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Coincidences and fixed points of strictly contractive and Lipschitz maps (**)

1 - Strict contractive conditions and common fixed point theorems

Introduction.

Let (X, d) be a metric space and $A : X \to X$. As a significant generalization of the well-known Banach contraction principle, M. Edelstein [**15**] (see also [**3**], [**6**], [**45**], [**65**], [**78**], [**83**]) obtained the following theorem.

Theorem E. *A map* $A : X \to X$ *satisfying:*

(E.1) $$d(Ax, Ay) < d(x, y) \text{ for all distinct } x, y \text{ in } X,$$

has a unique fixed point provided that X *is compact.*

Theorem E for a self-map, was generalized in various ways by Achari [**2**], Bailey [**5**], Chang and Zhong [**7**], Chatterjee and Ray [**9**], Chen and Yeh [**10**], Chen and Shih [**11**], Ciric [**13**], Das [**14**], Fisher [**16**]-[**21**], Jain and Dixit [**28**], Janos [**29**], Khan [**41**], [**42**], Maiti and Ghosh [**46**], Sehgal [**67**], Singh [**77**], Tan and Minh [**79**], Wong [**81**], Yeh [**82**] and others. For an excellent survey of the basic development of contractive maps, one may refer to Rhoades [**62**] (see also [**44**], [**55**], [**63**], [**64**]).

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Papers proposing extensions and generalizations of Theorem E to two self-maps are amply numerous (see, for instance, [**4**], [**8**], [**22**], [**23**], [**30**]-[**33**], [**40**], [**49**], [**50**], [**61**], [**69**], [**70**]). The following fixed point theorem essentially due to Naimpally et al. [**49**] is obtained for a pair of commuting and not necessarily continuous maps on a compact metric space.

Theorem NSW. *Let A and S be commuting maps from a metric space X to itself such that:*

(NSW.1) $$A(X) \subset S(X); \text{ and}$$

(NSW.2) $$d(Ax, Ay) < d(Sx, Sy), Sx \neq Sy, x, y \in X.$$

If $S(X)$ is compact then A and S have a unique common fixed point.

We remark that similar results satisfying conditions (NSW.1) and (NSW.2), appear earlier in [**30**], [**31**] with the additional hypotheses of continuity of maps A and S and the requirement of compactness of the space in full strength.

Sessa [**68**] defined weak commutativity as a generalization of commuting maps. Jungck [**34**] further generalized weak commutativity by introducing the notion of compatibility (also called asymptotic commutativity cf. [**27**] and [**80**]). For an excellent comparison of various weaker forms of commuting maps one may refer to Singh and Tomar [**76**] (see also Murthy [**47**] and Pathak and Khan [**58**]).

In due course of time, several common fixed point theorems were obtained by weakening either the commutativity requirement or strict contractive conditions for three and four maps on compact spaces (see, for instance, [**24**]-[**27**], [**35**], [**39**], [**43**], [**48**], [**75**]).

Definition 1 [**68**]. *Self-maps A and S of a metric space (X, d) are weakly commuting at a point $x \in X$ whenever $d(ASx, SAx) \leq d(Ax, Sx)$. They are weakly commuting on X if they commute weakly at each point $x \in X$.*

Definition 2 [**34**]. *Self-maps A and S of a metric space (X, d) are compatible (also called asymptotically commuting) if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. They are compatible maps of type (A) [**36**] if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.*

Clearly, weakly commuting maps are compatible. However, compatible maps need not be weakly commuting (see [**76**], Ex. 2.2, p.147).

Notice that A and S will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent (see also [**1**], p. 183, [**52**] and [**53**], p. 328).

Definition 3 [**38**], (see also [**76**]). *Self-maps A and S of a metric space* (X, d) *are weakly compatible if they commute at their coincidence points, that is, if* $ASx = SAx$, *whenever* $Ax = Sx$ *for* $x \in X$.

We cite here the following result of Singh and Mishra [**75**], Cor. 4.2, obtained under very tight conditions.

Theorem SM. *Let* A, B, S *and* T *be continuous self-maps of a compact metric space* (X, d) *satisfying:*

(SM.1) $AX \subset TX$ *and* $BX \subset SX$;

(SM.2) *the pair* (A, S) *is compatible; and*

(SM.3) $d(Ax, By) < \max(Mxy)$, *when* $\max(Mxy) > 0$,

where $Mxy = \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}$.

Then:

(Ia) *A and S have a common fixed point;*

(Ib) *B and T have a coincidence;*

(Ic) A, B, S *and T have a common fixed point provided that B and T are weakly compatible.*

Theorem SM improves the result of Jungck [**35**], Th. 3.2, and Kang and Kim [**39**], Th. 3.4, by requiring weak compatibility [**38**] in lieu of compatibility of the pair (B, T).

The concept of compatible maps has proven useful in the context of metric fixed point theory. However, the study of noncompatible maps is also interesting, and Pant [**51**]-[**54**] and Aamri and Moutawakil [**1**] have recently started work along these lines.

Definition 4 [**51**]. *Self-maps A and S of a metric space* (X, d) *are R-weakly commuting at a point* $x \in X$ *if* $d(ASx, SAx) \le Rd(Ax, Sx)$ *for some* $R > 0$. *They are pointwise R-weakly commuting on X if given* $x \in X$ *there exists an* $R > 0$ *such that* $d(ASx, SAx) \le Rd(Ax, Sx)$.

We remark that weak commutativity, compatibility, compatibe maps of type (A), *R-weak commutativity, pointwise R-weak commutativity and weak compatibility* [**37**], [**38**] *are equivalent at their coincidences* (cf. [**72**], [**73**], [**76**]).

The following result is due to Pant and Pant [**53**] (see also [**54**]).

Theorem PP. *Let* (A, S) *and* (B, T) *be pointwise R-weakly commuting self-maps of* X *satisfying the conditions* (SM.1);

(PP.1) *one of the pairs* (A, S) *or* (B, T) *is noncompatible; and*

(PP.2) $d(Ax, By) < \max(mxy)$, *when* $\max(mxy) > 0$,

where $mxy = \{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2, [d(Ax, Ty) + d(By, Sx)]/2\}$, *for all* $x, y \in X$ *and* $1 \leq k < 2$. *Then* A, B, S *and* T *have a unique common fixed point provided the range of one of the maps is a complete subspace of* X.

We make use of the (EA) property, introduced by Aamri and Moutawakil [**1**], to obtain our results without using the continuity of the maps involved and completeness of the space. Our results improve several known results (see [**12**], [**36**], [**37**], [**56**]-[**60**], [**74**]).

Fixed Point Theorems. Throughout this paper, let Y be an arbitrary nonempty set, (X, d) a metric space and $C(A, S) = \{u : Au = Su\}$, the collection of coincidence points of A and S.

Definition 5 [**71**]. *Let* A *and* S *be maps on* Y *with values in* X. *Then* A *and* S *satisfy the* (EA) *property if there exists a sequence* $\{x_n\}$ *in* Y *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ *for some* $t \in X$.

If we take $Y = X$ then we get the definition of (EA) property (also called tangential maps by Sastry and Murthy [**66**]) for two self-maps of X studied by Aamri and Moutawakil [**1**].

The following is our first result for noncompatible maps.

Theorem 1.1. *Let* A *and* S *be noncompatible self-maps of a metric space* (X, d) *such that*

(1) $\overline{AX} \subset SX$;

(2) $d^2(Ax, Ay) < \max\{d^2(Sx, Sy) + d(Sx, Ax).d(Sy, Ay)$
$+a[d(Sx, Ax).d(Sy, Ax) + d(Sy, Ay).d(Sx, Ay)], d(Sx, Ay).d(Sy, Ax)\}, 1/2 \leq a < 1$,
when the right hand side of (2) *is non zero.*

Then $C(A, S)$ *is nonempty. Further,* A *and* S *have a unique common fixed point provided that* A *and* S *commute at (some)* $u \in C(A, S)$.

Proof. Since A and S are noncompatible, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent. Since $t \in \overline{AX}$ and $\overline{AX} \subset SX$, there exists a point $u \in X$ such that $t = Su$. Suppose $Au \neq Su$, then by (2),

$$d^2(Au, Ax_n) < \max\{d^2(Su, Sx_n) + d(Su, Au).d(Sx_n, Ax_n)$$
$$+a[d(Su, Au).d(Sx_n, Au) + d(Sx_n, Ax_n).d(Su, Ax_n)], d(Su, Ax_n).d(Sx_n, Au)\}.$$

Making $n \to \infty$ yields $d^2(Au, Su) \leq ad^2(Au, Su) < d^2(Au, Su)$, and $Au = Su$.

Consequently $C(A, S)$ is nonempty.

Further, the commutativity of A and S at u implies $AAu = ASu = SAu = SSu$, and by (2),

$$d^2(Au, AAu) < \max\{d^2(Su, SAu) + d(Su, Au).d(SAu, AAu)$$
$$+a[d(Su, Au).d(SAu, Au) + d(SAu, AAu).d(Su, AAu)], d(Su, AAu).d(SAu, Au)\}$$
$$= d^2(Au, AAu).$$

So Au is a common fixed point of A and S. The uniqueness of the common fixed point follows easily.

In view of the above proof, we have another version of Theorem 1.1.

Theorem 1.1 Bis. *Let $A, S : X \to X$ be such that conditions* (1) *and* (2) *hold. If maps A and S satisfy the* (EA) *property then the conclusions of Theorem* 1.1 *are true.*

Recently, Pant and Pant [**53**] (see also [**54**]) obtained a common fixed point theorem for a pair of noncompatible, pointwise R-weakly commuting self-maps of a metric space X satisfying the condition:

(P*) $$d(Ax, Ay) < \max\{d(Sx, Sy), k[d(Ax, Sx) + d(Ay, Sy)]/2, [d(Ax, Sy) +d(Ay, Sx)]/2\},$$

where $1 \leq k < 2$.

The following example shows that the maps A and S satisfy the condition (2) (cf. Theorem 1.1) but not (P*).

Example 1.2. *Let $X = [2, 20]$ be endowed with the usual metric and $Ax = 2$ if $x = 2$ or $x > 5$, $Ax = 5$ if $2 < x \leq 4$, $Ax = 10$ if $4 < x \leq 5$, and $S2 = 2$, $Sx = 8$ if $2 < x \leq 4$, $Sx = 12$ if $4 < x \leq 5$, $Sx = (x + 1)/3$ if $5 < x < 10$, $Sx = 5$ if $10 \leq x < 15$, $Sx = x - 5$ if $x \geq 15$.*

Then A and S have a unique common fixed point $x = 2$. *We consider the sequence* $\{x_n = 5 + 1/n : n \geq 1\}$ *to see that the maps A and S are noncompatible. Also,* $AX = \{2, 5, 10\}, SX = [2, 11/3) \cup \{5, 8, 12\} \cup [10, 15]$, *and* $A\overline{X} \subset SX$. *Further, A and S satisfy the condition* (2) *(cf. Theorem* 1.1*). On the other hand, the condition* (P*) *is not satisfied for* $x \in (2, 4], y \in (4, 5]$. *Notice that A and S are discontinuous (even) at* $x = 2$.

Now we present a fixed point theorem for a quadruplet of maps on an arbitrary set with values in a metric space which generalizes, among others, the results of Jungck et al. [**36**], Jungck and Rhoades [**37**], Pathak [**56**], Pathak et al. [**57**], Pathak and Khan [**58**], Popa [**59**] and Prasad [**60**].

Theorem 1.3. *Let X be a metric space and* $A, B, S, T : Y \to X$ *such that*

(3) $\overline{AY} \subset TY$ *and* $\overline{BY} \subset SY$;

(4) *one of the pairs* (A, S) *or* (B, T) *satisfies the* (EA) *property;*

(5) $d^2(Ax, By) < \max\{d^2(Sx, Ty) + d(Sx, Ax).d(Ty, By), d(Sx, By).d(Ty, Ax), a[d(Sx, By).d(Ty, By) + d(Sx, Ax).d(Ty, Ax)]\}, 1/2 \leq a < 1$, *when the right hand side of* (5) *is non-zero.*

Then $C(A, S)$ *and* $C(B, T)$ *are nonempty. Further, if* $Y = X$, *then*

(I) *A and S have a common fixed point provided that A and S commute at (some)* $u \in C(A, S)$*;*

(II) *B and T have a common fixed point provided that B and T commute at (some)* $w \in C(B, T)$;

(III) A, B, S, *and T have a unique common fixed point provided that* (I) *and* (II) *are true.*

Proof. If the pair (B, T) satisfies the (EA) property, then there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Since $\overline{BY} \subset SY$, for each x_n, there exists y_n in Y such that $Bx_n = Sy_n$, and $Sy_n \to t$ as well. We show that $Ay_n \to t$. If not, there exist a subsequence $\{Ay_m\}$ of $\{Ay_n\}$, a positive integer M, and a real number $r > 0$ such that for some positive integer $m \geq M$, we have $d(Ay_m, t) \geq r, d(Ay_m, Bx_m) \geq r$, and by (5),

$$\begin{aligned} d^2(Ay_m, Bx_m) &< \max\{d^2(Sy_m, Tx_m) + d(Sy_m, Ay_m).d(Tx_m, Bx_m), \\ &\quad d(Sy_m, Bx_m).d(Tx_m, Ay_m), \\ &\quad a[d(Sy_m, Bx_m).d(Tx_m, Bx_m) + d(Sy_m, Ay_m).d(Tx_m, Ay_m)]\} \\ &= ad^2(Ay_m, Bx_m) < d^2(Ay_m, Bx_m), \end{aligned}$$

a contradiction, and $Ay_n \to t$.

Since $t \in \overline{BY}$ and $\overline{BY} \subset SY$, there exists an element $u \in Y$ such that $t = Su$. To show that $Au = Su$, we suppose otherwise and use the condition (5) to get

$$d^2(Au, Bx_n) < \max\{d^2(Su, Tx_n)$$
$$+d(Su, Au).d(Tx_n, Bx_n), d(Su, Bx_n).d(Tx_n, Au),$$
$$a[d(Su, Bx_n).d(Tx_n, Bx_n) + d(Su, Au).d(Tx_n, Au)]\}.$$

Making $n \to \infty, d^2(Au, Su) \leq ad^2(Au, Su)$, yielding $Au = Su$. This proves that $C(A, S)$ is nonempty.

Since $\overline{AY} \subset TY$, there exists an element $w \in Y$ such that $Au = Tw$. If $Tw \neq Bw$, then by (5),

$$d^2(Au, Bw) < \max\{d^2(Su, Tw) + d(Su, Au).d(Tw, Bw), d(Su, Bw).d(Tw, Au),$$
$$a[d(Su, Bw).d(Tw, Bw) + d(Su, Au).d(Tw, Au)]\} = d^2(Au, Bw).$$

Consequently $Tw = Au = Bw$, and $C(B, T)$ is nonempty.

Now let $Y = X$.

If A and S commute at their coincidence point u, then $AAu = ASu = SAu = SSu$, and by (5),

$$d^2(Au, AAu) < \max\{d^2(SAu, Tw)$$
$$+d(SAu, AAu).d(Tw, Bw), d(SAu, Bw).d(Tw, AAu),$$
$$a[d(SAu, Bw).d(Tw, Bw) + d(SAu, AAu).d(Tw, AAu)]\} = d^2(Au, AAu).$$

This proves (I). The proof of (II) is analogous, and the proof of (III) is immediate.

To appreciate the generality of Theorem 1.3 consider the following result of Jungck et al. [**36**] obtained for the pairs of compatible maps of type (A) on a complete metric space wherein $\phi : (R^+)^5 \to R^+$ is an upper semi-continuous and non-decreasing function.

Theorem JMC. *Let (X, d) be a complete metric space and A, B, S and T be self-maps of X satisfying the conditions* (SM.1) *and the following:*

(JMC.2) *the pairs (A, S) and (B, T) are compatible of type* (A); *and*

(JMC.3) $d^2(Ax, By) \leq \phi(M(x, y))$ for all $x, y \in X$ *where* $M(x, y) = \{d^2(Sx, Ty), d(Sx, Ax).d(Ty, By), d(Sx, By).d(Ty, Ax), d(Sx, Ax).d(Ty, Ax), d(Sx, By).d(Ty, By)\}$.

Then A, B, S and T have a unique common fixed point provided that one of A, B, S or T is continuous.

The following example establishes the superiority of Theorem 1.3 over the Theorem PP and Theorem JMC.

Example 1.4. *Let* $X = [2, 20]$ *be endowed with the usual metric and*
$A2 = 2, Ax = 3$ *if* $2 < x \leq 4$ *or* $x > 15, Ax = 12$ *if* $4 < x \leq 15$,
$Bx = 2$ *if* $x = 2$ *or* $x > 15, Bx = 6$ *if* $2 < x \leq 5, Bx = 14$ *if* $5 < x \leq 15$,
$S2 = 2, Sx = 6$ *if* $2 < x \leq 4$ *or* $x > 15, Sx = 15$ *if* $4 < x \leq 5, Sx = 14$ *if* $5 < x \leq 15$
and $T2 = 2, Tx = 12$ *if* $2 < x \leq 5, Tx = 17$ *if* $5 < x \leq 15, Tx = x - 13$ *if* $15 < x \leq 17, Tx = 3$ *if* $x > 17$.
Then A, B, S *and* T *have a unique common fixed point* $x = 2$ *and all the conditions of Theorem* 1.3 *are satisfied. To see that the conditions* (PP.2) *and* (JMC.3) *of Theorem* PP *and Theorem* JMC *respectively are not satisfied; for example, consider* $x \in (2, 4]$, $y \in (5, 15]$. *Notice that all the maps are discontinuous* (*even*) *at* $x = 2$.

In case $S = T$ in Theorem 1.3, we obtain a slightly improved version which we state below.

Theorem 1.4. *Let* X *be a metric space and* $A, B, S : X \to X$ *such that* (5) *with* $S = T$, *and*
(6) $\overline{AY} \cup \overline{BY} \subset SY$;
(7) *one of the pairs* (A, S) *or* (B, S) *satisfies the* (EA) *property.*

Then A, B and S have a coincidence. Further, if S commutes with each of A and B at their coincidences, then A, B and S have a unique common fixed point.

2 - Strictly contractive Lipschitz type maps and common fixed point theorems

This section is devoted to some coincidence and fixed point theorems for the maps satisfying strictly contractive Lipschitz type conditions. Our results, obtained without continuity of the maps and completeness of the space, generalize the results of Pant [**52**] and Singh and Kumar [**71**].

The following is our first result of this section for noncompatible maps.

Theorem 2.1. *Let* A *and* S *be noncompatible self-maps of a metric space* (X, d) *such that*
(i) $\overline{AX} \subset SX$;
(ii) $$d^2(Ax, Ay) < k\{d^2(Sx, Sy) + d(Sx, Ax).d(Sy, Ay) + d(Sx, Ay).d(Sy, Ax) + ad(Sx, Ax).d(Sy, Ax) + d(Sy, Ay).d(Sx, Ay)\},$$

for all $x, y \in X$, when the right hand side is non-zero, $k \geq 0$ and $a \geq 0$ is chosen such that $ka < 1$. Then $C(A,S)$ is nonempty. Further, A and S have a common fixed point provided that A and S commute at (some) $u \in C(A,S)$ and one of the following holds:

(iii) $d(Ax, A^2x) \neq \max\{d(Sx,SAx), d(Ax,Sx), d(A^2x,SAx), d(Ax,SAx), d(Sx,A^2x)\}$ *whenever the right-hand side is nonzero for $x \in C(A,S)$;*

(iv) $d(Sx,S^2x) \neq \max\{d(Ax,ASx), d(Sx,Ax), d(S^2x,ASx), d(Sx,ASx), d(Ax,S^2x)\}$ *whenever the right-hand side is nonzero for $x \in C(A,S)$.*

Proof. Since A and S are noncompatible, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent. Since $t \in \overline{AX}$ and $\overline{AX} \subset SX$, there exists a point $u \in X$ such that $t = Su$. Suppose $Au \neq Su$, then by (ii),

$$d^2(Au, Ax_n) < k\{d^2(Su, Sx_n) + d(Su, Au).d(Sx_n, Ax_n) + d(Su, Ax_n).d(Sx_n, Au)$$
$$+ad(Su, Au).d(Sx_n, Au) + d(Sx_n, Ax_n).d(Su, Ax_n)\}.$$

Making $n \to \infty$ yields $d^2(Au, Su) \leq kad^2(Au, Su) < d^2(Au, Su)$, and $Au = Su$. Consequently $C(A,S)$ is nonempty. Further, the commutativity of A and S at u implies $AAu = ASu = SAu = SSu$. So using (iii) or (iv) for $x = u$, we immediately see that $Au = Su$ is a common fixed point of A and S.

In view of the above proof, we have the following theorem.

Theorem 2.1 BIS. *Let $A, S : X \to X$ be such that maps A and S satisfy the* (EA) *property and the conditions* (i) *and* (ii) *of Theorem* 2.1 *hold. Then $C(A,S)$ is nonempty. Further, all other conclusions of Theorem* 2.1 *are also true.*

The main result of Singh and Kumar [**71**] is obtained under the conditions (i), (iii), and (iv) with (ii) replaced by the condition:

(S.1*) $d(Ax,Ay) \leq kd(Sx,Sy) + \max\{ad(Ax,Sx) + d(Ay,Sy), ad(Ax,Sy) + d(Ay,Sx)\}$,

for all $x, y \in X$ where, $k \geq 0, 0 \leq a < 1$.

The following example demonstrates the generality of Theorems 2.1 and 2.1 BIS.

Example 2.2. *Let $X = [2,20]$ be endowed with the usual metric and $Ax = x$ if $2 \leq x \leq 3$, $Ax = 6$ if $3 < x \leq 4$, $Ax = 8$ if $x > 4$, and $Sx = x$ if $2 \leq x \leq 3$, $Sx = 7$ if $3 < x \leq 7$, $Sx = (x+1)/2$ if $7 < x \leq 11$ or $14 \leq x \leq 20$, $Sx = 11$ if $11 < x \leq 14$. Then A and S satisfy all the conditions of Theorems* 2.1 *and* 2.1 BIS *and A and S have infinitely many common fixed points. Notice that there is a discontinuity at* *their common fixed point* $x = 3$. *It is also verified that condition* (S.1*) *is not satisfied for* $x \in (3, 4], y \in (4, 7]$, *since in this situation*

$$d(Ax, Ay) = 2 > 1 + a = kd(Sx, Sy) + \max\{ad(Ax, Sx) + d(Ay, Sy),$$

$$ad(Ax, Sy) + d(Ay, Sx)\}, \quad where \;\; k \geq 0, 0 \leq a < 1.$$

The following coincidence theorem is obtained for a quadruplet of maps on an arbitrary set with values in a metric space.

Theorem 2.3. *Let* X *be a metric space and* $A, B, S, T : Y \to X$ *such that*

(i*) $\overline{AY} \subset TY$ and $\overline{BY} \subset SY$;

(ii*) one of the pairs (A, S) or (B, T) satisfies the (EA) property;

(iii*) $d^2(Ax, By) < k \max\{d^2(Sx, Ty) + d(Sx, Ax).d(Ty, By) + d(Sx, By).d(Ty, Ax),$
$a[d(Sx, Ax).d(Ty, Ax) + d(Sx, By).d(Ty, By)]\}$

for all $x, y \in Y$, *when the right hand side is non-zero,* $k \geq 0$ *and* $a \geq 0$ *is chosen such that* $ka < 1$.

Then $C(A, S)$ *and* $C(B, T)$ *are nonempty. Further, if* $Y = X$, *then*

(I*) A *and* S *have a common fixed point provided that* A *and* S *commute at (some)* $u \in C(A, S)$ *and one of* (iii) *and* (iv) *holds;*

(II*) B *and* T *have a common fixed point provided that* B *and* T *commute at (some)* $w \in C(B, T)$ *and one of the following holds:*

(iv*) $d(Bx, B^2x) \neq \max\{d(Tx, TBx), d(Bx, Tx), d(B^2x, TBx), d(Bx, TBx), d(Tx, B^2x)\}$ *whenever the right-hand side is nonzero for* $x \in C(B, T)$;

(v*) $d(Tx, T^2x) \neq \max\{d(Bx, BTx), d(Tx, Bx), d(T^2x, BTx), d(Tx, BTx), d(Bx, T^2x)\}$ *whenever the right-hand side is nonzero for* $x \in C(B, T)$;

(III*) A, B, S *and* T *have a common fixed point provided* (I*) *and* (II*) *are true.*

Proof. If the pair (B, T) satisfies the (EA) property, then there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Since $\overline{BY} \subset SY$, for each x_n, there exists y_n in Y such that $Bx_n = Sy_n$, and $Sy_n \to t$ as well. We show that $Ay_n \to t$. If not, there exist a subsequence $\{Ay_m\}$ of $\{Ay_n\}$, a positive integer M, and a real number $r > 0$ such that for some positive integer $m \geq M$, we have
$d(Ay_m, t) \geq r, d(Ay_m, Bx_m) \geq r$, and by (iii*),

$$d^2(Ay_m, Bx_m) < k \max\{d^2(Sy_m, Tx_m) + d(Sy_m, Ay_m).d(Tx_m, Bx_m)$$

$$+ d(Sy_m, Bx_m).d(Tx_m, Ay_m), a[d(Sy_m, Ay_m).d(Tx_m, Ay_m)$$

$$+ d(Sy_m, Bx_m).d(Tx_m, Bx_m)]\} = kad^2(Ay_m, Bx_m) < d^2(Ay_m, Bx_m),$$

a contradiction, and $Ay_n \to t$.

Since $t \in \overline{BY}$ and $\overline{BY} \subset SY$, there exists an element $u \in Y$ such that $t = Su$. To show that $Au = Su$, we suppose otherwise and use the condition (iii*) to get

$$d^2(Au, Bx_n) < k \max\{d^2(Su, Tx_n) + d(Su, Au).d(Tx_n, Bx_n)$$
$$+d(Su, Bx_n).d(Tx_n, Au),$$
$$a[d(Su, Au).d(Tx_n, Au) + d(Su, Bx_n).d(Tx_n, Bx_n)]\}.$$

Making $n \to \infty$, $d^2(Au, Su) \leq kad^2(Au, Su)$, yielding $Au = Su$.

This proves that $C(A, S)$ is nonempty.

Since $\overline{AY} \subset TY$, there exists an element $w \in Y$ such that $Au = Tw$. If $Tw \neq Bw$, then by (iii*),

$$d^2(Au, Bw) < k \max\{d^2(Su, Tw) + d(Su, Au).d(Tw, Bw) + d(Su, Bw).d(Tw, Au),$$
$$a[d(Su, Au).d(Tw, Au) + d(Su, Bw).d(Tw, Bw)]\}$$
$$= kad^2(Au, Bw) < d^2(Au, Bw)$$

Consequently $Tw = Au = Bw$, and $C(B, T)$ is nonempty.
Now let $Y = X$.
The commutativity of A and S at u implies $AAu = ASu = SAu = SSu$. So using (iii) or (iv) for $x = u$, we immediately see that Au is a common fixed point of A and S. This proves (I*). A similar argument shows that Bw is a common fixed point of B and T, proving (II*). Now (III*) is immediate.

In case $S = T$ in Theorem 2.3, we obtain a slightly improved version which we state below.

Theorem 2.4. *Let* (X, d) *be a metric space and* $A, B, S : Y \to X$ *such that* (iii*) *with* $S = T$, *and*

(vi) $\overline{AY} \cup \overline{BY} \subset SY$;

(vii) *one of the pairs* (A, S) *or* (B, S) *satisfies the* (EA)*-property.*

Then:

(I) *maps* A, B *and* S *have a coincidence point u (say);*

(II) *maps* A, B *and* S *have a common fixed point* $z(= Au = Bu = Su)$ *provided that* $Y = X$ *and* S *commutes with each of* A *and* B *at* u *and one of* (iii), (iv) *or the following holds:*

(viii) $d(Bx, B^2x) \neq \max\{d(Sx, SBx), d(Bx, Sx), d(B^2x, SBx), d(Bx, SBx), d(Sx, B^2x)\}$ *whenever the right-hand side is nonzero for* $x \in C(B, S)$;

(ix) $d(Sx, S^2x) \neq \max\{d(Bx, BSx), d(Sx, Bx), d(S^2x, BSx), d(Sx, BSx), d(Bx, S^2x)\}$ *whenever the right-hand side is nonzero for* $x \in C(B, S)$.

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Abstract

The first section of this paper obtains coincidence and fixed point theorems for strictly contractive type maps on metric spaces. The second section is devoted to the study of existence of common fixed points of maps satisfying strictly Lipschitz type conditions. The main tool is the (*EA*)*-property for a pair of maps on an arbitrary set with values in a metric space. This helps us to avoid the continuity of maps and completeness or compactness of the space.*

* * *