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MARIA CARMELA DE BONIS and CONCETTA LAURITA (*)

Nyström methods for Cauchy Singular Integral Equations. A survey. (**)

1 - Introduction

The Cauchy singular integral equations (CSIEs)

$$(1)\qquad a(y)F(y)+\frac{b(y)}{\pi}\int_{-1}^{1}\frac{F(x)}{x-y}dx+\int_{-1}^{1}k(x,y)F(x)dx=g(y),\quad |y|<1,$$

where a, b, k and g are known functions and F is the unknown, appear in several problems of applied sciences like crack theory, wing theory, elasticity and fluid flow problems (see for example [**1, 20, 30, 49, 52**]). The general theory on this topic has been developed in the fundamental books [**18, 31, 44, 49**].

In this paper we will consider equation (1) with constant coefficients $a, b \in \mathbb{R}$ such that $a^2+b^2=1, b\neq 0$.

It is well-known that (see [**49**]), even if g and k are regular functions, the solution F of (1) can be unbounded at one or both the endpoints ± 1. Thus the solution is searched in the following form

$$F(x)=f(x)v^{\alpha,\beta}(x),$$

(*) Department of Mathematics and Computer Sciences, University of Basilicata, Viale dell'Ateneo Lucano 10, 85100 Potenza, Italy, e-mail: mariacarmela.debonis@unibas.it, concetta.laurita@unibas.it

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where $f(x)$ is a smooth function and $v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is a Jacobi weight. The exponents $-1 < \alpha, \beta < 1$ are given by

$$\alpha = M - \frac{1}{2\pi i} \log\left(\frac{a+ib}{a-ib}\right)$$

and

$$\beta = N + \frac{1}{2\pi i} \log\left(\frac{a+ib}{a-ib}\right),$$

where M and N are integers chosen so that the index $\chi = -(\alpha+\beta) = -(M+N) \in \{-1, 0, 1\}$. Letting

$$(A^{\alpha,\beta} f)(y) = af(y)v^{\alpha,\beta}(y) + \frac{b}{\pi} \int_{-1}^{1} \frac{f(x)}{x-y} v^{\alpha,\beta}(x)dx \tag{2}$$

and

$$(Kf)(y) = \int_{-1}^{1} k(x,y) f(x) v^{\alpha,\beta}(x)dx,$$

equation (1) can be rewritten as follows

$$(A^{\alpha,\beta} f)(y) + (Kf)(y) = g(y).$$

$A^{\alpha,\beta}$ is called the *dominant operator* while K is called the *perturbation operator*.

The aim of this paper is to make a short survey on the numerical treatment of Cauchy singular integral equations with constant coefficients on the interval $[-1,1]$. We are going to describe some direct methods (quadrature and collocation methods), but our attention will be mainly focused on the indirect methods. In particular Nyström methods applied to the equivalent regularized Fredholm integral equation will be shown with the related convergence results and error estimates in spaces of continuous functions with weighted uniform norm. We will also consider the problem of the well-conditioning of the involved linear systems which is crucial for the computation of the approximate solution.

The paper is organized as follows. In Section 2 we introduce some function spaces that we will consider in sections 4 and 5 and in Section 3 we show the mapping properties of the operator $A^{\alpha,\beta}$. In Section 4 we present a brief outline of some direct methods, while Section 5 deals with the Nyström-type methods. Finally, in Section 6 we give some numerical tests on the described methods.

2 - Functions spaces

For $A \subseteq [-1,1]$, let $L^p(A)$ be the space of all measurable functions f such that

$$\|f\|_{L^p(A)} = \left(\int_A |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty, \quad 1 \le p < +\infty .$$

With $v(x) = v^{\rho,\theta}(x) = (1-x)^\rho (1+x)^\theta$, $\rho, \theta > -\dfrac{1}{p}$, a Jacobi weight, we set $f \in L^p_v(A)$ if and only if $fv \in L^p(A)$, $1 \le p < +\infty$. We equip the space $L^p_v(A)$ with the norm

$$\|f\|_{L^p_v(A)} = \left(\int_A |f(x)v(x)|^p dx \right)^{\frac{1}{p}} < +\infty, \quad 1 \le p < +\infty .$$

If $A = [-1,1]$, for brevity, we use the following notations $L^p_v = L^p_v([-1,1])$ and $\|fv\|_p = \|f\|_{L^p_v} = \|f\|_{L^p_v([-1,1])}$.

When $p = +\infty$ we define, for $\rho, \theta > 0$,

$$L^\infty_v = C_v = \left\{ f \in C^0((-1,1)) \ : \ \lim_{|x| \to 1} (fv)(x) = 0 \right\},$$

where $C^0(A)$ is the collection of the continuous functions in $A \subseteq [-1,1]$. In case $\rho = 0$ (respectively, $\theta = 0$) C_v consists of all continuous functions on $(-1,1]$ (respectively, $[-1,1)$) such that

$$\lim_{x \to -1} (fv)(x) = 0 \quad \left(\text{respectively, } \lim_{x \to 1} (fv)(x) = 0 \right).$$

In the case where $\rho = \theta = 0$, we set $C_v = C^0([-1,1])$. We equip the space C_v with the norm

$$\|f\|_{C_v} = \|fv\|_\infty = \max_{|x| \le 1} |(fv)(x)| .$$

Finally, for $A \subset [-1,1]$, we will set $\|f\|_{L^\infty_v(A)} = \sup_{x \in A} |f(x)v(x)|$.

A subspace of L^p_v, $1 \le p \le \infty$, is the Zygmund space $Z^p_r(v)$ defined as

$$Z^p_r(v) = \left\{ f \in L^p_v \ : \ \sup_{t > 0} \frac{\Omega^s_\varphi(f,t)_{v,p}}{t^r} < +\infty, \ s > r > 0 \right\}, \quad s \in \mathbb{N},$$

where [**17**]

$$\Omega^s_\varphi(f,t)_{v,p} = \sup_{0 < h \le t} \|(\Delta^s_{h\varphi} f)\|_{L^p_v(I_{h,s})},$$

$I_{h,s} = [-1 + 4s^2h^2, 1 - 4s^2h^2], 0 < t < 1$, $\varphi(x) = \sqrt{1-x^2}$ and

$$\Delta^s_{h\varphi} f(x) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} f\left(x + h\varphi(x)\left(\frac{s}{2} - i\right)\right).$$

We equip it with the following norm

$$\|f\|_{Z^p_r(v)} = \|fv\|_p + \sup_{t>0} \frac{\Omega^s_\varphi(f,t)_{v,p}}{t^r}\,.$$

In case $p = \infty$ we will set $\Omega^s_\varphi(f,t)_v = \Omega^s_\varphi(f,t)_{v,\infty}$ and $Z_r(v) = Z^\infty_r(v)$. For $v(x) \equiv 1$ we will denote by Z^p_r the space $Z^p_r(v)$ and then, in particular, $Z_r = Z^\infty_r$.

Denoting by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by

$$E_m(f)_v = \inf_{P \in \mathbb{P}_m} \|(f-P)v\|_\infty$$

the error of best weighted approximation of a function $f \in C_v$ by means of polynomials in $\mathbb{P}_m$, for all functions $f \in Z_r(v)$, we have [**17,** p. 94]

$$E_m(f)_v \le \frac{\mathcal{C}}{m^r} \|f\|_{Z_r(v)}, \quad \mathcal{C} \ne \mathcal{C}(m,f)\,. \tag{3}$$

Here and in the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C} \ne \mathcal{C}(a,b,\dots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \dots$. If $A, B \ge 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that

$$\frac{B}{\mathcal{C}} \le A \le \mathcal{C}B.$$

3 - Mapping properties of the operator $A^{\alpha,\beta}$

Denoting by $\{p^{\rho,\theta}_m\}_{m\in\mathbb{N}}$ the sequence of the polynomials which are orthonormal with respect to the Jacobi weight $v^{\rho,\theta}$, the operator $A^{\alpha,\beta}$ defined by (2) satisfies [**53,** Chapter 9]

$$A^{\alpha,\beta} p^{\alpha,\beta}_m = -\frac{b}{\sin(\pi\alpha)} p^{-\alpha,-\beta}_{m-\chi} \; (p^{-\alpha,-\beta}_{-1} = 0), \quad m = 0,1,2,\dots, \quad \chi \in \{-1,0,1\}. \tag{4}$$

Moreover, using the Fourier sum in the system of the Jacobi polynomials, it was proved that $A^{\alpha,\beta}$ is a continuous map in the pair of spaces $(L^2_{\sqrt{v^{\alpha,\beta}}}, L^2_{\sqrt{v^{-\alpha,-\beta}}})$ (see [**3, 5, 22, 23, 27, 29, 44, 48, 53, 54**] and the references therein). On the other hand, the operator $\widehat{A}^{\alpha,\beta}$ defined as

$$(\widehat{A}^{\alpha,\beta}f)(y) = af(y)v^{-\alpha,-\beta}(y) - \frac{b}{\pi}\int_{-1}^{1} \frac{f(x)}{x-y} v^{-\alpha,-\beta}(x)dx \tag{5}$$

is the adjoint operator of $A^{\alpha,\beta}$ and satisfies

$$\widehat{A}^{\alpha,\beta} p_m^{-\alpha,-\beta} = -\frac{b}{\sin(\pi\alpha)} p_{m+\chi}^{\alpha,\beta}, \quad m = 0,1,\ldots, \quad \chi \in \{-1,0,1\}. \tag{6}$$

Therefore, for $\chi = 0$, $\widehat{A}^{\alpha,\beta} : L^2_{\sqrt{v^{-\alpha,-\beta}}} \to L^2_{\sqrt{v^{\alpha,\beta}}}$ is the two-sided inverse of $A^{\alpha,\beta}$, i.e.,

$$\widehat{A}^{\alpha,\beta} = (A^{\alpha,\beta})^{-1}.$$

For $\chi = 1$, $\widehat{A}^{\alpha,\beta}$ is a right inverse and for $\chi = -1$, $\widehat{A}^{\alpha,\beta}$ is a left inverse.

In spaces of continuous functions equipped with uniform norms, $A^{\alpha,\beta}$ is an unbounded operator and, for long time, only partial results were available in literature on this topic [**6, 7, 24, 37, 38**]. More recently in [**40**] and [**15**], using the de la Vallée Poussin polynomials in place of the Fourier sums, the operators $A^{\alpha,\beta}$ were studied in pairs of Zygmund spaces $Z_r(v^{\rho,\theta})$ (see the definition in Section 2) and also in the more general case of Besov spaces. In particular, in [**40, 15**] the Authors proved the following results: for all $r > 0$ and $0 < \alpha < 1$,

1. $A^{\alpha,-\alpha} : Z_r(v^{\alpha,0}) \to Z_r(v^{0,\alpha})$ is a continuous and invertible map and its two-sided inverse is the continuous operator $\widehat{A}^{\alpha,-\alpha} : Z_r(v^{0,\alpha}) \to Z_r(v^{\alpha,0})$;
2. $A^{-\alpha,\alpha-1} : Z_r \to Z_r(v^{\alpha,1-\alpha})$ is a continuous map and its right inverse is the continuous map $\widehat{A}^{-\alpha,\alpha-1} : Z_r(v^{\alpha,1-\alpha}) \to Z_r$. Moreover it results

$$\widehat{A}^{-\alpha,\alpha-1} A^{-\alpha,\alpha-1} f = f - \frac{\displaystyle\int_{-1}^{1} f(x)v^{-\alpha,\alpha-1}(x)dx}{\displaystyle\int_{-1}^{1} v^{-\alpha,\alpha-1}(x)dx}, \quad \forall f \in Z_r. \tag{7}$$

4 - Direct methods

In this section we give a brief description of some of the direct methods for the numerical solution of the equation

$$(A^{\alpha,\beta} + K)f = g. \tag{8}$$

In the development of direct methods, particular attention, both computationally and theoretically, was given to those procedures which use polynomials as trial functions (Galerkin, collocation and quadrature methods), the so-called polynomial approximation methods. They are based on the application of the well-known properties (4) and (6) of the Jacobi polynomials and the literature about them is very wide. We recommend the interested reader to consult [**2, 3, 6, 8, 13, 21, 24, 25, 26, 27, 28, 29, 33, 34, 42, 43, 46**] and all the references given there.

We premise some notations. In the sequel $L_n^{\rho,\theta}$ will denote the Lagrange interpolation operator based on the zeros $x_k^{\rho,\theta} = x_{n,k}^{\rho,\theta}$, $k = 1, \dots, n$ ($x_1^{\rho,\theta} < x_2^{\rho,\theta} < \cdots < x_n^{\rho,\theta}$), of the orthonormal polynomial $p_n^{\rho,\theta}$, i.e.

$$L_n^{\rho,\theta}(f, x) = \sum_{k=1}^{n} f(x_k^{\rho,\theta}) l_{n,k}^{\rho,\theta}(x)$$

where $l_{n,k}^{\rho,\theta}$ is the k-th fundamental Lagrange polynomial.

Moreover, $\lambda_k^{\rho,\theta} = \lambda_{n,k}^{\rho,\theta}$, $k = 1, \dots, n$, stands for the related Christoffel numbers, i.e

$$\lambda_k^{\rho,\theta} = \int_{-1}^{1} l_{n,k}^{\rho,\theta}(x)\, v^{\rho,\theta}(x) dx.$$

4.1 - *Collocation methods*

The *collocation method* consists in approximating (8) by the finite dimensional equation

$$L_{m-\chi}^{-\alpha,-\beta}(A^{\alpha,\beta} + K) f_m = L_{m-\chi}^{-\alpha,-\beta} g, \tag{9}$$

and in looking for a polynomial solution $f_m \in \mathbb{P}_{m-1}$ having the following form

$$f_m(y) = \sum_{k=0}^{m-1} a_k p_k^{\alpha,\beta}(y). \tag{10}$$

Let us note that, in virtue of (4), the previous equation can be rewritten as follows

$$(A^{\alpha,\beta} + L_{m-\chi}^{-\alpha,-\beta} K) f_m = L_{m-\chi}^{-\alpha,-\beta} g.$$

Now, studying equation (9) in the space $L^2_{\sqrt{v^{\alpha,\beta}}}$, we require that the coefficients of the polynomial f_m satisfy the following equation

$$\left\| L_{m-\chi}^{-\alpha,-\beta}\big((A^{\alpha,\beta} + K) f_m - g\big) \sqrt{v^{-\alpha,-\beta}} \right\|_2 = 0.$$

Consequently, by applying the Gaussian quadrature rule with respect to the weight $v^{-\alpha,-\beta}$, we get

$$\sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}} \left[(A^{\alpha,\beta} + K) f_m(x_{m-\chi,i}^{-\alpha,-\beta}) \right] = \sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}\, g(x_{m-\chi,i}^{-\alpha,-\beta}) \quad i = 1, \dots, m-\chi,$$

i.e, taking into account (10) and (4), the linear system

$$\sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}\sum_{k=0}^{m-1}\left[-\frac{b}{\sin\pi\alpha}p_{k-\chi}^{-\alpha,-\beta}(x_{m-\chi,i}^{-\alpha,-\beta})+(Kp_k^{\alpha,\beta})(x_{m-\chi,i}^{-\alpha,-\beta})\right]a_k \tag{11}$$

$$=\sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}g(x_{m-\chi,i}^{-\alpha,-\beta}),\quad i=1,\ldots,m-\chi\,,$$

in the unknowns a_k, $k=0,\ldots,m-1$.

For $\chi=1$, in order to define the solution of (8) uniquely, the additional condition

$$\int_{-1}^{1} f(x)v^{\alpha,\beta}(x)dx=0$$

has to be considered. In this case the equation

$$a_0=0 \tag{12}$$

is added to (11).

Let us observe that, in order to compute the entries of the matrix of system (11), the integrals $(Kp_k^{\alpha,\beta})(x_{m-\chi,i}^{-\alpha,-\beta})$, $k=0,1,\ldots,m-1$, $i=1,\ldots,m-\chi$, usually called *modified moments*, have to be computed.

For classical weakly singular kernels like

$$k(x,y)=|x-y|^{\mu},\quad \text{with} -1<\mu<0,$$

or

$$k(x,y)=\log|x-y|\quad (\mu=0),$$

they can be exactly evaluated by means of recurrence formulas (see for example [**39**]).

Moreover, if $\mu>-\dfrac{1}{2}$ and $g\in Z_s^2(v^{-\alpha,-\beta})$ with $s>\dfrac{1}{2}$, then, from the estimate

$$\|(K-L_m^{-\alpha,-\beta}K)f\|_{L^2_{\sqrt{v^{-\alpha,-\beta}}}}\leq\frac{\mathcal{C}}{m^{1+\mu}}\|f\|_{L^2_{\sqrt{v^{\alpha,\beta}}}},\quad \mathcal{C}\neq\mathcal{C}(m,f),$$

it follows that, for a sufficiently large m, system (11) is unisolvent and the polynomial f_m corresponding to its unique solution satisfies

$$\|f-f_m\|_{L^2_{\sqrt{v^{\alpha,\beta}}}}\leq\frac{\mathcal{C}}{m^r}\|f\|_{Z_r^2(\sqrt{v^{\alpha,\beta}})},\quad \mathcal{C}\neq\mathcal{C}(m,f),$$

with $r=\min(1+\mu,s)$ if $-1<\mu<0$, and $r=s$ if $\mu=0$ (see [**34**]).

Furthermore it was also proved that the linear system (11) is well conditioned (see [**34**]).

When the exact computation of the modified moments $(Kp_k^{\alpha,\beta})(x_{m-\chi,i}^{-\alpha,-\beta})$ is not possible and the kernel $k(x,y)$ is sufficiently smooth, one can approximate them by using a suitable Gaussian quadrature rule.

In this case, in place of (9), we consider the following finite dimensional equation (see [**33, 35**])

$$L_{m-\chi}^{-\alpha,-\beta}(A^{\alpha,\beta}+\widetilde{K}_m)f_m = L_{m-\chi}^{-\alpha,-\beta}g \tag{13}$$

with

$$(\widetilde{K}_m f_m)(y) = \int_{-1}^{1} L_m^{\alpha,\beta}(k_y,x)f_m(x)v^{\alpha,\beta}(x)dx$$

and $k_y(x) = k(x,y) = k_x(y)$.

Then, in order to construct the polynomial solution f_m of (13) in the form (10), we can proceed as above and obtain the unknown coefficients a_k, $k = 0,1,\dots,m-1$, by solving the linear system

$$\sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}\sum_{k=0}^{m-1}\left[-\frac{b}{\sin\pi\alpha}p_{k-\chi}^{-\alpha,-\beta}(x_{m-\chi,i}^{-\alpha,-\beta})+\sum_{j=1}^{m}\lambda_{m,j}^{\alpha,\beta}k(x_{m,j}^{\alpha,\beta},x_{m-\chi,i}^{-\alpha,-\beta})p_k^{\alpha,\beta}(x_{m,j}^{\alpha,\beta})\right]a_k \tag{14}$$

$$= \sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}g(x_{m-\chi,i}^{-\alpha,-\beta}), \quad i = 1,\dots,m-\chi,$$

with the additional condition (12) for $\chi = 1$.
This modified procedure is usually referred to as *discrete collocation method*.

By assuming the kernel $k(x,y)$ of the integral operator K such that

$$\sup_{|y|\le 1}\|k_y\|_{Z_r^2(\sqrt{v^{\alpha,\beta}})} < \infty, \quad \sup_{|x|\le 1}\|k_x\|_{Z_r^2(\sqrt{v^{-\alpha,-\beta}})} < \infty, \tag{15}$$

and $g \in Z_r^2(\sqrt{v^{-\alpha,-\beta}})$ with $r > \dfrac{1}{2}$, from

$$\|(K - L_m^{-\alpha,-\beta}\widetilde{K}_m)f\|_{L^2_{\sqrt{v^{-\alpha,-\beta}}}} \le \frac{\mathcal{C}}{m^r}\|f\|_{L^2_{\sqrt{v^{\alpha,\beta}}}}$$

one can also deduce (see [**33, 35**]) that, for a sufficiently large m, system (14) is unisolvent and well conditioned and that the polynomial f_m corresponding to its solution satisfies the following estimate

$$\|f - f_m\|_{L^2_{\sqrt{v^{\alpha,\beta}}}} \le \frac{\mathcal{C}}{m^r}\|f\|_{Z_r^2(\sqrt{v^{\alpha,\beta}})}, \quad \mathcal{C} \ne \mathcal{C}(m,f). \tag{16}$$

A generalization of the described methods can be obtained by replacing in (11) (respectively, in (14)) the collocation nodes $x_{m-\chi,i}^{-\alpha,-\beta}$ by the zeros $x_{m-\chi,i}^{\rho,\theta}$ of the polynomial $p_{m-\chi}^{\rho,\theta}$ which is orthonormal with respect to a Jacobi weight $v^{\rho,\theta}$ such that

$$\left(\sqrt{v^{\alpha,\beta}v^{\rho,\theta}\varphi}\right)^{\pm 1} \in L^2, \quad \varphi(x) = \sqrt{1-x^2},$$

and by substituting the Christoffel numbers $\lambda_{m-\chi,i}^{-\alpha,-\beta}$ by $\lambda_{m-\chi}(v^{-\alpha,-\beta}, x_{m-\chi,i}^{\rho,\theta})$, being

$$\lambda_{m-\chi}(v^{-\alpha,-\beta}, x) = \left[\sum_{k=0}^{m-\chi-1}\left(p_k^{-\alpha,-\beta}(x)\right)^2\right]^{-1}$$

the $m-\chi$-th Christoffel function related to $v^{-\alpha,-\beta}$ (see [**34, 35**]). Let us recall that for $v^{\rho,\theta} = v^{-\alpha,-\beta}$ it results $\lambda_{m-\chi}(v^{-\alpha,-\beta}, x_{m-\chi,i}^{-\alpha,-\beta}) = \lambda_{m-\chi,i}^{-\alpha,-\beta}$.

In the general case $v^{\rho,\theta} \neq v^{-\alpha,-\beta}$, the linear system is deduced from equation (9) (respectively, (13)) by applying a Marcinkiewicz inequality (see [**34**]) instead of the Gaussian rule.

4.2 - *Quadrature methods*

The *quadrature method* can be considered a procedure equivalent, in some sense, to the above described discrete collocation method. The method consists in approximating the unknown solution f of (8) by means of a polynomial of degree $m-1$ of the following kind

$$f_m(y) = \sum_{k=1}^{m} \xi_k l_{m,k}^{\alpha,\beta}(y) \tag{17}$$

satisfying the finite dimensional equation (13).

Considering equation (13) in the space $L^2_{\sqrt{v^{\alpha,\beta}}}$, we require that the coefficients of the polynomial f_m in (17) satisfy the following equation

$$\left\| L_{m-\chi}^{-\alpha,-\beta}\left((A^{\alpha,\beta} + \widetilde{K}_m)f_m - g\right)\sqrt{v^{-\alpha,-\beta}}\right\|_2 = 0. \tag{18}$$

Since, applying relation (4), one can show (see, for instance, [**44**, p. 448]) that

$$(A^{\alpha,\beta}f_m)(x_{m-\chi,i}^{-\alpha,-\beta}) = \frac{b}{\pi}\sum_{k=1}^{m}\lambda_{m,k}^{\alpha,\beta}\frac{f_m(x_{m,k}^{\alpha,\beta})}{x_{m,k}^{\alpha,\beta} - x_{m-\chi,i}^{-\alpha,-\beta}},$$

if we set

$$\xi_k = \left(\sqrt{\lambda_{m,k}^{\alpha,\beta}}\right)^{-1}\eta_k, \quad k = 1, \ldots, m,$$

then condition (18) is satisfied if and only if the array $(\eta_k)_{k=1,\ldots,m}$ is the solution of the following linear system

$$\sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}\sum_{k=1}^{m}\sqrt{\lambda_{m,k}^{\alpha,\beta}}\left[\frac{b}{\pi(x_{m,k}^{\alpha,\beta}-x_{m-\chi,i}^{-\alpha,-\beta})}+k(x_{m,k}^{\alpha,\beta},x_{m-\chi,i}^{-\alpha,-\beta})\right]\eta_k \tag{19}$$

$$=\sqrt{\lambda_{m-\chi,i}^{-\alpha,-\beta}}g(x_{m-\chi,i}^{-\alpha,-\beta}),\quad i=1,\dots,m-\chi.$$

Let us observe that the distance between the zeros $x_{m,k}^{\alpha,\beta}$ and $x_{m-\chi,i}^{-\alpha,-\beta}$, $\forall k=1,\dots,m$, $\forall i=1,\dots,m-\chi$, is large enough to avoid the numerical cancellation [**41**]. In case $\chi=1$, the equation

$$\sum_{k=1}^{m}\sqrt{\lambda_{m,k}^{\alpha,\beta}}\eta_k=0$$

is added to (19).

It was proved in [**35**] that if $g\in Z_r^2(\sqrt{v^{-\alpha,-\beta}})$ and $k(x,y)$ satisfies (15) with $r>\dfrac{1}{2}$, the polynomial f_m constructed by solving (19) and using (17), satisfies the error estimate (16). Moreover, (19) is a well conditioned linear system (its condition number is independent of its dimension m) as well as (14).

Let us remark that the equivalence between the two procedures (14),(10) and (19),(17) can be easily proved taking into account the well known relation

$$l_{m,k}^{\alpha,\beta}=\lambda_{m,k}^{\alpha,\beta}\sum_{j=0}^{m-1}p_j^{\alpha,\beta}(x_{m,k}^{\alpha,\beta})p_j^{\alpha,\beta}$$

and the consequent one

$$a_k=\sum_{j=1}^{m}\lambda_{m,j}^{\alpha,\beta}p_k^{\alpha,\beta}(x_{m,j}^{\alpha,\beta})\xi_j\,.$$

Nevertheless, the first method is more expensive from a computational point of view because of the evaluation of the orthornormal Jacobi polynomials $p_k^{\alpha,\beta}$ and $p_{k-\chi}^{-\alpha,-\beta}$ both in constructing the entries of the matrix of system (14) and in computing f_m by means of (10).

Comparing the above described methods, it is clear that they are equivalent in terms of convergence order. The collocation method, among the three procedures, is more expensive from the computational point of view but can also be used when the kernel is weakly singular.

The exposure of the two methods is essentially the one presented in [**34, 35**], where, using ideas proposed in [**3, 6, 8, 24, 27, 33, 42**], special attention was devoted to the conditioning of the involved linear systems.

Finally, we remark that, in case $\chi = -1$, such systems are overdetermined (they have $m+1$ equations and m unknowns) and the choice of the equation that has to be dropped is not clear.

5 - Indirect methods

Multiplying both sides of (8) on the left by $\widehat{A}^{\alpha,\beta}$, we get the equation

$$f + \widehat{A}^{\alpha,\beta} Kf = \widehat{A}^{\alpha,\beta} g + b_\chi \,, \tag{20}$$

where $b_\chi = 0$ for $\chi = -(\alpha+\beta) \in \{-1, 0\}$ and b_χ is a fixed real number for $\chi = 1$. In the cases where $\chi = 0$ and $\chi = 1$, (8) and (20) are equivalent. For $\chi = -1$, sufficient conditions on the kernel k and the known term g can be established in order to obtain that (8) and (20) are equivalent [**30, 52**].
If $\widehat{A}^{\alpha,\beta} K$ is compact in the space in which we are looking for the solution of (8), then (20) is called the regularized Fredholm equation of (8) and it satisfies the Fredholm alternative. Therefore it is possible to compute the solution of (8) by solving the equivalent regularized Fredholm equation (20).

The literature about the indirect methods is not very wide. Among the other here we mention [**16, 19, 39**]. In [**19**] a Nyström method is applied in the special case $\alpha = \beta = -\dfrac{1}{2}$ and its uniform convergence is proved provided that the input functions k and g have a continuous derivative.

In virtue of recent results on the mapping properties of the dominant operator $A^{\alpha,\beta}$, in [**39, 16**] the more general cases $\chi \in \{0, 1\}$ has been investigated in suitable weighted spaces $C_{v^{\rho,\theta}}$. The case $\chi = 0$ has been treated in both the papers, while the case $\chi = 1$ only in the second one.

Applying projection methods, polynomial approximations, convergent to the exact solution, are computed by solving well-conditioned linear systems. In particular, concerning the case $\chi = 0$, in [**39**] equation (8) is considered in the space $C_{v^{\alpha,0}}, 0 < \alpha < 1$, and two different linear systems are used for $0 < \alpha < \dfrac{1}{2}$ and $\alpha \geq \dfrac{1}{2}$, while in [**16**] equation (8) is studied in $C_{v^{\alpha+\gamma,\delta}}$, where γ and δ satisfy suitable assumptions, and this leads to solve a unique system for any value of $0 < \alpha < 1$. Anyway the results on the condition numbers of the respective linear systems and on the convergence of both approaches are equivalent.

The numerical treatment of CSIEs with negative index ($\chi = -1$) was investigated in [**14**].

In the sequel, for $\chi \in \{0, 1\}$, equation (8) is considered in spaces of continuous functions with weighted uniform norm. This is more in line with the numerical treatment where punctual errors are often shown.

The solution of (8) is computed solving the equivalent regularized Fredholm integral equation (20) by means of a Nyström-type method. We premise a brief description of such method giving results concerning its stability and convergence.

5.1 - *Nyström method*

In order to fix the ideas let us consider the Fredholm integral equation of the second kind

$$f(y) + \int_{-1}^{1} h(x, y) f(x) v^{\alpha,\beta}(x) dx = g(y), \quad |y| < 1, \tag{21}$$

where the kernel h and the right-hand side g are known functions.

If we set

$$(Kf)(y) = \int_{-1}^{1} h(x, y) f(x) v^{\alpha,\beta}(x) dx, \tag{22}$$

equation (21) can be rewritten as

$$(I + K)f = g, \tag{23}$$

where I denotes the identity operator. We will study (23) in the space $C_{v^{\rho,\theta}}$, $\rho, \theta \geq 0$, where $v^{\rho,\theta}$ is a suitable Jacobi weight.

If $h(x, y)$ is continuous with respect to both the variables, we replace (23) by the new equation

$$(I + K_m)f_m = g \tag{24}$$

where the operator K_m is defined as follows

$$(K_m f)(y) = \sum_{k=1}^{m} \lambda_k^{\alpha,\beta} h(x_k^{\alpha,\beta}, y) f(x_k^{\alpha,\beta}), \quad y \in [-1, 1]. \tag{25}$$

We recall that $x_k^{\alpha,\beta}$ and $\lambda_k^{\alpha,\beta}$ are the nodes and weights of the Gauss-Jacobi quadrature formula with respect to the weight $v^{\alpha,\beta}$, respectively.

Multiplying both sides of (24) by the weight function $v^{\rho,\theta}$ and evaluating them in the quadrature knots, we obtain the following linear system

$$(26)\qquad a_i + v^{\rho,\theta}(x_i^{\alpha,\beta}) \sum_{k=1}^{m} \frac{\lambda_k^{\alpha,\beta}}{v^{\rho,\theta}(x_k^{\alpha,\beta})} h(x_k^{\alpha,\beta}, x_i^{\alpha,\beta}) a_k = g(x_i^{\alpha,\beta}) v^{\rho,\theta}(x_i^{\alpha,\beta}),\ i = 1, \dots, m,$$

in the unknowns $a_i = f_m(x_i^{\alpha,\beta}) v^{\rho,\theta}(x_i^{\alpha,\beta})$, $i = 1, \dots, m$.

Then the Nyström interpolating function f_m defined as

$$f_m(x) = g(x) - \sum_{k=1}^{m} \frac{\lambda_k^{\alpha,\beta}}{v^{\rho,\theta}(x_k^{\alpha,\beta})} h(x_k^{\alpha,\beta}, x) a_k$$

is a solution of (24) in $C_{v^{\rho,\theta}}$ if and only if the array $(a_k)_{k=1,\dots,m}$ is solution of system (26) (see for example [**2**, p. 101]).

Letting

$$\|T\| = \|T\|_{C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}} = \sup_{\|f\|_{C_{v^{\rho,\theta}}}=1} \|Tf\|_{C_{v^{\rho,\theta}}},$$

for any linear operator $T : C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}$, the following theorem establishes the convergence and the stability of the Nyström method (see for example [**2**, Theorem 4.1.1, p. 106]).

Theorem 5.1. *Assume that the operators K and K_m defined in* (22) *and* (25)*, respectively, satisfy*

$$(27)\qquad \|Kf\|_{C_{v^{\rho,\theta}}} \le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}}, \quad \mathcal{C} \ne \mathcal{C}(f),$$

$$(28)\qquad \sup_m \|K_m\| < +\infty,$$

$$(29)\qquad \lim_{m\to+\infty} \|(K - K_m)f\|_{C_{v^{\rho,\theta}}} = 0, \quad \forall f \in C_{v^{\rho,\theta}},$$

and

$$(30)\qquad \lim_{m\to+\infty} \|(K - K_m)K_m\| = 0.$$

Further assume that the integral equation (23) *is uniquely solvable for any given $g \in C_{v^{\rho,\theta}}$. Then for a sufficiently large m, say $m > m_0$, the approximate inverses $(I + K_m)^{-1}$ exist and are uniformly bounded*

$$\|(I + K_m)^{-1}\| \le \frac{1 + \|(I + K)^{-1}\|\|K_m\|}{1 - \|(I + K)^{-1}\|\|(K - K_m)K_m\|} \le \mathcal{C},$$

for a suitable constant $\mathcal{C} \ne \mathcal{C}(m)$. Moreover the solutions f of (23) *and f_m of* (24) *satisfy*

$$\|f - f_m\|_{C_{v^{\rho,\theta}}} \sim \|(K - K_m)f\|_{C_{v^{\rho,\theta}}}. \tag{31}$$

Finally, the matrix A_m *of system* (26) *verifies*

$$\mathrm{cond}(A_m) \leq \mathrm{cond}(I + K_m), \tag{32}$$

where $\mathrm{cond}(A_m) = \|A_m\| \|A_m^{-1}\|$ *denotes its condition number in uniform norm.*

In the next corollary we give sufficient conditions on the known functions assuring that the previous theorem holds true.

Corollary 5.1. *Let* $v^{\rho,\theta}$ *be a Jacobi weight such that*

$$0 \leq \rho < \alpha + 1, \quad 0 \leq \theta < \beta + 1. \tag{33}$$

Assume that $h_x(y) \equiv h(x,y) \equiv h_y(x)$ *and* g *verify the conditions*

$$\sup_{|x|\leq 1} \|h_x\|_{Z_r(v^{\rho,\theta})} < +\infty\,, \tag{34}$$

$$\sup_{|y|\leq 1} v^{\rho,\theta}(y)\|h_y\|_{Z_r} < +\infty \tag{35}$$

and

$$g \in Z_r(v^{\rho,\theta}) \tag{36}$$

for some $r > 0$*. If the integral equation* (23) *has in* $C_{v^{\rho,\theta}}$ *the unique solution* f *for a fixed* g*, then, for a sufficiently large* m *(say* $m > m_0$*), equation* (24) *has* f_m *as unique solution and*

$$\|f - f_m\|_{C_{v^{\rho,\theta}}} \leq \frac{\mathcal{C}}{m^r}\|f\|_{Z_r(v^{\rho,\theta})}, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{37}$$

Moreover the matrix of system (26) *satisfies* (32).

Proof. In order to prove (37) and (32), we proceed to verify that the hypotheses of Theorem 5.1 are fulfilled. We first prove (27). We have

$$\begin{aligned}|(Kf)(y)|v^{\rho,\theta}(y) &= \int_{-1}^{1} |v^{\rho,\theta}(y)h_x(y)||(fv^{\rho,\theta})(x)|v^{\alpha-\rho,\beta-\theta}(x)dx \\ &\leq \|f\|_{C_{v^{\rho,\theta}}} \sup_{|x|\leq 1}\|h_x\|_{C_{v^{\rho,\theta}}} \int_{-1}^{1} v^{\alpha-\rho,\beta-\theta}(x)dx.\end{aligned}$$

Then under the assumptions (33) and (34), (27) follows.

Now we prove (28). We first note that

$$|(K_m f)(y)|v^{\rho,\theta}(y) \le \|f\|_{C_{v^{\rho,\theta}}} v^{\rho,\theta}(y) \sum_{k=1}^{m} \lambda_k^{\alpha,\beta} \frac{|h(x_k^{\alpha,\beta}, y)|}{v^{\rho,\theta}(x_k^{\alpha,\beta})}$$

$$\le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}} v^{\rho,\theta}(y)\|h_y\|_\infty \sum_{k=1}^{m} \frac{\lambda_k^{\alpha,\beta}}{v^{\rho,\theta}(x_k^{\alpha,\beta})}.$$

Taking into account that $\lambda_k^{\alpha,\beta} \sim \Delta x_k^{\alpha,\beta} v^{\alpha,\beta}(x_k^{\alpha,\beta})$, $k = 1, \dots, m$, with $\Delta x_k^{\alpha,\beta} = x_{k+1}^{\alpha,\beta} - x_k^{\alpha,\beta} (x_{m+1}^{\alpha,\beta} = 1)$, and that $1 \pm x_k^{\alpha,\beta} \sim 1 \pm x$, $\forall x \in [x_k^{\alpha,\beta}, x_{k+1}^{\alpha,\beta}]$ (see [**50**]), we get

(38) $$\sum_{k=1}^{m} \frac{\lambda_k^{\alpha,\beta}}{v^{\rho,\theta}(x_k^{\alpha,\beta})} \le \mathcal{C} \int_{-1}^{1} v^{\alpha-\rho,\beta-\theta}(x)dx.$$

Thus we have

$$\|K_m f\|_{C_{v^{\rho,\theta}}} \le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}} \sup_{|y|\le 1} v^{\rho,\theta}(y)\|h_y\|_\infty \int_{-1}^{1} v^{\alpha-\rho,\beta-\theta}(x)dx$$

and hypotheses (33) and (35) imply (28).

By standard computations it is possible to deduce

$$|[(K - K_m)f](y)|v^{\rho,\theta}(y) \le \mathcal{C} v^{\rho,\theta}(y) E_{2m-2}(h_y f)_{v^{\rho,\theta}}$$

$$\le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}} v^{\rho,\theta}(y) E_{m-1}(h_y)_{v^{0,0}}$$

$$+ 2v^{\rho,\theta}(y)\|h_y\|_\infty E_{m-1}(f)_{v^{\rho,\theta}}.$$

Then, using (35) together with (3), we get

(39) $$\|(K - K_m)f\|_{C_{v^{\rho,\theta}}} \le \frac{\mathcal{C}}{m^r}\|f\|_{C_{v^{\rho,\theta}}} + \mathcal{C}E_{m-1}(f)_{v^{\rho,\theta}}$$

and, for all $f \in C_{v^{\rho,\theta}}$, (29) follows.

In order to prove (30), we replace f by $K_m f$ into (39). Thus we have

$$\|(K - K_m)K_m f\|_{C_{v^{\rho,\theta}}} \le \frac{\mathcal{C}}{m^r}\|K_m f\|_{C_{v^{\rho,\theta}}} + \mathcal{C}E_{m-1}(K_m f)_{v^{\rho,\theta}}$$

and, taking into account (28), it remains only to estimate $E_{m-1}(K_m f)_{v^{\rho,\theta}}$. But

$$|\Delta_{h\varphi}^s (K_m f)(y)|v^{\rho,\theta}(y) \le \|f\|_{C_{v^{\rho,\theta}}} \sum_{k=1}^{m} |v^{\rho,\theta}(y)\Delta_{h\varphi}^s h_{x_k^{\alpha,\beta}}(y)| \frac{\lambda_k^{\alpha,\beta}}{v^{\rho,\theta}(x_k^{\alpha,\beta})}$$

and, taking the supremum on $y \in [-1 + 4s^2h^2, 1 - 4s^2h^2]$ first and then the supremum on $0 < h \le t$, we get

$$\Omega_\varphi^s(K_m f, t)_{v^{\rho,\theta}} \le \|f\|_{C_{v^{\rho,\theta}}} \sum_{k=1}^{m} \Omega_\varphi^s(h_{x_k^{\alpha,\beta}}, t)_{v^{\rho,\theta}} \frac{\lambda_k^{\alpha,\beta}}{v^{\rho,\theta}(x_k^{\alpha,\beta})}.$$

Moreover, using (33), (34) and (38), we deduce

$$\Omega_\varphi^s(K_m f, t)_{v^{\rho,\theta}} \le \mathcal{C} t^r \|f\|_{C_{v^{\rho,\theta}}}$$

and then

$$E_m(K_m f)_{v^{\rho,\theta}} \le \frac{\mathcal{C}}{m^r} \|f\|_{C_{v^{\rho,\theta}}},$$

recalling (3). The proof of (30) is now complete.

Since all the hyphoteses of Theorem 5.1 are fulfilled, (32) holds true and we can use (31) to prove (37). We first note that $Kf \in Z_r(v^{\rho,\theta})$ for all $f \in C_{v^{\rho,\theta}}$. In fact, we have

$$|\Delta_{h\varphi}^s(Kf)(y)| v^{\rho,\theta}(y) = v^{\rho,\theta}(y) \left| \int_{-1}^{1} (\Delta_{h\varphi}^s h_x)(y)(f v^{\rho,\theta})(x) v^{\alpha-\rho,\beta-\theta}(x) dx \right|$$

$$\le \|f\|_{C_{v^{\rho,\theta}}} \int_{-1}^{1} v^{\rho,\theta}(y) |(\Delta_{h\varphi}^s h_x)(y)| v^{\alpha-\rho,\beta-\theta}(x) dx.$$

and, taking the supremum on $y \in [-1 + 4s^2h^2, 1 - 4s^2h^2]$ first and then the supremum on $0 < h \le t$, we get

$$\Omega_\varphi^s(Kf, t)_{v^{\rho,\theta}} \le \|f\|_{C_{v^{\rho,\theta}}} \int_{-1}^{1} \Omega_\varphi^s(h_x, t)_{v^{\rho,\theta}} \; v^{\alpha-\rho,\beta-\theta}(x) dx$$

$$\le \mathcal{C} t^r \|f\|_{C_{v^{\rho,\theta}}} \sup_{|x|\le 1} \|h_x\|_{Z_r(v^{\rho,\theta})}$$

$$\le \mathcal{C} t^r \|f\|_{C_{v^{\rho,\theta}}},$$

by (33) and (34), i.e. $Kf \in Z_r(v^{\rho,\theta})$, for all $f \in C_{v^{\rho,\theta}}$. Then, under the assumption (36), the unique solution f of equation (23) belongs to $Z_r(v^{\rho,\theta})$, too. Consequently (37) easily follows from (31), (39) and (3). □

In the sequel we will give applications of Corollary 5.1.

5.2 - *CSIEs with index* 0

Consider the equation

$$(A^{\alpha,-\alpha}f)(y) + \int_{-1}^{1} k(x,y)f(x)v^{\alpha,-\alpha}(x)dx = g(y)\,, \tag{40}$$

where $A^{\alpha,-\alpha}$ is defined in (2) with $0 < \alpha < 1$.

Assuming that $g \in Z_r(v^{0,\alpha})$ and $k_x \in Z_r(v^{0,\alpha})$, $r \in \mathbb{R}^+$, uniformly with respect to x, we can obtain an equivalent Fredholm equation multiplying (40) from the left by the operator $\widehat{A}^{\alpha,-\alpha}$ defined in (5).

Taking into account that (see Section 3) $\widehat{A}^{\alpha,-\alpha}A^{\alpha,-\alpha}F = F$, $\forall F \in Z_r(v^{\alpha,0})$, under the assumptions on g and k_x we get

$$f(y) + \int_{-1}^{1} (\widehat{A}^{\alpha,-\alpha}k_x)(y)f(x)v^{\alpha,-\alpha}(x)dx = (\widehat{A}^{\alpha,-\alpha}g)(y). \tag{41}$$

Moreover, setting

$$\Psi(x,y) = (\widehat{A}^{\alpha,-\alpha}k_x)(y), \quad G(y) = (\widehat{A}^{\alpha,-\alpha}g)(y)\,,$$

we rewrite (41) as follows

$$f(y) + \int_{-1}^{1} \Psi(x,y)f(x)v^{\alpha,-\alpha}(x)dx = G(y). \tag{42}$$

We consider (42) in $C_{v^{\alpha+\gamma,\delta}}$ with $(\alpha+\gamma)$ and δ satisfying (33) (with $\beta = -\alpha$), i.e., with γ and δ s.t.

$$0 \le \gamma < 1, \qquad 0 \le \delta < 1 - \alpha. \tag{43}$$

Applying the above described Nyström method to (42), we get the system

$$\sum_{k=1}^{m} \left[\delta_{i,k} + \lambda_k^{\alpha,-\alpha} \frac{v^{\alpha+\gamma,\delta}(x_i)}{v^{\alpha+\gamma,\delta}(x_k)} \Psi(x_k,x_i)\right] a_k = G(x_i)v^{\alpha+\gamma,\delta}(x_i), \quad i = 1,2,\dots,m, \tag{44}$$

where $x_k = x_{m,k}^{\alpha,-\alpha}$, $k = 1,\dots,m$, are the zeros of $p_m^{\alpha,-\alpha}$ and $\lambda_k^{\alpha,-\alpha} = \lambda_{m,k}^{\alpha,-\alpha}$, $k = 1,\dots,m$, are the corresponding Christoffel numbers, and we construct the weighted Nyström interpolating function

$$v^{\alpha+\gamma,\delta}(y)f_m(y) = v^{\alpha+\gamma,\delta}(y)\left[G(y) - \sum_{k=1}^{m} \frac{\lambda_k^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(x_k)} \Psi(x_k,y)a_k\right]. \tag{45}$$

If Ψ and G verify in $C_{v^{\alpha+\gamma,\delta}}$ the assumptions (34)-(35) and (36), respectively, then we can apply Corollary 5.1. As a consequence of the equivalence [**40**]

$$\|g\|_{Z_r(v^{0,\alpha})} \sim \|\widehat{A}^{\alpha,-\alpha} g\|_{Z_r(v^{\alpha,0})} ,$$

in [**16**] the following lemma was proved.

Lemma 5.1. *If* $0<\alpha<1$ *and* $\gamma,\delta\geq 0$ *then we have*

$$\|G\|_{Z_r(v^{\alpha+\gamma,\delta})} \leq \mathcal{C}\|g\|_{Z_r(v^{0,\alpha})}, \tag{46}$$

$$\sup_{|x|\leq 1}\|\Psi_x\|_{Z_r(v^{\alpha+\gamma,\delta})} \leq \mathcal{C}\sup_{|x|\leq 1}\|k_x\|_{Z_r(v^{0,\alpha})}, \tag{47}$$

$$\sup_{|y|\leq 1} v^{\alpha+\gamma,\delta}(y)\|\Psi_y\|_{Z_r} \leq \mathcal{C}\left[\sup_{|y|\leq 1} v^{\gamma,\delta+\alpha}(y)\|k_y\|_{Z_r} + \sup_{|y|\leq 1} v^{\gamma,\delta+\alpha}(y)\left\|\frac{\partial}{\partial y}k_y\right\|_{Z_r}\right], \tag{48}$$

with $r>0$, $\Psi_x(y)=\Psi(x,y)=\Psi_y(x)$ *and* $\mathcal{C}\neq\mathcal{C}(\Psi,G,x,y)$.

Thus, if the right-hand sides of (46)-(48) are finite, then the functions Ψ and G of (42) satisfy in $C_{v^{\alpha+\gamma,\delta}}$ the assumptions (34)-(35) and (36) of Corollary 5.1, respectively, and we can deduce the following proposition

Proposition 5.1. *Assume that the original equation* (40) *is uniquely solvable in* $C_{v^{\alpha+\gamma,\delta}}$ *and that the kernel* k *and the known term* g *of* (40) *satisfy*

$$\|g\|_{Z_r(v^{0,\alpha})} < +\infty,$$

$$\sup_{|x|\leq 1}\|k_x\|_{Z_r(v^{0,\alpha})} < +\infty,$$

$$\sup_{|y|\leq 1} v^{\gamma,\delta+\alpha}(y)\|k_y\|_{Z_r} + \sup_{|y|\leq 1} v^{\gamma,\delta+\alpha}(y)\left\|\frac{\partial}{\partial y}k_y\right\|_{Z_r} < +\infty,$$

with γ,δ *according to* (43). *Then, for a* m *sufficiently large (say* $m>m_0$*), system* (44) *admits* $(a_1,\ldots,a_m)$ *as unique solution and the condition number in uniform norm of its matrix of coefficients* A_m *satisfies*

$$\sup_m \mathrm{cond}(A_m) < +\infty.$$

Moreover, the corresponding Nyström interpolating function f_m *defined by* (45) *converges to the exact solution f and*

$$\|f - f_m\|_{C_{v^{\alpha+\gamma,\delta}}} \le \frac{\mathcal{C}}{m^r} \|f\|_{Z_r(v^{\alpha+\gamma,\delta})},$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Of course to solve system (44) it is necessary to compute the quantities

$$\Psi(x_k, x_i) = a\ k(x_k, x_i) v^{-\alpha,\alpha}(x_i) - \frac{b}{\pi} \int_{-1}^{1} \frac{k(x_k, t)}{t - x_i} v^{-\alpha,\alpha}(t) dt$$

and

$$G(x_i) = a\ g(x_i) v^{-\alpha,\alpha}(x_i) - \frac{b}{\pi} \int_{-1}^{1} \frac{g(t)}{t - x_i} v^{-\alpha,\alpha}(t) dt.$$

We can see that the only difficulty consists in the computation of the Hilbert transforms. When their analytical expressions are not available, the last ones can be computed using one of the several methods available in literature, which are based on Gaussian rules, on product rules [**9, 10, 11, 12, 45**] or on suitable transformation of the integrand [**4, 32, 36, 47, 51, 55**].

Here, for completeness, we propose to replace $\Psi(x_k, x_i)$ and $G(x_i)$, $i, k = 1, \ldots, m$, by

$$\Psi_m(x_k, x_i) = (\widehat{A}^{\alpha,-\alpha} L_m^{-\alpha,\alpha} k_{x_k})(x_i) \tag{49}$$

and

$$G_m(x_i) = (\widehat{A}^{\alpha,-\alpha} L_m^{-\alpha,\alpha} g)(x_i), \tag{50}$$

respectively, where $L_m^{-\alpha,\alpha}$ denotes the Lagrange interpolation operator based on the zeros $t_j = t_{m,j}^{-\alpha,\alpha}$, $j = 1, \ldots, m$, of the polynomial $p_m^{-\alpha,\alpha}$. Moreover, in the sequel, $\lambda_j^{-\alpha,\alpha} = \lambda_{m,j}^{-\alpha,\alpha}$, $j = 1, \ldots, m$, are the corresponding Christoffel numbers.

Now, taking into account that, in virtue of (6),

$$\left[\widehat{A}^{\alpha,-\alpha}\left(\frac{p_m^{-\alpha,\alpha}}{\cdot - t_j}\right)\right](y) = -\frac{b}{\sin \pi\alpha} \frac{p_m^{\alpha,-\alpha}(y) - p_m^{\alpha,-\alpha}(t_j)}{y - t_j}$$

and

$$p_m^{\alpha,-\alpha}(t_j) = \frac{\sin \pi\alpha}{\pi} (p_m^{-\alpha,\alpha})'(t_j) \lambda_j^{-\alpha,\alpha}$$

hold true, we obtain

$$(\widehat{A}^{\alpha,-\alpha} l_{m,j}^{-\alpha,\alpha})(x_i) = \frac{b}{\pi} \frac{\lambda_j^{-\alpha,\alpha}}{x_i - t_j}.$$

Consequently, (49) and (50) become

$$\Psi_m(x_k, x_i) = \frac{b}{\pi}\sum_{j=1}^{m} \frac{k(x_k, t_j)}{x_i - t_j}\lambda_j^{-\alpha,\alpha}$$

and

$$G_m(x_i) = \frac{b}{\pi}\sum_{j=1}^{m} \frac{g(t_j)}{x_i - t_j}\lambda_j^{-\alpha,\alpha}.$$

Then, we replace system (44) by

$$\sum_{k=1}^{m}\left[\delta_{i,k} + \lambda_k^{\alpha,-\alpha}\frac{v^{\alpha+\gamma,\delta}(x_i)}{v^{\alpha+\gamma,\delta}(x_k)}\Psi_m(x_k, x_i)\right]\bar{a}_k = G_m(x_i)v^{\alpha+\gamma,\delta}(x_i),\ i = 1, \dots, m, \tag{51}$$

and we construct the following approximation of the weighted Nyström interpolating function $v^{\alpha+\gamma,\delta}f_m$:

$$v^{\alpha+\gamma,\delta}(y)f_m^*(y) = v^{\alpha+\gamma,\delta}(y)\left[G_m(y) - \sum_{k=1}^{m}\frac{\lambda_k^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(x_k)}\Psi_m(x_k, y)\bar{a}_k\right], \tag{52}$$

where

$$\Psi_m(x_k, y) = (\widehat{A}^{\alpha,-\alpha}L_m^{-\alpha,\alpha}k_{x_k})(y)$$

and

$$G_m(y) = (\widehat{A}^{\alpha,-\alpha}L_m^{-\alpha,\alpha}g)(y).$$

Using (6) again, we can write

$$\Psi_m(x_k, y) = -\frac{b}{\sin \pi\alpha}\sum_{j=1}^{m} k(x_k, t_j)\lambda_j^{-\alpha,\alpha}\sum_{i=0}^{m-1} p_i^{-\alpha,\alpha}(t_j)p_i^{\alpha,-\alpha}(y)$$

and

$$G_m(y) = -\frac{b}{\sin \pi\alpha}\sum_{j=1}^{m} g(t_j)\lambda_j^{-\alpha,\alpha}\sum_{i=0}^{m-1} p_i^{-\alpha,\alpha}(t_j)p_i^{\alpha,-\alpha}(y).$$

Note that the polynomial solution f_m^* defined in (52) coincides with the solution f_m obtained applying the discrete collocation method (10), (14) and with the approximate solution f_m^{**} of the projection method proposed in [**16**, pp. 1356-1359].

It is possible to prove (see [**16**]) that the matrix of the coefficients of system (51) is invertible and well-conditioned and that

$$\|f - f_m^*\|_{C_{v^{\alpha+\gamma,\delta}}} = \mathcal{O}\left(\frac{\log^2 m}{m^r}\right).$$

Thus, the proposed approximations introduce an additional $\log^2 m$ factor in the error.

5.3 - *CSIEs with index* 1

With respect to the equation

$$(A^{-\alpha,\alpha-1}f)(y) + \int_{-1}^{1} k(x,y)f(x)v^{-\alpha,\alpha-1}(x)dx = g(y)\,, \tag{53}$$

where $A^{-\alpha,\alpha-1}$ is defined in (2) with $0 < \alpha < 1$, we assume $g \in Z_r(v^{\alpha,1-\alpha})$ and $k_x \in Z_r(v^{\alpha,1-\alpha})$, $r > 0$, uniformly with respect to x. Multiplying (53) from the left by the operator $\widehat{A}^{-\alpha,\alpha-1}$ defined in (5) and taking into account (7), (53) becomes

$$f(y) + \int_{-1}^{1} (\widehat{A}^{-\alpha,\alpha-1}k_x)(y)f(x)v^{-\alpha,\alpha-1}(x)dx = (\widehat{A}^{-\alpha,\alpha-1}g)(y) + c\,, \tag{54}$$

being

$$c = \frac{\int_{-1}^{1} f(x)v^{-\alpha,\alpha-1}(x)dx}{\int_{-1}^{1} v^{-\alpha,\alpha-1}(x)dx} \in \mathbb{R}\,. \tag{55}$$

Equation (53) cannot be uniquely solvable, since the index of the operator in the spaces under consideration is equal to 1. However (53) together with the additional condition (55), for a given constant c, is equivalent to (54). Letting

$$\Gamma(x,y) = (\widehat{A}^{-\alpha,\alpha-1}k_x)(y) \quad \text{and} \quad G_1(y) = (\widehat{A}^{-\alpha,\alpha-1}g)(y)\,,$$

and assuming $c = 0$, (54) can be rewritten as

$$f(y) + \int_{-1}^{1} \Gamma(x,y)f(x)v^{-\alpha,\alpha-1}(x)dx = G_1(y). \tag{56}$$

We consider (56) in $C_{v^{\rho,\theta}}$ and we choose ρ, θ replacing α by $-\alpha$ and β by $\alpha - 1$ in (33). Thus, we take

$$0 \le \rho < 1 - \alpha, \qquad 0 \le \theta < \alpha. \tag{57}$$

Proceeding as in the previous case, we solve the system

$$\sum_{k=1}^{m}\left[\delta_{i,k}+\ \lambda_k^{-\alpha,\alpha-1}\frac{v^{\rho,\theta}(x_i)}{v^{\rho,\theta}(x_k)}\Gamma(x_k,x_i)\right]a_k=G_1(x_i)v^{\rho,\theta}(x_i),\quad i=1,2,\dots,m, \tag{58}$$

where $x_k=x_{m,k}^{-\alpha,\alpha-1}$, $k=1,\dots,m$, are the zeros of $p_m^{-\alpha,\alpha-1}$ and $\lambda_k^{-\alpha,\alpha-1}=\lambda_{m,k}^{-\alpha,\alpha-1}$, $k=1,\dots,m$, are the corresponding Christoffel numbers, and we construct the weighted Nyström interpolating function

$$v^{\rho,\theta}(y)f_m(y)=v^{\rho,\theta}(y)\left[G_1(y)-\sum_{k=1}^{m}\frac{\lambda_k^{-\alpha,\alpha-1}}{v^{\rho,\theta}(x_k)}\Gamma(x_k,y)a_k\right]. \tag{59}$$

Taking into account the equivalence [**15**]

$$\|g\|_{Z_r(v^{\alpha,1-\alpha})}\sim\|\widehat{A}^{-\alpha,\alpha-1}g\|_{Z_r},$$

as in case $\chi=0$, we can deduce the next proposition

Proposition 5.2. *Assume that the original equation* (53) *is uniquely solvable in $C_{v^{\rho,\theta}}$ with ρ,θ satisfying* (57). *If*

$$\|g\|_{Z_r(v^{\alpha,1-\alpha})}<+\infty,$$

$$\sup_{|x|\le1}\|k_x\|_{Z_r(v^{\alpha,1-\alpha})}<+\infty,$$

$$\sup_{|y|\le1}v^{\alpha+\rho,1-\alpha+\theta}(y)\|k_y\|_{Z_r}+\sup_{|y|\le1}v^{\alpha+\rho,1-\alpha+\theta}(y)\left\|\frac{\partial}{\partial y}k_y\right\|_{Z_r}<+\infty,$$

then, for a sufficiently large m (say $m>m_0$), the matrix A_m of the coefficients of system (58) *is invertible and its condition number in uniform norm satisfies*

$$\sup_m \operatorname{cond}(A_m)<+\infty.$$

Moreover, the corresponding Nyström interpolating function f_m defined by (59) *converges to the unique solution f of* (53) *with $c=0$ and we have*

$$\|f-f_m\|_{C_{v^{\rho,\theta}}}\le\frac{\mathcal{C}}{m^r}\|f\|_{Z_r(v^{\rho,\theta})},$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$.

Proceeding as above, system (58) can be replaced by

$$\sum_{k=1}^{m}\left[\delta_{i,k}+\lambda_k^{-\alpha,\alpha-1}\frac{v^{\rho,\theta}(x_i)}{v^{\rho,\theta}(x_k)}\Gamma_{m-1}(x_k,x_i)\right]\bar{a}_k=G_{1,m-1}(x_i)v^{\rho,\theta}(x_i), \tag{60}$$

$$i=1,\dots,m,$$

where

$$\Gamma_{m-1}(x_k, x_i) = \frac{b}{\pi} \sum_{j=1}^{m-1} \frac{k(x_k, t_j)}{x_i - t_j} \lambda_j^{\alpha,1-\alpha},$$

$$G_{1,m-1}(x_i) = \frac{b}{\pi} \sum_{j=1}^{m-1} \frac{g(t_j)}{x_i - t_j} \lambda_j^{\alpha,1-\alpha},$$

with $t_j = t_{m-1,j}^{\alpha,1-\alpha}$, $j = 1, \dots, m-1$, the zeros of the Jacobi polynomial $p_{m-1}^{\alpha,1-\alpha}$ and $\lambda_j^{\alpha,1-\alpha} = \lambda_{m-1,j}^{\alpha,1-\alpha}$, $j = 1, \dots, m-1$, the corresponding Christoffel numbers. Note that system (60) coincides with the linear system (3.38) in [**16**].

Moreover, we construct the following approximation of the weighted Nyström interpolating function in (59):

$$v^{\rho,\theta}(y) f_m^*(y) = v^{\rho,\theta}(y) \left[G_{1,m-1}(y) - \sum_{k=1}^{m} \frac{\lambda_k^{-\alpha,\alpha-1}}{v^{\gamma,\delta}(x_k)} \Gamma_{m-1}(x_k, y) \bar{a}_k \right], \tag{61}$$

where

$$\Gamma_{m-1}(x_k, y) = -\frac{b}{\sin \pi\alpha} \sum_{j=1}^{m-1} k(x_k, t_j) \lambda_j^{\alpha,1-\alpha} \sum_{i=0}^{m-2} p_i^{\alpha,1-\alpha}(t_j) p_{i+1}^{-\alpha,\alpha-1}(y)$$

and

$$G_{1,m-1}(y) = -\frac{b}{\sin \pi\alpha} \sum_{j=1}^{m-1} g(t_j) \lambda_j^{\alpha,1-\alpha} \sum_{i=0}^{m-2} p_i^{\alpha,1-\alpha}(t_j) p_{i+1}^{-\alpha,\alpha-1}(y).$$

Also in this case it is not hard to prove that the matrix of the coefficients of system (60) is invertible and well-conditioned. Moreover, the following estimate

$$\|f - f_m^*\|_{C_{v^{\rho,\theta}}} = \mathcal{O}\left(\frac{\log^2 m}{m^r} \right)$$

holds (see [**16**]).

6 - Numerical examples

In the following examples, we compare the numerical results obtained by using the above described methods.

For the discrete collocation (DCM) and quadrature methods (QM) we first solve systems (14) and (19), respectively, and then construct the approximate solution f_m using (10) and (17), respectively. For the Nyström method (NM), we construct the approximate solution f_m^* by (51) and (52) if the index χ of the equation is 0 and by (60) and (61) if the index χ of the equation is 1.

When we don't know the exact solutions of the integral equations, we will think as exact their approximate solutions obtained for $m = 512$ and in all the tables we will report only the digits which are correct according to them.

Moreover, in every table we will denote by cond_2 and cond_∞ the condition numbers in spectral norm and in uniform norm, respectively, related to the matrix of the solved system.

Example 6.1. *Consider the following integral equation of index* 0

$$\frac{\sqrt{2}}{2} f(y) v^{\frac{1}{4},-\frac{1}{4}}(y) - \frac{\sqrt{2}}{2\pi} \int_{-1}^{1} \frac{f(x)}{x-y} v^{\frac{1}{4},-\frac{1}{4}}(x) dx + \frac{\sqrt{2}}{2} \int_{-1}^{1} x^3 y^2 f(x) v^{\frac{1}{4},-\frac{1}{4}}(x) dx$$

$$= 1 - \frac{11\sqrt{2}}{1024} \pi y^2 \left(8\sqrt{2} - \cot\frac{\pi}{8} - \cot\frac{3\pi}{8} + \cot\frac{5\pi}{8} + \cot\frac{7\pi}{8} \right)$$

The exact solution is the function $f(x) = 1$.

All three methods for $m = 8$ *give an approximation of the exact solution with* 15 *exact decimal digits.*

Table 1. *Example* 6.1, $m = 8$.

	cond_2	cond_∞
DCM	1.402486506056423	8.858019540413084
QM	1.402486506056423	3.164816989158717
NM $\left(\gamma = \delta = \frac{7}{10}\right)$	2.181297493982877	3.180129408984790

Example 6.2. *The equation*

$$\cos\frac{\pi}{10} f(y) v^{\frac{1}{10},-\frac{1}{10}}(y) - \frac{\sin\frac{\pi}{10}}{\pi} \int_{-1}^{1} \frac{f(x)}{x-y} v^{\frac{1}{10},-\frac{1}{10}}(x) dx$$

$$+ \frac{1}{4} \int_{-1}^{1} \frac{|x-y|^{\frac{7}{2}}}{(5+x^2+y^2)^2} f(x) v^{\frac{1}{10},-\frac{1}{10}}(x) dx = \sin(1+y)$$

has index 0. *Its exact solution in unknown.*

In Table 2 we show, for increasing values of m, the values of the approximate solutions computed using all three methods (the Nyström method with $\gamma = \delta = \dfrac{3}{5}$) in the points $y = 0.1$ and $y = -0.8$.

Table 2. *Example* 6.2.

m	$y = 0.1$	$y = -0.8$
8	0.974870	0.332143
16	0.9748705	0.33214344
32	0.97487058	0.332143444
64	0.974870580	0.3321434446
128	0.9748705808	0.332143444657
256	0.97487058081	0.332143444657

Table 3. *Example* 6.2, $m = 256$.

	cond_2	cond_∞
DCM	1.035325516880034	229.6917914275608
QM	1.035325516880032	4.102618625748687
NM $\left(\gamma = \delta = \dfrac{3}{5}\right)$	1.057242525446276	1.098035226074213

Example 6.3. *The exact solution of the following integral equation of index* 1

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{x-y} v^{-\frac{1}{2},-\frac{1}{2}}(x)dx + \int_{-1}^{1} (xy^2 + y) f(x) v^{-\frac{1}{2},-\frac{1}{2}}(x)dx$$

$$= \frac{2}{\pi}\left(1 + \frac{y^2}{\sqrt{1-y^2}} \log \frac{\left|\sqrt{1-y^2} - y + 1\right|}{\left|\sqrt{1-y^2} + y - 1\right|}\right) + \frac{4}{3}\, y^2$$

is the function $f(y) = y|y|$.

All three methods for $m = 257$ give an approximation of the exact solution with 7 *exact decimal digits.*

Table 4. *Example* 6.3, $m = 257$.

	cond_2	cond_∞
DCM	2.887363876722051	226.6553606239541
QM	2.656207875403805	119.5858577072562
NM $\left(\rho = \theta = \frac{2}{5}\right)$	28.59091239949910	28.01088901364780

Example 6.4. *Consider the following integral equation of index* 1

$$\cos\frac{3\pi}{4} f(y) v^{-\frac{3}{4},-\frac{1}{4}}(y) + \frac{\sin\frac{3\pi}{4}}{\pi} \int_{-1}^{1} \frac{f(x)}{x-y} v^{-\frac{3}{4},-\frac{1}{4}}(x)dx$$

$$+\frac{1}{8}\int_{-1}^{1} |x-y|^{\frac{9}{2}} \log|x-y| f(x) v^{-\frac{3}{4},-\frac{1}{4}}(x)dx = (y^3+4)\arctan y$$

whose exact solution is unknown.

In Table 5 *we show, for increasing values of* m, *the values of the approximate solutions computed all three methods* (*the Nyström method with* $\rho = \frac{1}{5}$, $\theta = \frac{1}{2}$) *in the points* $y = 0.5$ *and* $y = -0.9$.

Table 5. *Example* 6.4.

m	$y = 0.5$	$y = -0.9$
8	−1.22	2.98
16	−1.2275062	2.98697
32	−1.22750621	2.986972374
64	−1.2275062149	2.98697237494
128	−1.22750621493	2.98697237494

Table 6. *Example* 6.4, $m = 128$.

	cond_2	cond_∞
DCM	2.243480177993310	117.9538660942301
QM	2.724403411295303	117.3450514994343
NM $\left(\rho = \frac{1}{5}, \theta = \frac{1}{2}\right)$	23.33370704403387	9.529793005924597

As you can see in tables 1, 3, 4 and 6, if the approximate solution is searched in a weighted L^2 space, then DCM and QM lead to the resolution of well conditioned linear systems, while if one needs to solve the CSIE in a weighted uniform space, the linear systems connected with the NM appear, in general, better conditioned.

In the following two examples we consider CSIEs having weakly singular kernels. In these cases only the collocation method is tested, therefore the approximate solution f_m is computed by solving system (11) and using (10). The integrals of type $Kp_k^{\alpha,\beta}(y)$, $k = 0, \ldots, m-1$, $y \in (-1, 1)$, are computed by using suitable recurrence relations (see, for example, [**39**]).

Example 6.5. *This example deals with the so called generalized airfoil equation*

$$-\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{x-y} v^{\frac{1}{2},-\frac{1}{2}}(x)dx + \int_{-1}^{1} \log|x-y| f(x) v^{\frac{1}{2},-\frac{1}{2}}(x)dx = ye^y$$

of index 0*, whose exact solution is unknown.*

Table 7. *Example* 6.5.

m	$f_m(0.4)$	$f_m(-0.8)$	cond_2
8	0.789	0.98	2.581412260211701
16	0.7890289492	0.9832512417	2.581438477458795
32	0.789028949270289	0.983251241767435	2.581438477458794

Example 6.6. *The equation*

$$\cos\frac{4\pi}{10} f(y) v^{-\frac{4}{10},-\frac{3}{5}}(y) + \frac{\sin\frac{4\pi}{10}}{\pi}\int_{-1}^{1} \frac{f(x)}{x-y} v^{-\frac{4}{10},-\frac{3}{5}}(x)dx$$
$$+ \int_{-1}^{1} |x-y|^{-\frac{1}{4}} f(x) v^{-\frac{4}{10},-\frac{3}{5}}(x)dx = e^{3y}$$

has index 1*. Its exact solution in unknown.*

Table 8. *Example* 6.6.

m	$f_m(0.2)$	$f_m(-0.4)$	cond_2
8	−1.00	−0.84	9.038635235280157
16	−1.00	−0.84	9.048980850979259
32	−1.007	−0.8430	9.049978487702072
64	−1.0079	−0.8430	9.050073679168758
128	−1.00793	−0.843080	9.050082631780384
256	−1.00793213	−0.8430803	9.050083462837785

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Abstract

In this paper the Authors make a short survey on the numerical treatment of CSIEs with constant coefficients on the interval $[-1,1]$. *They describe some direct methods but their attention is mainly focused on the indirect methods, in particular on the Nyström method.*

* * *