cup\{G \in \mathcal{C} \mid G \cap A \neq \emptyset\}$. Then there exists, for some m, a $(\mathcal{J}_m)$open neighbourhood W of x meeting only a finite number of sets $V_1, V_2..., V_k$ of $\mathcal{C}$. Let $V_i \subset V_{y_i}$, $y_i \in A$, $i = 1, 2, ...k$. Then $U = W \cap \left(\cap_{i=1}^{k} U_{y_i}\right) \in \mathcal{J}_l$ and $V \in \mathcal{J}_l$ where $l = \max\{m, n, n_{y_1}, n_{y_2}, ..., n_{y_k}\}$. Since $x \in U$, $A \subset V$ and $U \cap V = \emptyset$, X is (ω)regular. □

Using this theorem and proceeding as above we can show that X is (ω)normal if it is (ω)paracompact.

Added remarks in the light of referee's comments:

1) The possibility of an analogue of Michael's theorem ([**9**], p. 831) on regular topological spaces in the (ω)setting remains as an open question. This is a sort of converse of Theorem 4.6. We will consider it in a separate paper. For (ω)paracompactness, the existence of a $(\mathcal{J}_n)$open refinement for any (ω)open cover is a stronger condition. So a stronger (ω)regularity notion might be needed to prove the analogue of Michael's theorem.

2) If $\mathcal{J} = \bigcup_n \mathcal{J}_n$ then $(X, \mathcal{J})$ is not a topological space and even it is not an Alexandroff space [**1**] which is a generalization of a topological space requiring only countable union of open sets to be open. In fact, an arbitrary (or countable) union of sets $\in \mathcal{J}$ may not belong to $\mathcal{J}$. But taking advantage of the topologies $\mathcal{J}_n$ we can, however, get many properties of $(X, \{\mathcal{J}_n\})$, close to that of a topological space which are not necessarily possessed by an Alexandroff space.

3) A possible field of application of the new topological notions presented in this paper seems to be in digital topology and in topologies inspired by computer science.

Acknowledgments. We are thankful to the referee for his very positive criticism.

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Abstract

In this paper we introduce and investigate the notion of a (ω)*topological space which is a set equipped with an increasing sequence of topologies on it.*

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