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Singular integral equations and contact problems in Kirchhoff plates resting on irregular linear supports (**)

1 - Introduction

This paper deals with some difficulties encountered when a contact problem between a Kirchhoff (thin) plate, discontinuously supported along arcs or segments, is formulated in terms of (singular) integral equations.

Two main approaches have been employed for solving these contact problems, *e.g.* Strozzi & Vaccari (2001). The two methods may be classified as differential and integral approach.

According to the differential method, the plate deflection is assumed as the primary unknown. The boundary conditions along the supported edge are imposed in terms of the plate deflection and its derivatives, and the reaction forces and couples (or moments) are subsequently computed from the deflection field, *e.g.* Williams (1951), Buchwald (1957), Werner & Peter (1966).

In the integral equation method, a preventative selection of the suitable types of (equivalent) reaction forces and couples is effected, and the plate deflections are

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formulated as integral transformations (based upon suitable Green functions) of the unknown reaction forces. Each reaction component, through the associated integral transform having a specific kernel, defines a corresponding plate deflection component. In particular the shear force and the twisting moment have different integral transform kernels: the second one is the derivative of the first one. Therefore, to consider a reaction composed only by a shear force, or by a twisting moment, or by the sum of the two, in general is not irrelevant.

If the contact reactions are correctly chosen, then this integral representation satisfies the biharmonic equation as well as all the contact problem boundary conditions, except those expressing the requirement that the plate deformed shape matches with the support profile. These compatibility conditions are then reduced to singular integral equations whose solution defines the unknown reaction forces, *e.g.* Grigolyuk & Tolkachev (1987), p. 381, Monegato & Strozzi (2001a), Monegato & Strozzi (2002), Monegato & Strozzi (2005a). Once these have been determined, the plate deflection can be computed everywhere by means of its integral representation.

When the differential method is employed, the equivalent shear force (*e.g.* Timoshenko & Woinowsky-Krieger (1959), p. 84) along the plate support is computed once the solution in terms of plate deflection has been evaluated. Consequently, the possible circumstance that the equivalent shear force possesses non integrable endpoint singularities and that infinite concentrated forces occur at the contact endpoints to equilibrate the shear force, *e.g.* Werner & Peter (1966), constitutes a physical unreality rather than a mathematical difficulty.

Conversely, when the integral equation approach is employed, the force reactions, being the solution of integral equations, must generally respect functional restrictions, recalled in Section 3, which exclude non integrable functions. Therefore it follows that a preventative knowledge of the singularity strength of the contact reaction is necessary to choose whether to include a shear force and/or a twisting moment among the contact reactions. Indeed, if the component of the contact reactions are not chosen properly, the plate deflection given by its integral transform representation turns out to be not a solution of the original problem, in the sense that the boundary profile defined by this representation does not match the support profile.

The concepts of equivalent shear force and equivalent twisting moment are discussed in Section 2.

The Williams (1952) asymptotic method, addressed in Section 3 of this paper, is in general an efficient tool for forecasting the (algebraic) singularity strength of the contact reaction without requiring the contact problem to be entirely solved.

To further illustrate the concepts presented in sections 2 and 3, these are applied to two different contact problems: in Section 4 an infinite plate resting on a finite segment support is considered, while in Section 5 the case of a circular plate simply supported along two antipodal edge arcs is examined.

2 - The Poisson-Kirchhoff paradox, the equivalent shear force and the equivalent twisting moment

This section mainly deals with the Poisson-Kirchhoff paradox in the Kirchhoff theory of flexure of plates, that is, with the contraction of the three boundary conditions favoured by Poisson to only two edge conditions, as proposed by Kirchhoff, *e.g.* Love (1944), p. 458, Timoshenko & Woinowsky-Krieger (1959), p. 84, Panovko (1985), pp. 58-71, Vijayakumar (1988).

An account is presented in the following on the historical background of the classical Kirchhoff plate theory, with particular regard to the boundary conditions. It is known that this theory rests on two main requirements. First, the plate deflections must fulfil the biharmonic equation

$$\Delta^2 w = \frac{q}{D} \tag{2.1}$$

where the symbol Δ indicates the Laplacian operator, w represents the plate deflection, D is the plate flexural rigidity, and q the intensity of the distributed transverse load, *e.g.* Timoshenko & Woinowsky-Krieger (1959), p. 82. Secondly, only two independent distributed tractions or couples may be assigned along the plate periphery, namely a bending moment and an equivalent shear force, whereas a twisting moment is not regarded as an independent variable. In fact, in the classical Kirchhoff plate theory the combined effects of an imposed distributed shear force and of a distributed twisting moment are summarised by the equivalent shear force, defined as a combination of the shear force with the derivative of the twisting moment along the direction of the plate border, *e.g.* Timoshenko & Woinowsky-Krieger (1959), p. 84.

The equivalent shear force concept may be rationalized in two main ways. First, following Kirchhoff (1850), a variational approach may be employed to show that the total work and, hence, the plate deflections remain unchanged if the simultaneous presence of a distributed shear force and of a distributed twisting moment is summarised by a distributed equivalent shear force alone, accompanied by two concentrated transverse forces acting at the extremities of the loaded edge portion, whose intensity equals the value of the twisting moment at the ends of the loaded edge part, *e.g.* Timoshenko & Woinowsky-Krieger (1959), p. 84, Frangi & Guiggiani (1999). That the plate deflections remain unmodified upon contraction of the two above applied stress resultants is a relevant observation, since in contact problems the deformability of the contacting bodies plays a fundamental role in the definition of the contact reaction profile. This first rationalization of the equivalent shear force concept may therefore be classified as displacement-oriented.

Secondly, following Thomson & Tait (1962), p. 188, it may be shown that the system constituted by a distributed shear force and by a twisting moment is statically equivalent to a distributed equivalent shear force, so that the two stress fields set up in a plate modelled as a three-dimensional body by the two statically equivalent loadings become often indistinguishable at a distance from the plate border comparable to the plate thickness, *e.g.* Thomson & Tait (1962), p. 192, Gregory & Wan (1988). This second rationalization of the equivalent shear force concept may therefore be ranked as stress-oriented.

As it has been pointed out in Strozzi & Monegato (2008), the Kirchhoff equivalent shear force concept may be employed to change a description of the contact reaction along a linear segment into an alternative, equivalent description, and vice versa. In the first description, the contact reaction is expressed in terms of a distributed shear force with non integrable endpoint singularities, accompanied by equilibrating infinite endpoint concentrated forces. In the second description, the contact reaction is expressed in terms of a distributed twisting moment with integrable endpoint singularities, accompanied by finite endpoint concentrated forces.

In Strozzi & Monegato (2008) it has been underlined that the second description exhibits mathematical advantages deriving from the property that the distributed reaction expressed in terms of twisting moment exhibits integrable endpoint singularities. It has also been shown that this pure twisting moment plays the role of an "equivalent twisting moment" condensing the shear force effects in terms of plate deflection into the twisting moment, and not vice versa.

A more detailed discussion on the above aspect is presented in the following. Since in contact problems involving a Kirchhoff plate resting on a linear irregular support the equivalent shear force may exhibit non integrable endpoint singularities, *e.g.* Werner & Peter (1966), in an integral formulation capable of producing a correct solution even in the presence of non integrable equivalent shear force, the contact reaction cannot be described in terms of equivalent shear force. It is concluded that, in the solution procedure of a plate contact problem expressed in terms of an integral approach, it is necessary to abandon the equivalent shear force concept if the contact reaction is expected to be formed by a non integrable shear force, as proposed by Grigolyuk & Tolkachev (1987), p. 381. Once the solution has been found, that is, once the contact reaction has been determined, the equivalent shear force concept is again applicable, and it may be used to interpret in terms of shear force a contact reaction expressed in terms of distributed twisting moment, or vice versa.

Concerning the physical meaning of a contact reaction described in terms of a distributed twisting moment, in Grigolyuk & Tolkachev (1987), this contact term is regarded as a pure abstract entity, whereas in Strozzi & Monegato (2008) a physical meaning is attributed to such quantities, although only in some cases.

3 - Support endpoint singularities: the Williams asymptotic method

The previous discussions evidences the importance of forecasting the strength of the singularities affecting the contact reaction. The displacement-based Williams asymptotic method, Williams (1952), Soutas-Little (1999), p. 184, Barber (2002), p. 141, Szilard (2004), p.155, Lee & Barber (2006), is an effective tool for predicting the endpoint algebraic strength of the singularities of the reaction forces and moments in a Kirchhoff plate resting on irregular linear supports. One of the prerogatives of the Williams method is that it forecasts the singularity strength without requiring the complete solution of the contact problem under scrutiny. According to the Williams approach, the plate deflection is chosen of the form

$$w(r,\theta) = r^{\lambda+1}F(\theta;\lambda)$$

and inserted into the homogeneous biharmonic equation to obtain the following expression:

$$w(r,\theta) = r^{\lambda+1}[b_1 \sin(\lambda+1)\theta + b_2 \cos(\lambda+1)\theta + b_3 \sin(\lambda-1)\theta + b_4 \cos(\lambda-1)\theta], \tag{3.1}$$

where the origin of the polar coordinates is centered on one endpoint of the linear support. The imposition of (four) proper boundary conditions to (3.1) leads to a set of four homogeneous algebraic equations, which are linear in the four coefficients b_i of (3.1). The existence of non-trivial solutions requires that the determinant of the coefficients b_i must vanish, a condition that permits the eigenvalues of the exponent λ to be determined and, consequently, the singularity strength to be quantified.

Although this approach refers only to the homogeneous biharmonic equation, it is commonly used to detect the strength of the support endpoint singularity also for the non homogeneous one.

For the infinite plate supported along a central segment, that will be examined in the next section, the characteristic equation is $sin(2\pi\lambda) = 0$, and the eigenvalue of interest, that is, that falling in the range $0 < Re(\lambda) < 1$, Williams (1952), is $\lambda = 1/2$. For this eigenvalue, the twisting and bending moments are null along the supported portion, a results that indicates that they are free from singularities. Instead, the shear force (and, therefore, the equivalent shear force) exhibits a non integrable singularity of the kind $1/(x\sqrt{x})$ at the supported segment endpoints. The vertical equilibrium condition of a semicylindrical infinitesimal element describing the plate portion around the origin of the polar coordinates attest that an infinite concentrated force acts at the contact tip. The above findings agree with Werner & Peter (1966). A thorough review of the few contributions examining in detail the endpoint singularities encountered at the tips of discontinuous supports is carried out in Stahl & Keer (1972).

Once the distributed shear force with non integrable endpoint singularity is changed into a distributed twisting moment according to an integration by parts (see [**24**]), the singularity strength of the twisting moment is of the kind $1/\sqrt{x}$ and, therefore, integrable. The contact reaction integrability is a necessary condition to derive singular integral equation formulations, based on contact reaction Hilbert transforms, according to the theory developed by Söhngen and Tricomi that shall be briefly recalled next.

Söhngen in 1939 examined the solution of the equation

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{u(x)}{x-y}dx = f(y), \qquad -1 < y < 1 \tag{3.2}$$

where, here and in the following, integrals of this type are defined in the Cauchy principal value sense.

The standard definition of Cauchy principal value integral is (see for example Muskhelishvili (1953))

$$\int_{-1}^{1}\frac{u(x)}{x-y}dx = \lim_{\varepsilon\to 0}\left[\int_{-1}^{y-\varepsilon} + \int_{y+\varepsilon}^{1}\right]\frac{u(x)}{x-y}dx, \qquad -1 < y < 1. \tag{3.3}$$

One also has

$$\int_{-1}^{1}\frac{u(x)}{x-y}dx = \frac{d}{dy}\int_{-1}^{1}u(x)\log|x-y|dx. \tag{3.4}$$

A sufficient condition for the existence of such integral is the Hölder continuity of $u(x)$ in a neighborhood of y.

By assuming $f(y) \in L_1$ to be piecewise Hölder continuous, with a finite number of breakpoints, and $\sqrt{1-y^2}f^2(y) \in L_1$, Söhngen was able to derive the following inversion formula

$$u(x) = -\frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{1-y^2}\,\frac{f(y)}{x-y}dy + \frac{c}{\sqrt{1-x^2}}, \tag{3.5}$$

where c denotes an arbitrary constant, and show that the solution $u(x)$ satisfies the same assumptions made above on $f(x)$. To define uniquely $u(x)$ one has to impose an extra condition.

Tricomi in 1951 enlarged the class of functions for which (3.5) holds. In particular he proved that:

If $f \in L_p,\ p > 4/3$, *then* (3.2) *has only solutions of form* (3.5), *which necessarily belong to* L_q, *for any* $1 < q < 4/3$.

He also remarked that if equation (3.2) has a solution in L_p, for some $p > 1$, then this must necessarily have the form (3.5).

Later, in 1954 Söhngen eliminates every doubt about the validity of (3.5), even when $f \in L_p,\ 1 < p \leq 4/3$. Indeed, using also some of Tricomi's results he was able to prove that:

(i) *if* $f \in L_p,\ p > 1,$ *then* $u \in L_q$ *for some* $q > 1$;
(ii) *if* $f \in L_p,\ 1 < p \leq 2$ *then* $u \in L_q$ *for all* $q < p$.

Mapping properties of the finite Hilbert transform when acting on weighted Hölder and L_p spaces have been carried out by Muskhelishvili (1953) and by Khvedelidze (1956), respectively.

4 - Singular integral formulation of a simple contact problem

In order to illustrate the mathematical differences encountered when describing the contact reaction in terms of equivalent shear force or of equivalent twisting moment, in this section an example of an integral formulation of a contact problem between an infinite Kirchhoff plate and a segment support is worked out in detail.

The following function, representing the mechanical response of a plate subject to a transverse concentrated force, is known to hold for an infinite plate, *e.g.* Grigolyuk & Tolkachev (1987), Szilard (2004), p. 65, where w is the plate deflection, r the radius measured from the point of application of the concentrated transverse force P, θ the angular coordinate, and D the plate flexural rigidity

$$w(r, \theta) = \frac{P}{8\pi D} r^2 \ln\ r\ . \tag{4.1}$$

This function is the classical fundamental polar solution of the plate biharmonic equation (2.1).

If the plate is supported along a line given by a function $\delta(x)$ defined on the segment $[-1, 1]$, and the distributed contact reaction is assumed to be described by an integrable distributed shear force $f(z)$, the governing integral equation one can derive using (4.1) has the form, *e.g.* Strozzi & Monegato (2008)

$$\int_{-1}^{1} f(z)(x-z)^2 \ln|x-z| dz = h(x) - 8\pi D\delta(x) + c \tag{4.2}$$

where x and z are Cartesian coordinates in the support direction and whose origin coincides with the support midpoint, $h(x)$ is a known function depending on the kind of plate loading, and c is an unknown constant support profile translation, to be determined together with the function $f(z)$. Therefore the above equation ought to be interpreted as follows.

Definition 4.1. *An integrable function $f(z)$ represents the unknown shear force reaction iff it satisfies equation* (4.2), *for some value of the constant c.*

If the non integrable endpoint singularities forecast by the Williams' method are correct, then the above (4.2) integral formulation is not defined. Moreover, in this case also the subsequent differentiations would not be allowed. On the other hand, if (4.2) has an integrable solution, than this is necessarily the correct shear force reaction for the original contact problem. Therefore, since Williams' technique has been obtained to describe the behavior of the solution of the homogeneous biharmonic equation, not that of the non homogeneous one, in the following analysis it is assumed that the representations (4.2) and (4.3) below hold.

In this case, since it is impossible to examine and solve directly equation (4.2), recalling (3.4) and the results known for the singular integral equations of form (3.2), listed in Section 3, assuming for simplicity that the third derivatives of $h(x)$ and $\delta(x)$ are Hölder continuous in $[-1,1]$, equation (4.2) can be differentiated three times with respect to x, thus obtaining

$$2\int_{-1}^{1} \frac{f(z)}{x-z} dz = \frac{d^3[h(x) - 8\pi D\delta(x)]}{dx^3} \quad . \tag{4.3}$$

All the solutions of equation (4.2) are also solutions of equation (4.3), but the vice versa is not necessarily true. Moreover, recalling the Söngen formula recalled in Section 3, for the unknown $f(x)$ of (4.3) the following representation holds:

$$f(x) = \frac{K}{\sqrt{1-x^2}} - \frac{1}{2\pi^2}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1} \frac{d^3[h(z) - 8\pi D\delta(z)]}{dz^3}\frac{\sqrt{1-z^2}}{(x-z)} dz \tag{4.4}$$

where the constant K is arbitrary. Because of the smoothness assumptions we have made on $h(x)$ and $\delta(x)$, and the well known relationship

$$\int_{-1}^{1} \frac{\sqrt{1-z^2}}{x-z} = \pi x$$

the integral in (4.4) is certainly a continuous function for $x \in [-1, 1]$. Therefore

$$f(x) = \frac{K}{\sqrt{1-x^2}} + \frac{f_o(x)}{\sqrt{1-x^2}} \tag{4.5}$$

where $f_o \in C[-1, 1]$. This, as expected, contradicts the non integrable behavior forecast by Williams' approach; therefore in general the function $f(x)$ given by (4.4) cannot be a solution of the original contact problem, unless the true endpoint singularities are of integrable type, i.e., the function $f(x)$ defined by (4.4) is a solution of equation (4.2).

Indeed, as it will be shown next, for functions $h(x)$ and $\delta(x)$ having no special symmetries, the corresponding function $f(x)$ does not satisfy equation (4.2). This in turn implies that also this latter does not have a solution, since this should necessarily satisfy also (4.3). Nevertheless, when the functions $h(x)$ and $\delta(x)$ are both symmetric, by choosing properly the constant K, the corresponding $f(x)$ defined by (4.4) turns out to be the (unique) solution of (4.2). This means that in this particular case Williams' approach fails to detect the correct endpoint singular behavior.

To prove the above statements, it is first noted that the meaning of equation (4.3) differs from that of (4.2), since (4.3) imposes that the third derivative of the original left-hand side equals the third derivative of the original right-hand side. Consequently, when we substitute the solution (4.4) of (4.3) back into the left-hand side of the original equation (4.2), the original left-hand side is equal to the original right-hand side apart from a polynomial expression up to the second degree included.

The coefficients of this polynomial are not arbitrary, but they possess a precise expression that is computed by substituting expression (4.4) into (4.2).

When the first term of expression (4.4) is inserted into (4.2), the following integral is encountered

$$\int_{-1}^{1} \frac{(x-z)^2 \ln|x-z|}{\sqrt{1-z^2}} dz = \frac{\pi}{4}(1 - 2\ln 2) + \frac{\pi}{2}(3 - 2\ln 2)x^2 \quad . \tag{4.6}$$

This polynomial expression is symmetric, which agrees with the fact that the corresponding contact pressure is symmetric.

When the second term of expression (4.4) is introduced into (4.2), the following integral is met

$$(4.7)\qquad \int_{-1}^{1} \frac{(x-z)^2 \ln|x-z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y)-8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy \quad .$$

Since the right hand side of (4.2) is assumed to have continuous third derivatives in $[-1,1]$, and that of (4.6) is a (second degree) polynomial, to make the computation feasible this integral is differentiated three times with respect to the variable x:

$$(4.8)\qquad \frac{d^3}{dx^3} \int_{-1}^{1} \frac{(x-z)^2 \ln|x-z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y)-8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy$$

$$= 2 \int_{-1}^{1} \frac{1}{(x-z)\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y)-8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy$$

$$= -2\pi^2 \frac{d^3[h(x)-8\pi D\delta(x)]}{dx^3} \quad .$$

Consequently

$$(4.9)\qquad \int_{-1}^{1} \frac{(x-z)^2 \ln|x-z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y)-8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy$$

$$= -2\pi^2 [h(x) - 8\pi D\delta(x)] + A + Bx + Cx^2 \quad .$$

To compute the constants A, B and C appearing in (4.9), the integral of (4.9) is recast as

$$(4.10)\qquad \int_{-1}^{1} \frac{(x-z)^2 \ln|x-z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y)-8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy$$

$$= \int_{-1}^{1} \frac{d^3[h(y)-8\pi D\delta(y)]}{dy^3} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{(x-z)^2 \ln|x-z|}{\sqrt{1-z^2}(z-y)} dz$$

$$= -2\pi^2 [h(x) - 8\pi D\delta(x)] + A + Bx + Cx^2 \quad .$$

This order of integration exchange is justified by the result reported, for example, in Tricomi (1985), p. 170.

For the particular value $x = 0$, expression (4.10) becomes

$$\int_{-1}^{1} \frac{z^2 \ln|z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy \tag{4.11}$$

$$= \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{z^2 \ln|z|}{\sqrt{1-z^2}(z-y)} dz$$

$$= -2\pi^2[h(0) - 8\pi D\delta(0)] + A$$

where the inner singular integral appearing in the second row of (4.11) is computed exactly in the Appendix, see (6.8). Finally, the constant A may be evaluated by computing the outer integral appearing in the second row of (4.11), since $h(y)$ and $\delta(y)$ are known. Details are omitted for brevity.

To compute the constant B, equation (4.10) is differentiated once with respect to x, to obtain

$$\int_{-1}^{1} \frac{(x-z)[1 + 2\ln|x-z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy \tag{4.12}$$

$$= \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{(x-z)[1 + 2\ln|x-z|]}{\sqrt{1-z^2}(z-y)} dz$$

$$= -2\pi^2 \frac{d}{dx}[h(x) - 8\pi D\delta(x)] + B + 2Cx \ .$$

For the particular value $x = 0$, expression (4.12) becomes

$$\int_{-1}^{1} \frac{z[1 + 2\ln|z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy \tag{4.13}$$

$$= \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{z[1 + 2\ln|z|]}{\sqrt{1-z^2}(z-y)} dz$$

$$= 2\pi^2 \frac{d}{dx}[h(x) - 8\pi D\delta(x)]\Big|_{x=0} - B$$

where the inner singular integral appearing in the second row of (4.13) is computed exactly in the Appendix, see (6.9). Finally, the constant B may be evaluated by computing the outer integral appearing in the second row of (4.13) since $h(y)$ and $\delta(y)$ are given. When $h(y)$ and $\delta(y)$ are both symmetric functions, the above calculations become particularly simple, and

$$B = 2\pi^2 \frac{d}{dx}\left[h(x) - 8\pi D\delta(x)\right]\Big|_{x=0} = 0 \tag{4.14}$$

which describes a situation of plate indentation in the absence of tilting.

To compute the constant C, equation (4.10) is differentiated twice with respect to x, to obtain

$$\begin{aligned}
&\int_{-1}^{1} \frac{3 + 2\ln|x-z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy \\
&= \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{3 + 2\ln|x-z|}{\sqrt{1-z^2}(z-y)} dz \\
&= -2\pi^2 \frac{d^2}{dx^2}\left[h(x) - 8\pi D\delta(x)\right] + 2C \ .
\end{aligned} \tag{4.15}$$

For the particular value $x = 0$, expression (4.15) becomes

$$\begin{aligned}
&\int_{-1}^{1} \frac{3 + 2\ln|z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \frac{\sqrt{1-y^2}}{(z-y)} dy \\
&= \int_{-1}^{1} \frac{d^3[h(y) - 8\pi D\delta(y)]}{dy^3} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{3 + 2\ln|z|}{\sqrt{1-z^2}(z-y)} dz \\
&= -2\pi^2 \frac{d^2}{dx^2}\left[h(x) - 8\pi D\delta(x)\right]\Big|_{x=0} + 2C
\end{aligned} \tag{4.16}$$

where the inner singular integral appearing in the second row of (4.16) is computed exactly in the Appendix, see (6.10). Finally, the constant C may be evaluated by computing the outer integral appearing in the second row of (4.16).

Finally, by adding together, according to expression (4.4), the two terms (4.6) and (4.9), the following main result is obtained.

Theorem 4.2. *Under the assumptions made on the functions $h(x)$ and $\delta(x)$, the function $f(x)$ defined by* (4.4) *satisfies* (4.2) *with the right hand side replaced by*

$$h(x) - 8\pi D\delta(x) + A_o + B_o x + C_o x^2$$

where

$$A_o = K\frac{\pi}{4}(1-2\ln 2) - \frac{A}{2\pi^2}$$

(4.17)
$$B_o = -\frac{B}{2\pi^2}$$

$$C_o = K\frac{\pi}{2}(3-2\ln 2) - \frac{C}{2\pi^2}$$

with $A,\ B,\ C$ *being defined in* (4.9).

Thus in general equation (4.2) cannot be solved by means of (4.3) and therefore it does not have an integrable solution.

Corollary 4.3. *If* $h(x)$ *and* $\delta(x)$ *are symmetric and one takes*

$$K = \frac{C}{\pi^3(3-2\ln 2)}$$

then $B_o = C_o = 0$ *and, according to Definition* 4.1, $f(x)$ *is the unique solution of* (4.2), *where the constant* c *is given by* A_o *above, hence of the original contact problem.*

It is reminded that in the description of contact problems between Kirchhoff plates and irregular linear supports the shear force acts an equivalent shear force incorporating both the shear force and the twisting moment effects. The previous results testify that in general an incompatibility takes place between the equivalent shear force concept and the integral formulation of contact problems between Kirchhoff plates and irregular linear supports if the reaction force possesses non integrable endpoint singularities.

The above incompatibility emerges from the work of Yang (1968), in which a contact reaction expressed in terms of a line load alone is assumed along the internal support of a Kirchhoff rectangular plate. The singularity of the reaction force is found to be of the kind $1/\sqrt{x}$, a feature that disagrees with the body of the available results. In Stahl & Keer (1972) doubts are expressed about the correctness of the results of Yang (1968) "since the singularity is not in agreement with" the body of the literature studies. In the recent study of Sompornjaroensuk & Kiattikomol the presence of a stronger, non integrable singularity is confirmed.

In the following the alternative situation is examined in which the plate is again supported along a segment of length 2, but the distributed contact reaction is assumed to be described by a distributed twisting moment m, as in Monegato & Strozzi (2001b).

The function representing the mechanical response of a plate subjected to a concentrated moment M is obtained by differentiating the fundamental function referring to a concentrated force, *e.g.* Grigolyuk & Tolkachev (1987), p. 384. The governing integral equation, obtained by differentiating (4.1), possesses the form, *e.g.* Strozzi & Monegato (2008)

$$\int_{-1}^{1} m(z)(x-z)[1+2\ln|x-z|]dz = h(x) - 8\pi D\delta(x) + c \tag{4.18}$$

where c is a translation constant, not necessarily given a priori. The definition of $m(z)$ is very similar to that given in Definition 4.1 for $f(z)$.

By differentiating equation (4.18) twice (and not three times, as in the previous situation) with respect to x, thus assuming that the functions $h(x)$ and $\delta(x)$ have only the second derivatives Hölder continuous, the following canonical form is obtained:

$$2\int_{-1}^{1} \frac{m(z)}{x-z}dz = \frac{d^2[h(x) - 8\pi D\delta(x)]}{dx^2} \quad . \tag{4.19}$$

The analytical solution of the canonical Cauchy singular integral equation (4.19) in a space of proper integrable functions is

$$m(x) = \frac{K}{\sqrt{1-x^2}} - \frac{1}{2\pi^2}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1} \frac{d^2[h(z) - 8\pi D\delta(z)]}{dz^2}\frac{\sqrt{1-z^2}}{(x-z)}dz \tag{4.20}$$

which, as in the previous case, is of the type

$$m(x) = \frac{K}{\sqrt{1-x^2}} + \frac{m_o(x)}{\sqrt{1-x^2}} \tag{4.21}$$

with $m_o \in C[-1,1]$.

It is however noted that the meaning of equation (4.19) differs from that of (4.18), since (4.19) imposes that the second derivative of the original left-hand side equals the second derivative of the original right-hand side. Consequently, when we substitute the solution (4.20) of (4.19) back into the left-hand side of the original equation (4.18), the original left-hand side is equal to the original right-hand side apart from a polynomial expression up to the first degree included. The coefficients of the polynomial function are not general, but they possess a precise expression that is computed by substituting expression (4.20) into (4.18).

When the first term of expression (4.20) is inserted into (4.18), the following integral is encountered

$$\int_{-1}^{1} \frac{(x-z)[1+2\ln|x-z|]}{\sqrt{1-z^2}} dz = \pi x(3-2\ln 2) \quad . \tag{4.22}$$

This expression is skew-symmetric, which agrees with the circumstance that the corresponding form of the distributed twisting moment is symmetric.

When the second term of expression (4.20) is introduced into (4.18), the following integral is met

$$\int_{-1}^{1} \frac{(x-z)[1+2\ln|x-z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \quad . \tag{4.23}$$

To compute this integral, it is convenient to differentiate it twice with respect to the variable x and to follow Tricomi (1985), p. 171

$$\begin{aligned} &\frac{d^2}{dx^2} \int_{-1}^{1} \frac{(x-z)[1+2\ln|x-z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \\ &= 2 \int_{-1}^{1} \frac{1}{(x-z)\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \\ &= -2\pi^2 \frac{d^2[h(x)-8\pi D\delta(x)]}{dx^2} \quad . \end{aligned} \tag{4.24}$$

Consequently

$$\begin{aligned} &\int_{-1}^{1} \frac{(x-z)[1+2\ln|x-z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \\ &= -2\pi^2[h(x)-8\pi D\delta(x)] + A + Bx \quad . \end{aligned} \tag{4.25}$$

To compute the constants A and B, integral (4.25) is recast as

$$\begin{aligned} &\int_{-1}^{1} \frac{(x-z)[1+2\ln|x-z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \\ &= \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{(x-z)[1+2\ln|x-z|]}{\sqrt{1-z^2}(z-y)} dz \\ &= -2\pi^2[h(x)-8\pi D\delta(x)] + A + Bx \quad . \end{aligned} \tag{4.26}$$

For the particular value $x = 0$, expression (4.26) becomes

$$\int_{-1}^{1} \frac{z[1+2\ln|z|]}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \tag{4.27}$$
$$= \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{z[1+2\ln|z|]}{\sqrt{1-z^2}(z-y)} dz$$
$$= 2\pi^2[h(0)-8\pi D\delta(0)] - A$$

where the inner singular integral appearing in the second row of (4.27) is computed exactly in the Appendix, see (6.9). The constant A may then be evaluated by computing the outer integral appearing in the second row of (4.27) once $h(y)$ and $\delta(y)$ are known.

To compute the constant B, equation (4.26) is differentiated once with respect to x, to obtain

$$\int_{-1}^{1} \frac{3+2\ln|x-z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \tag{4.28}$$
$$= \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{3+2\ln|x-z|}{\sqrt{1-z^2}(z-y)} dz$$
$$= -2\pi^2 \frac{d}{dx}[h(x)-8\pi D\delta(x)] + B \ .$$

For the particular value $x = 0$, expression (4.28) becomes

$$\int_{-1}^{1} \frac{3+2\ln|z|}{\sqrt{1-z^2}} dz \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \frac{\sqrt{1-y^2}}{(z-y)} dy \tag{4.29}$$
$$= \int_{-1}^{1} \frac{d^2[h(y)-8\pi D\delta(y)]}{dy^2} \sqrt{1-y^2} dy \int_{-1}^{1} \frac{3+2\ln|z|}{\sqrt{1-z^2}(z-y)} dz$$
$$= -2\pi^2 \frac{d}{dx}[h(x)-8\pi D\delta(x)]\bigg|_{x=0} + B$$

where the inner singular integral appearing in the second row of (4.29) is computed exactly in the Appendix, see (6.10). The constant B, representing the plate tilting, may then be evaluated by computing the outer integral appearing in the second row of (4.29) once $h(y)$ and $\delta(y)$ are known.

Theorem 4.4. *The function* $m(x)$ *given by* (4.20) *satisfies equation* (4.18) *up to the linear term* $A_o + B_o x$ *defined by*

$$A_o = -\frac{A}{2\pi^2} \tag{4.30}$$
$$B_o = -\frac{B}{2\pi^2} + K\pi(3 - 2\ln 2)$$

where $A,\ B$ *are defined in* (4.25).

Corollary 4.5. *If one takes*

$$K = \frac{1}{2\pi^3(3 - 2\ln 2)}$$

then $B_o = 0$ *and* $m(x)$ *is the unique solution of* (4.18).

The previous results show that the description of the contact reaction in terms of an equivalent twisting moment permits the incompatibility, that was previously noted between the equivalent shear force concept and the integral formulations (4.2) and (4.3) of contact problems, to be avoided when the reaction force possesses non integrable endpoint singularities. Therefore, if the contact reaction is described in terms of an equivalent distributed twisting moment and not of a distributed shear force, a (integrable) solution to the original problem is found.

5 - Contact reaction described in terms of two simultaneous independent components, namely a shear force and a twisting moment

In the previous section, the equivalent shear force and the equivalent twisting moment have been considered as two alternative descriptions of the contact reaction. In some contact problems, however, to derive a simple singular integral equation having as (unique) solution with integrable endpoint singularities, solving the original contact problem, the two above reaction components ought to be employed simultaneously and treated as independent entities. For instance, in Grigolyuk & Tolkachev (1987), p. 381, the two contact reactions have been regarded as independent unknowns in the description of an infinite plate resting on an irregular linear support, and a solution strategy taking advantage from the simultaneous presence of the two functions has been proposed.

In Monegato & Strozzi (2002) a solid circular plate of radius R, simply supported along two antipodal edge arcs ($r = R, -a \leq \theta \leq a$), is studied. In this case the Williams' asymptotic method predicts a shear force contact reaction $f(\theta)$ with (non integrable) endpoint singularities of the type $\cos\theta/(\sin^2 a - \sin^2\theta)^{\frac{3}{2}}$, while for the twisting moment $m(\theta)$ these are weaker: $\sin\theta/(\sin^2 a - \sin^2\theta)^{\frac{1}{2}}$. Therefore it is not possible to reduce the original problem to a singular integral equation having as unknown only the equivalent shear force, because this has non integrable endpoint singularities. Since the non integrable component of the shear force is (Kirchhoff or mathematically) equivalent to an integrable twisting moment, and, moreover, a continuous twisting moment is (Kirchhoff) equivalent to an integrable shear force, only this latter and the (weakly) singular component of the twisting moment need to be considered explicitly. While the former is an unknown function, the second must necessarily have the form

$$\frac{C_2 P}{2}\frac{\sin\theta}{\sqrt{\sin^2 a - \sin^2\theta}} \tag{5.1}$$

where C_2 is a constant to be determined. Therefore in Monegato & Strozzi (2002) the simple twisting moment (5.1), possessing the correct endpoint singularity strength, has been imposed, whereas the (equivalent) shear force has been regarded as the unknown function.

Using two particular Green functions for the biharmonic equation, the following integral representation for the plate periphery deflection $w(\theta) = w(R,\theta)$ with respect to the plate center and for $-a \le \theta \le a$ has been obtained:

$$\begin{aligned}\frac{w(\theta)}{A} = {} & 24(1+\upsilon)R\int_{-a}^{+a}[(1+\cos(\theta-\omega))\ln(1+\cos(\theta-\omega)) \\ & +(1-\cos(\theta-\omega))\ln(1-\cos(\theta-\omega))]f(\omega)\,d\omega \\ & -24(1+\upsilon)\int_{-a}^{+a}[\sin(\theta-\omega)\ln(1-\cos(\theta-\omega)) \\ & -\sin(\theta-\omega)\ln(1+\cos(\theta-\omega))]m(\omega)\,d\omega \\ & -12(1+\upsilon)^2 R\int_{-a}^{+a}\omega^2 f(\omega)\,d\omega - 24(1+\upsilon)^2\int_{-a}^{+a}\omega\, m(\omega)\,d\omega \\ & +48(1+\upsilon)\ln 2 - 2\pi^2(1+\upsilon)^2 + 3(1-\upsilon)^3 \\ & -12(1+\upsilon)^2\theta^2 R\int_{-a}^{+a}f(\omega)\,d\omega \\ & +12\pi(1+\upsilon)^2 R\int_{-a}^{\theta}[\theta-\omega-\sin(\theta-\omega)]f(\omega)\,d\omega\end{aligned} \tag{5.2}$$

$$- 12\pi(1+v)^2 R \int_{\theta}^{+a} [\theta - \omega - \sin(\theta - \omega)] f(\omega)\, d\omega$$

$$+ 12\pi(1+v)^2 R \int_{-a}^{\theta} [\theta - \omega - \sin(\theta - \omega)] f(\omega)\, d\omega \tag{5.2}$$

$$- 12\pi(1+v)^2 R \int_{\theta}^{+a} [\theta - \omega - \sin(\theta - \omega)] f(\omega)\, d\omega$$

where a is the angular semi-width of each of the two edge supports, v the Poisson's ratio, R the plate radius and A is a given constant.

Some simplifications allowed by the symmetry of $f(\theta)$ and by the skew-symmetry of $m(\theta)$ have been incorporated.

By imposing that along the supported boundary arcs the plate edge deflection $w(\theta)$ assumes the constant value δ measured with respect to the (unknown) plate center, i.e.,

$$w(\theta) = \delta \ , \tag{5.3}$$

we obtain a Fredholm integral equation of the first kind. This equation is however too complex to allow the existence of solutions to be discussed. A considerably simpler integral equation is derived by observing that, if the supported edge arc must remain flat, the second derivative of $w(\theta)$ must vanish and, therefore, a solution to the initial Fredholm integral equation must also be a solution to the following equation:

$$\frac{\mathrm{d}^2 w(\theta)}{\mathrm{d}\,\theta^2} + w(\theta) = \delta \ . \tag{5.4}$$

All solutions of the original integral equation (5.3), expressed in terms of the equivalent shear force f and the given moment m, are solutions of equation (5.4). Conversely, equation (5.4) embraces additional solutions, namely all those distributions of force f and moment m which produce a deflection w of the form

$$w(\theta) = \delta + C_1 \cos\theta \ , \tag{5.5}$$

which constitutes the general symmetric solution to the differential equation (5.4). It is therefore necessary to solve equation (5.4) subject to a constraint condition which eliminates all non constant deflections, that is, which imposes the vanishing of constant C_1 of (5.5). A suitable constraint condition is obtained by annulling the second derivative of $w(\theta)$ for $\theta = 0$:

$$\frac{1}{24(1+\upsilon)A}\left.\frac{\mathrm{d}^2 w(\theta)}{\mathrm{d}\,\theta^2}\right|_{\theta=0} = -\frac{C_1}{24(1+\upsilon)A} = 0$$

$$(5.6)\qquad \begin{aligned} &= 2R\int_{-a}^{+a}\cos\omega\,\ln\left(\tan\frac{\omega}{2}\right)f(\omega)\,d\omega - 2\int_{-a}^{+a}\sin\omega\,\ln\left(\tan\frac{\omega}{2}\right)m(\omega)\,d\omega \\ &+ 2\int_{-a}^{+a}\frac{1}{\tan\omega}\,m(\omega)\,d\omega + (1-\upsilon)\,R\int_{-a}^{+a}f(\omega)\,d\omega \\ &+ \pi(1+\upsilon)\,R\int_{0}^{+a}\sin\omega\,f(\omega)\,d\omega + \pi(1+\upsilon)\int_{0}^{+a}\cos\omega\,m(\omega)\,d\omega\,. \end{aligned}$$

Finally, the integral equation one derives from (5.4), after performing a further differentiation, takes the simpler form:

$$(5.7)\qquad \int_{-a}^{+a}\frac{f(\omega)}{\tan(\theta-\omega)}d\omega + \frac{\pi(1+\upsilon)}{2}\int_{-a}^{\theta}f(\omega)d\omega = \frac{(1+\upsilon)P}{4r_o}\left(\frac{\pi}{2}+\theta+\frac{C_2\sin\theta}{\sqrt{\sin^2 a - \sin^2\theta}}\right).$$

Due to this further differentiation, the imposed displacement δ disappears and a description in terms of forces alone is achieved. However, a plate global translational equilibrium condition must then be added, i.e., the condition that the resultant of the reaction force f along one support equilibrates half the central load P,

$$(5.8)\qquad \int_{-a}^{+a}f(\omega)d\omega = \frac{P}{2R}\,.$$

Condition (5.6) which removes the non-constant deflections becomes

$$(5.9)\qquad \begin{aligned} &\frac{C_2 P}{R}\left[\int_{-a}^{+a}\frac{\sin^2\omega\,\ln\left(\tan\frac{\omega}{2}\right)}{\sqrt{\sin^2 a - \sin^2\omega}}\,d\omega - \pi - \frac{\pi(1+\upsilon)}{2}\sin a\right] \\ &= 2\int_{-a}^{+a}\cos\omega\,\ln\left(\tan\frac{\omega}{2}\right)f(\omega)\,d\omega \\ &+ \pi(1+\upsilon)\int_{0}^{+a}\sin\omega\,f(\omega)\,d\omega + \frac{(1-\upsilon)P}{2R}\,. \end{aligned}$$

Introducing proper changes of variables and functions, the singular integral equation (5.7) takes the canonical form

$$\frac{\sqrt{1-t^2\sin^2\alpha}}{\sin\alpha}\frac{1}{\pi}\int_{-1}^{+1}\frac{1}{\sqrt{1-x^2}}\frac{u(x)}{(t-x)}\,dx+\frac{(1+\nu)}{2}\int_{-1}^{t}\frac{u(x)}{\sqrt{1-x^2}}\,dx=q(t)\ , \tag{5.10}$$

where

$$q(t)=\frac{(1+\nu)}{4\pi R}P\left[\frac{\pi}{2}+\arcsin\left(t\sin\alpha\right)+C_2\frac{t}{\sqrt{1-t^2}}\right]\ .$$

Condition (5.8) assumes the new form

$$\int_{-1}^{+1}\frac{u(x)}{\sqrt{1-x^2}}\,dx=\frac{P}{2R}\ . \tag{5.11}$$

Monegato & Strozzi (2002) have proved existence and uniqueness of the solution of equation (5.10), subject to conditions (5.9) and (5.11), in certain weighted L_2 Sobolev type Hilbert spaces. Moreover, in such spaces stability and convergence of a simple polynomial collocation method have been proved. Thus the following statement holds.

Theorem 5.1. *In the above mentioned Hilbert spaces, the unique solution of equation* (5.10), *subject to conditions* (5.9) *and* (5.10), *is also the unique solution of equation* (5.3).

Following the same approach discussed in this section, i.e. by considering the appropriate (integrable) contact reaction components, it is possible to impose to the circular plate different boundary conditions: for example clamped, or simply supported/clamped. Also in these cases the contact problem can be represented in terms of singular integral equations.

6 - Appendix

Useful singular integrals are presented in the following, where $-1<y<1$. Details on the techniques employed are omitted for brevity.

$$\int_{-1}^{1}\frac{1}{\sqrt{1-z^2}(z-y)}dz=0 \tag{6.1}$$

$$\int_{-1}^{1} \frac{z}{\sqrt{1-z^2}(z-y)} dz = \pi \tag{6.2}$$

$$\int_{-1}^{1} \frac{\ln|z|}{\sqrt{1-z^2}} dz = -\pi \log 2 \tag{6.3}$$

$$\int_{-1}^{1} \frac{z \ln|z|}{\sqrt{1-z^2}} dz = 0 \tag{6.4}$$

$$\int_{-1}^{1} \frac{\ln|z|\sqrt{1-z^2}}{(z-y)} dz = \pi\left[\sqrt{1-y^2}\left(\frac{y}{|y|}\frac{\pi}{2} - \arcsin y\right) + y \log 2\right] \tag{6.5}$$

$$\int_{-1}^{1} \frac{\ln|z|}{\sqrt{1-z^2}(z-y)} dz = \frac{\pi}{\sqrt{1-y^2}}\left(\frac{y}{|y|}\frac{\pi}{2} - \arcsin y\right) \tag{6.6}$$

$$\int_{-1}^{1} \frac{z \ln|z|}{\sqrt{1-z^2}(z-y)} dz = -\pi \log 2 + \frac{\pi y}{\sqrt{1-y^2}}\left(\frac{y}{|y|}\frac{\pi}{2} - \arcsin y\right) \tag{6.7}$$

$$\int_{-1}^{1} \frac{z^2 \ln|z|}{\sqrt{1-z^2}(z-y)} dz = -\pi y \log 2 + \frac{\pi y^2}{\sqrt{1-y^2}}\left(\frac{y}{|y|}\frac{\pi}{2} - \arcsin y\right) . \tag{6.8}$$

Integral (6.8) appears in expression (4.11). It is skew-symmetric with respect to y.

The following integral appears in expressions (4.16) and (4.29). It may be computed in closed form with the aid of integrals (6.2) and (6.7).

$$\begin{aligned} &\int_{-1}^{1} \frac{z[1+2\ln|z|]}{\sqrt{1-z^2}(z-y)} dz \\ &= \int_{-1}^{1} \frac{z}{\sqrt{1-z^2}(z-y)} dz + 2\int_{-1}^{1} \frac{z \ln|z|}{\sqrt{1-z^2}(z-y)} dz \\ &= \pi(1-2\log 2) + \frac{2\pi y}{\sqrt{1-y^2}}\left(\frac{y}{|y|}\frac{\pi}{2} - \arcsin y\right) . \end{aligned} \tag{6.9}$$

Integral (6.9) is symmetric with respect to y.

The following integral appears in expressions (4.22) and (4.22). It may be computed in closed form with the aid of integrals (6.1) and (6.6).

(6.10)
$$\int_{-1}^{1} \frac{3+2\ln|z|}{\sqrt{1-z^2}(z-y)} dz$$
$$= 3 \int_{-1}^{1} \frac{1}{\sqrt{1-z^2}(z-y)} dz + 2 \int_{-1}^{1} \frac{\ln|z|}{\sqrt{1-z^2}(z-y)} dz$$
$$= \frac{2\pi}{\sqrt{1-y^2}} \left(\frac{y}{|y|}\frac{\pi}{2} - \arcsin y \right) .$$

Integral (6.10) is symmetric with respect to y.

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Abstract

The construction of (boundary) singular integral equation formulations of contact problems for Kirchhoff (thin) plates is addressed. In particular, the need to forecast the singularity strength of the contact reaction is evidenced. It is shown that, when employing the integral formulation to describe contact problems between Kirchhoff plates and irregular linear supports, the equivalent shear force concept may be incompatible with the integral equation approach. In such circumstances the equivalent shear force concept has to be abandoned in favour of, or coupled with, an equivalent twisting moment approach. These concepts are described and applied to two particular contact problems.

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