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A generation theorem for hyperbolic equations with coefficients of bounded variation in time (**)

In this paper, we are interested in the nonautonomous abstract Cauchy problem

$$(0.1)\qquad \begin{aligned} &\dot u(t) = A(t)u(t), \quad \text{for } t \in [0, T],\\ &u(0) = x. \end{aligned}$$

There are many generation theorems giving sufficient conditions on A to be a generator of an evolution family $U(\cdot,\cdot)$. The solution of (0.1) is then given by $u(t) := U(t, 0)x$. Usually, these generation theorems have three assumptions. The first assumption is stability (or weaker quasi-stability) of the family $A(t)$, which guarantees nice behavior of compositions of semigroups $(e^{sA(t)})_{s\geq 0}$ generated by $A(t)$ on the Banach space X.

We say that $(A(t))_{t\in[0,T]}$ is *quasi-stable* if there exists an integrable function β such that

$$(0.2)\qquad \|e^{s_k A(t_k)} e^{s_{k-1}A(t_{k-1})} \dots e^{s_1 A(t_1)}\|_X \leq M e^{s_1\beta(t_1)+\dots+s_k\beta(t_k)}$$

holds for all $s_1, \dots, s_k \geq 0$, $0 \leq t_1 \leq \dots \leq t_k \leq T$. If β is independent of t in (0.2) then A is *stable*.

The second assumption wants that domains of $A(t)$ for various t's are not much different. In fact, we can assume that there are equal, or that they have a common subspace $Y \subset D(A(t))$ dense in X such that Y is $A(t)$-admissible. There are some

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sufficient conditions for $A(t)$-admissibility, which need existence of a family of isomorphisms $S(t)$ of Y onto X that have some regularity in t and $S(t)A(t)S(t)^{-1} = A(t) + B(t)$ with $B(t) \in B(X)$. Kato proved in [Kat70] that $S \in BV([0,T], B(Y,X))$ works.

Third, we have to assume some regularity of the mapping $t \mapsto A(t)$. For instance strong Lipschitz continuity ([Kat85]), norm continuity from Y to X ([Kat70], [Kat73]). Kobayasi takes A strongly continuous in [Kob79] but he assumes S to be strongly continuously differentiable. Ishii in [Ish82] needs A strongly uniform-measurable and $\|A(\,\cdot\,)\|$ upper integrable and S to be an indefinite strong integral of a strongly measurable function $\dot{S}$. Okazawa and Unai work in a Hilbert space and assume A to be norm continuous (see [OU93]) and Tanaka in [Tan99], [Tan04-1] has S constant and A strongly continuous and in [Tan04-2] A strongly integrable and S norm-continuous.

If $D(A(t)) = Y$ for all $t \in [0,T]$, then $S(t) = A(t)$ or $S(t) = A(t) + \lambda$ can be the wanted isomorphism of Y onto X, if it satisfies the time regularity conditions. However, stronger assumptions on S disallow to take $S = A + \lambda$. We present a generation theorem for the case $A \in BV([0,T], B(Y,X))$ which is not contained in the results mentioned above. We show the existence of an evolution family which is differentiable with exception of countably many points (let us mention that we cannot expect in general that the evolution family is differentiable in every point since it would be a contradiction with results by Colombini and Spagnolo, see [CS89]). Moreover, we generalize a criterion by Kato (see [Kat70], Proposition 3.4) on A to be stable.

The abstract results mentioned in the previous paragraph are proved in Section 1. Section 2 is devoted to applications to hyperbolic second order equations with coefficients of bounded variation

$$\frac{\partial^2 u}{\partial t^2} = m(t,x)\Delta u(t,x), \tag{0.3}$$

resp.

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\,\frac{\partial u}{\partial x_j}(t,x)\right) + \sum_{i=1}^{n} b_i(t,x)\,\frac{\partial u}{\partial x_i}(t,x) + c(t,x)u(t,x). \tag{0.4}$$

In Section 2 we work in Hilbert spaces. However, the abstract result can also be applied to (0.4) in L^p spaces in dimension 1, or to first order hyperbolic equations in L^p. In Hilbert spaces De Simon and Torelli [DT81] showed the existence of a solution in $W^{1,2}([0,T],L^2) \cap L^2([0,T],H^1)$ for (0.4) with $u(0) \in H^1$, $u'(0) \in L^2$. Here we show that the solution is in $C^1([0,T],L^2) \cap C([0,T],H^1)$ and if $u(0) \in H^2$, $u'(0) \in H^1$, then the solution is $C^1([0,T] \setminus N, H^1) \cap C([0,T] \setminus N, H^2)$, N countable and the second derivative of $u : [0,T] \to L^2$ exists with exception of countably many points. Moreover, the solution of (0.3) is bounded on $\mathbb{R}_+$ if $m \in BV(\mathbb{R}_+, L^\infty)$.

1 - Main results

In this section we present the main results of this paper. They are based on the following lemma, which is a generalization of Proposition 3.4 in Kato [Kat70]. This lemma allows to generalize a result on existence of an evolution family, since it is sufficient to assume $(S(t))_{t\geq 0}$ to be of bounded variation (not necessarily $\dot{S}$ strongly measurable and upper integrable), see Theorem 1.2.

In this paper, $\mathrm{Var}_s^t a$ means the variation of a from s to t, i.e.

$$Var_s^t a := \sup\left\{ \sum_{i=1}^{N} \|a(r_i) - a(r_{i-1})\| : s = r_0 < r_1 < \cdots < r_N = t, N \in \mathbb{N} \right\}.$$

By $\mathrm{Var}\, a$ we mean the function $t \mapsto \mathrm{Var}_0^t\, a$.

Now, we can formulate the lemma.

Lemma 1.1. *Let $X_t = (X, \|\cdot\|_t)$ be a family of Banach spaces and $a : [0,T] \to \mathbb{R}$ a function of bounded variation. Assume that*

$$\frac{\|x\|_t}{\|x\|_s} \leq e^{|a(t)-a(s)|}$$

holds for all $x \in X$ and $s, t \in [0,T]$. If $A(t) \in G(X_t, 1, \beta(t))$ and β is upper integrable on $[0,T]$, then $(A(t))_{t\in[0,T]}$ is quasi-stable with $M = e^{\mathrm{Var}_0^T a} \cdot e^{|a(0)-a(T)|}$.

Proof. Let $0 \leq t_1 \leq \cdots \leq t_n \leq T$ and define $t_{n+1} = T$, $t_0 = 0$. We can estimate

$$\begin{aligned}
\left\| \Pi_{i=1}^n (A(t_i) + \lambda_i)^{-1} x \right\|_T &\leq e^{|a(T)-a(t_n)|} \left\| \Pi_{i=1}^n (A(t_i) + \lambda_i)^{-1} x \right\|_{t_n} \\
&\leq e^{|a(T)-a(t_n)|} (\lambda_n - \beta(t_n))^{-1} \left\| \Pi_{i=1}^{n-1} (A(t_i) + \lambda_i)^{-1} x \right\|_{t_n} \\
\leq \Pi_{i=1}^{n+1} e^{|a(t_i)-a(t_{i-1})|} \Pi_{i=1}^n (\lambda_i - \beta(t_i))^{-1} \|x\|_0 &\leq e^{\Sigma_{i=1}^{n+1}|a(t_i)-a(t_{i-1})|} \Pi_{i=1}^n (\lambda_i - \beta(t_i))^{-1} \|x\|_0 \\
&\leq M \Pi_{i=1}^n (\lambda_i - \beta(t_i))^{-1} \|x\|_T
\end{aligned}$$

since $\|x\|_0 \leq e^{|a(0)-a(T)|} \|x\|_T$. □

If β is a constant function in Lemma 1.1, then the family $(A(t))_{t\in[0,T]}$ is stable.

Denote $\Delta = \{(t,s) \in \mathbb{R}^2 : \ 0 \leq s \leq t \leq T\}$. Let X, Y be uniformly convex Banach spaces, Y densely continuously embedded in X.

(i) The family $(A(t))_{t\in[0,T]}$ of closed linear operators in X is stable with constants β, M.

(ii) $Y \subset D(A(t))$ and there exists a family of uniformly convex Banach spaces $Y_t = (Y, \|\cdot\|_{Y,t})$ and $a : [0,T] \to \mathbb{R}$ a function of bounded variation such that

$$\frac{\|y\|_{Y,t}}{\|y\|_{Y,s}} \le e^{|a(t)-a(s)|} \tag{1.1}$$

holds for all $y \in Y$ and $T \ge s$, $t \ge 0$ and the parts $A(t)_Y$ of $A(t)$ in Y belong to $G(Y_t, 1, \beta_Y)$.

(iii) The mapping $t \mapsto A(t)$ is of bounded variation with values in $B(Y,X)$.

Theorem 1.2. *Let the assumptions* (i)–(iii) *hold. Then there exists a family of operators* $U(t,s) \in B(X)$, $(t,s) \in \Delta$ *such that*

(a) $U(t,s)$ *is strongly continuous in* X *in* s, t, $U(t,t) = I$ *and* $\|U(t,s)\|_X \le Me^{\beta(t-s)}$.

(b) $U(t,s) = U(t,r)U(r,s)$ *for* $s \le r \le t$.

(c) $U(t,s)Y \subset Y$ *and* $\|U(t,s)\|_Y \le M_Y e^{\beta_Y(t-s)}$.

(d) *For every* $y \in Y$ *there exists a countable set* $N_1 \subset [0,T]$ *such that the mappings* $s \mapsto U(t,s)y$ *and* $t \mapsto U(t,s)y$ *are continuous in* (t,s) *in the norm of* Y, *provided* $(t,s) \in \Delta$, $s \notin N_1$, $t \notin N_1$.

(e) *For every* $y \in Y$ *there exists a countable set* $N_2 \subset [0,T]$ *such that* $D_s U(t,s)y = -U(t,s)A(s)y$ *and* $D_t U(t,s)y = A(t)U(t,s)y$ *hold for all* $(t,s) \in \Delta$, $s \notin N_2$, $t \notin N_2$.

Proof. We will follow the proofs of Theorems 4.1, 5.1 and 5.2 in [Kat70] and we will show that the assumption of norm continuity and strong Lipschitz continuity is not necessary.

First of all, it follows from (ii) and Lemma 1.1 that $A(t)_Y$ form a stable family with constants β_Y, M_Y on Y. Now, we approximate $A(t)$ by step functions $A_n(t) := A(T\lfloor nt/T\rfloor/n)$. Then we have

$$\int_0^T \|A_n(t) - A(t)\|_{Y\to X}\, \mathrm{d}t \to 0,$$

since the integral is estimated by

$$\sum_{k=0}^{n-1} \frac{T}{n} \sup_{t\in(T\frac{k}{n}, T\frac{k+1}{n})} \|A(Tk/n) - A(t)\|_{Y\to X} \le \frac{T}{n}\mathrm{Var}_0^T(A) \to 0.$$

We define $U_n(t,s) := e^{(t-s)A}$ if t and s, $t > s$ are in the same interval where

$A_n = const = A$. For $s \in [T\frac{k}{n}, T\frac{k+1}{n}]$ and $t \in [T\frac{k+l}{n}, T\frac{k+l+1}{n}]$ we define $U_n(t,s)$ by

$$U_n(t,s) := e^{(t-T\frac{k+l}{n})A(T\frac{k+l}{n})}\left(\prod_{h=1}^{l-1} e^{(T\frac{1}{n})A(T\frac{k+l-h}{n})}\right)e^{(T\frac{k+1}{n}-s)A(T\frac{k}{n})}.$$

From the stability of $A(t)$ and $A(t)_Y$ it follows that

$$\|U_n(t,s)\|_X \le Me^{\beta(t-s)}, \qquad \|U_n(t,s)\|_Y \le M_Y e^{\beta_Y(t-s)}. \tag{1.2}$$

Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}U_n(t,s)y = A_n(t)U_n(t,s)y,$$

$$\frac{\mathrm{d}}{\mathrm{d}s}U_n(t,s)y = -U_n(t,s)A_n(s)y$$

provided $s \ne Tk/n$, $k = 0, 1, \ldots, n$, $y \in Y$.

We show that $\lim_{n\to\infty} U_n(t,s)y$ exists (in the norm of X) for every $y \in Y$. Differentiating $U_n(t,s)U_m(s,r)y$ with respect to s and then integrating from r to t we obtain the identity

$$U_n(t,r)y - U_m(t,r)y = \int_r^t U_n(t,s)[A_n(s) - A_m(s)]U_m(s,r)y \; \mathrm{d}s.$$

Hence,

$$\|U_n(t,r)y - U_m(t,r)y\|_X \le \int_r^t \|U_n(t,s)\|_X \|A_n(s) - A_m(s)\|_{Y\to X} \|U_m(s,r)y\|_Y \; \mathrm{d}s$$

$$\le MM_Y e^{\bar\beta(t-r)}\|y\|_Y \int_r^t \|A_n(s) - A_m(s)\|_{Y\to X} \; \mathrm{d}s \to 0$$

as n, m tend to ∞ ($\bar\beta = \max(\beta, \beta_Y)$). So, $U_n(t,r)y$ converges uniformly in r, t. We denote the limit by $U(t,s)y$. It follows from estimates (1.2) that U can be extended to a bounded operator on X. It is obvious that U satisfies (a) and (b).

We will show (c) and that for $y \in Y$, $U(t,s)y$ is weakly continuous in Y. Let $y \in Y$. First of all, since $\|U_n(t,s)\|_Y$ are uniformly bounded and Y is reflexive, there exists a weakly convergent subsequence of $\{U_n(t,s)y\}$. Its limit is $U(t,s)y$. Hence, $U(t,s)y \in Y$ and the growth estimate holds. Take $s_i \to s$ and $t_j \to t$. Since $U(s_i, t_j)$ is bounded, it follows that any subsequence of $\{U(s_i,t_j)y\}$ has a weakly convergent subsequence and its limit is $U(t,s)y$. Hence, $U(t,s)y$ is weakly continuous in t and s in Y.

We will show the strong continuity in Y. Denote N_a the set of points where $t \mapsto \mathrm{Var}_0^t\, a$ is not continuous. Then N_a is countable. If we replace the interval $[0,T]$ in

Lemma 1.1 by $[s,t]$, we obtain

$$\|U(t,s)\|_{Y,r} \le e^{2|a(r)-a(t)|}\|U(t,s)\|_{Y,t} \le e^{2|a(r)-a(t)|}e^{2\mathrm{Var}_s^t a}e^{\beta_Y(t-s)}.$$

Hence, $\limsup_{t,s\to r}\|U(t,s)\|_{Y,r} \le 1$ if $r \notin N_a$. Let $y \in Y$. Since $U(t,s)y \to y$ weakly as t, $s \to r$, we obtain $U(t,s)y \to y$ strongly by uniform convexity of Y_r and the fact that $\|y\|_{Y,r} \ge \liminf \|U(t,s)y\|_{Y,r} \ge \|y\|_{Y,r}$.

Let $s \ge r$. Then we have $\|U(t,s)y - U(t,r)y\|_Y \le \|U(t,s)\|_Y \cdot \|y - U(s,r)y\|_Y \to 0$ as $s \to r$ or $r \to s$, $s \ge r$ for all $y \in Y$. We have proved that $s \mapsto U(t,s)y$ is continuous in $s \in (0,t) \setminus N_a$. Similarly for t, $r \ge s$, $\|U(t,s)y - U(r,s)y\|_Y \le \|(U(t,r)-I)U(r,s)y\|_Y \to 0$ as $t \to r$, $t \ge r$ and the right continuity of $t \mapsto U(t,s)y$ follows for $t \in (s,T) \setminus N_a$. The left continuity for all t except a countable set follows from the monotonicity of the mapping

$$t \mapsto e^{-2\mathrm{Var}_s^t a - \beta_Y(t-s)}\|U(t,s)y\|_{Y,t}. \tag{1.3}$$

Since this mapping is nonincreasing (as we show later), it is continuous for all $t \in [s,T] \setminus N_y$, where N_y is countable. So, for $t \in (s,T) \setminus (N_y \bigcup N_a)$ we have

$$\|U(t',s)y\|_{Y,t'} \to \|U(t,s)y\|_{Y,t} \quad \text{as} \quad t' \to t.$$

Since

$$|\|U(t',s)y\|_{Y,t'} - \|U(t',s)y\|_{Y,t}| \to 0 \quad \text{as} \quad t' \to t$$

for $t \in (s,T) \setminus N_a$ by Lemma 1.1, we have

$$\|U(t',s)y\|_{Y,t} \to \|U(t,s)y\|_{Y,t} \quad \text{as} \quad t' \to t$$

for all $t \in (s,T) \setminus (N_y \bigcup N_a)$. Since $U(t',s)y \to U(t,s)y$ weakly in Y and $\|\cdot\|_{Y,t}$ is uniformly convex, we have $U(t',s)y \to U(t,s)y$ strongly in Y.

To complete the proof of (d) it remains to show monotonicity of (1.3). By assumption (ii) we have for $t > t'$

$$\begin{aligned}\|U(t,s)y\|_{Y,t} &\le \|U(t,t')\|_{Y,t} \cdot \|U(t',s)\|_{Y,t} \le e^{\mathrm{Var}_{t'}^t a + \beta_Y(t-t')} \cdot e^{|a(t)-a(t')|}\|U(t',s)\|_{Y,t'}\\ &\le e^{2\mathrm{Var}_{t'}^t a + \beta_Y(t-t')}\|U(t',s)\|_{Y,t'}.\end{aligned}$$

Hence,

$$\begin{aligned}e^{-2\mathrm{Var}_s^t a - \beta_Y t}\|U(t,s)y\|_{Y,t} &\le e^{2(\mathrm{Var}_{t'}^t a - \mathrm{Var}_s^t a) + \beta_Y(t-t'-t)}\|U(t',s)\|_{Y,t'}\\ &= e^{-2\mathrm{Var}_s^{t'} a - \beta_Y t'}\|U(t',s)y\|_{Y,t'}.\end{aligned}$$

The strong continuity in Y norm is proved.

Now, we will show that U satisfies (e). Let us fix $s \in (0,T)$ such that A is right norm continuous in s and define a constant family A' of operators by $A'(r) = A(s)$ for all $r \in [0, T]$. It holds that $U'(t, r) = e^{(t-r)A(s)}$ where $U'(t, r)$ is the evolution family generated by A'. Then we have

$$\|U'(t,s)y - U(t,s)y\|_X \leq MM_Y e^{\bar{\beta}(t-s)} \|y\|_Y \int_s^t \|A'(r) - A(r)\|_{X\to Y} \, \mathrm{d}r.$$

Since the right-hand side is $o(t-s)$ as $t \downarrow s$, we have

$$D_t^+ U(t,s)y = D_t^+ U'(t,s)y = A(s)y$$

at $t = s$. If $t > s$ we have

$$D_t^+ U(t,s)y = \lim_{h\downarrow 0} \frac{U(t+h,s)y - U(t,s)y}{h}$$

$$= \lim_{h\downarrow 0} \frac{U(t+h,t) - U(t,t)}{h} U(t,s)y = A(t)U(t,s)y.$$

To show the existence of $D_s^- U(t,s)y$, let us fix $s \in (0, T)$ such that A is left norm continuous in s and define $A'(r) = A(s)$ for all $r \in [0, T]$. Then we obtain $D_s^- U(t,s)y = -U(t,s)A(s)y$ by arguments similar to the above ones.

Now, we will prove $D_s^+ U(t,s)y = -U(t,s)A(s)y$. Let A be right norm continuous in s and $t > s$. Then

$$D_s^+ U(t,s)y = \lim_{h\downarrow 0} \frac{U(t,s+h)y - U(t,s)y}{h} =$$

$$= \lim_{h\downarrow 0} U(t,s+h)\, \frac{y - U(s+h,s)y}{h} = -U(t,s)A(s)y.$$

To show $D_t^- U(t,s)y = A(t)U(t,s)$ let us compute

$$D_t^- U(t,s)y = \lim_{h\downarrow 0} \frac{U(t,s)y - U(t-h,s)y}{h} = \lim_{h\downarrow 0} \frac{U(t,t-h) - I}{h} U(t-h,s)y.$$

We have

$$\left\| \frac{U(t,t-h) - I}{h} U(t-h,s)y + A(t)U(t,s)y \right\|_X$$

$$\leq \left\| \frac{U(t,t-h) - I}{h} \right\|_{Y\to X} \cdot \|U(t-h,s)y - U(t,s)y\|_Y + \left\| \left[\frac{U(t,t-h) - I}{h} + A(t) \right] U(t,s)y \right\|_X.$$

The last norm on the right-hand side tends to zero as $h \downarrow 0$, since $D_s^- U(t,s) = -U(t,s)A(s) = -A(t)$ at $s = t$. The second norm tends to zero by strong continuity of $U(t,s)$ in $B(Y)$. It remains to show that the first expression in norm is bounded. It follows from the fact that

$$U_n(t,t-h)y - y = -\int_t^{t-h} U_n(t,s)A_n(s)y \, \mathrm{d}s$$

and

$$\|U(t,t-h)y - y\|_X \le Me^{\beta h} \int_{t-h}^{t} \|A(s)\|_{Y\to X} \, \mathrm{d}s \|y\|_Y \le h \cdot Me^{\beta h} \sup_{s\in[0,T]} \|A(s)\|_{Y\to X} \|y\|_Y.$$

The proof is complete. □

In fact, better description of countable sets N_1, N_2 follows from the proof of Theorem 1.2. We collect them in the following proposition. This proposition yields better properties of the evolution family U in case of a continuous or A norm continuous.

Proposition 1.3. *The evolution family $U(t,s)$ from Theorem* 1.2 *satisfies.*

(1) *$U(t,s)y$ is right (left) continuous in Y-norm in $s = s_0$ if* Var a *is right (left) continuous in s_0.*

(2) *$U(t,s)y$ is right continuous in Y-norm in $t = t_0$ if* Var a *is right continuous in t_0.*

(3) *$U(t,s)y$ is left continuous in Y-norm in $t = t_0$ if* Var a *is left continuous in t_0 and $t_0 \notin N_y$.*

(4) *$D_s^- U(t,s)y = -U(t,s)A(s)$ holds in $s = s_0$ if A is left continuous in s_0 in $B(Y,X)$.*

(5) *$D_s^+ U(t,s)y = -U(t,s)A(s)$ holds in $s = s_0$ if A is right continuous in s_0 in $B(Y,X)$ and* Var a *is right continuous in s_0.*

(6) *$D_t^+ U(t,s)y = A(t)U(t,s)$ holds in $t = t_0$ if A is right continuous in t_0 in $B(Y,X)$.*

(7) *$D_t^- U(t,s)y = A(t)U(t,s)$ holds in $t = t_0$ if A is left continuous in t_0 in $B(Y,X)$,* Var a *is left continuous in t_0 and $t_0 \notin N_y$.*

If the domain $D(A(t)) = Y$ for all $t \in [0,T]$, then the assumptions of Theorem 1.2 can be weakened in the following way.

(i') There exists a family of uniformly convex Banach spaces $X_t = (X, \|\cdot\|_t)$ and $a : [0,T] \to \mathbb{R}$ a function of bounded variation such that

$$\frac{\|x\|_t}{\|x\|_s} \le e^{|a(t)-a(s)|} \tag{1.4}$$

holds for all $x \in X$ and $T \ge s, t \ge 0$ and $A(t) \in G(X_t, 1, \beta)$.

Corollary 1.4. *Let* (i') *and* (iii) *hold. Then the assertions of Theorem* 1.2 *remain valid.*

Proof. According to Lemma 1.1, (i') implies (i). We can assume without loss of generality that $\beta < 0$ (otherwise we take $A(t) - \lambda$ instead of $A(t)$ for some $\lambda > \beta$). For $y \in Y$ and $t \in [0, T]$ define $\|y\|_{Y,t} := \|A(t)y\|_t$. Since $A(t)$ are isometric isomorphisms of $Y_t := (Y, \|\cdot\|_{Y,t})$ onto X_t, we have immediately that Y_t are uniformly convex, inequality (1.4) holds and $A(t)_Y \in G(Y_t, 1, \beta)$. Hence, the assumptions of Theorem 1.2 are satisfied. □

2 - Hyperbolic equations of second order

In this section we present two examples where the generation Theorem 1.2 can be applied. In both examples, Ω is a bounded open subset of $\mathbb{R}^n$ with C^2-boundary. As a first example assume the following equation (non-autonomous non-homogeneous wave equation)

$$\frac{\partial^2 u}{\partial t^2} = m(t,x)\Delta u(t,x), \qquad x \in \Omega \tag{2.1}$$

where $m \in BV([0,T], L^\infty(\Omega))$, $m(t,x) \geq c > 0$ for all $t \in [0,T]$, $x \in \Omega$. Such equation appears for instance during investigation of an integrodifferential equation coming from a model of heat flow in materials with memory, see Barta [Bar07].

Denote

$$(C(t)u)(x) := m(t,x)\Delta u(x), \qquad D(C(t)) = D := H_0^1(\Omega) \cap H^2(\Omega).$$

Moreover, we define

$$A(t) = \begin{pmatrix} 0 & I \\ C(t) & 0 \end{pmatrix}, \qquad D(A(t)) = D \times H_0^1(\Omega),$$

$X = H_0^1(\Omega) \times L^2(\Omega)$, $Y = D \times H_0^1(\Omega)$. We apply Corollary 1.4 to show that there exists an evolution family, which is strongly continuous and differentiable for all t, s with countably many exceptions.

We first show that (i') holds. Define a family of equivalent scalar products on L^2 by

$$(u,v)_t := \int_\Omega u(x)v(x)\,\frac{1}{m(t,x)}\,\mathrm{d}x.$$

Then $C(t)$ is self-adjoint with respect to $(\cdot,\cdot)_t$. We now apply Lemma 1.1 to show that

$A(t)$ is a stable family. We define equivalent scalar products on X by

$$\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_t=\int_\Omega \nabla u_1\nabla v_1\ \mathrm{d}x+\int_\Omega u_2v_2\frac{1}{m(t,x)}\ \mathrm{d}x.$$

According to Stone's theorem (see, e.g., [EN00], Theorem II.3.24), an operator generates a group of contractions if it is skew-adjoint. We show that $A(t)$ is skew-adjoint with respect to $(\cdot,\cdot)_t$. In fact,

$$(A(t)u,v)_t=\left(\begin{pmatrix}u_2\\C(t)u_1\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_t=\int_\Omega \nabla u_2\nabla v_1\ \mathrm{d}x+\int_\Omega \Delta u_1v_2\ \mathrm{d}x=$$

$$-\int_\Omega u_2\Delta v_1\ \mathrm{d}x-\int_\Omega \nabla u_1\nabla v_2\ \mathrm{d}x=\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}-v_2\\-C(t)v_1\end{pmatrix}\right)_t=(u,-A(t)v)_t.$$

So, $A(t)\in G(X_t,1,0)$.

If we show that the mapping $t\mapsto\|\cdot\|_t$ is of bounded variation, assumption (i') will be verified. We have

$$|\,\|u\|_t^2-\|u\|_{t'}^2|\le\int_\Omega u_2^2\left|\frac{1}{m(t)}-\frac{1}{m(t')}\right|\ \mathrm{d}x\le\sup_{x\in\Omega}\left|\frac{m(t,x)-m(t',x)}{m(t,x)m(t',x)}\right|\cdot\|u\|_X^2.$$

Moreover, there is $K>0$ such that

$$|\,\|u\|_t+\|u\|_{t'}|\ge\frac{1}{K}\|u\|_X.$$

Hence,

$$|\,\|u\|_t-\|u\|_{t'}|=\frac{|\,\|u\|_t^2-\|u\|_{t'}^2|}{|\,\|u\|_t+\|u\|_{t'}|}\le\frac{K}{c^2}\sup_{x\in\Omega}|m(t,x)-m(t',x)|\cdot\|u\|_X.$$

Denote

$$M(t):=\frac{K}{c^2}\mathrm{Var}_0^t m:=\frac{K}{c^2}\sup\left\{\sum_{k=1}^n\|m(t_k)-m(t_{k-1})\|_\infty:\ 0=t_0\le t_1\le\dots\le t_n=t\right\},$$

where $0<c=\inf_{t,x}m(t,x)$. Then

$$|\,\|u\|_t-\|u\|_{t'}|\le\frac{K}{c^2}\cdot\|m(t)-m(t')\|_\infty\cdot\|u\|_X\le|M(t)-M(t')|\cdot\|u\|_X.$$

It follows that

$$\log\frac{\|u\|_t}{\|u\|_{t'}}=\log\left(\frac{|\,\|u\|_t-\|u\|_{t'}|}{\|u\|_{t'}}+1\right)\le\frac{|\,\|u\|_t-\|u\|_{t'}|}{\|u\|_{t'}}\le|M(t)-M(t')|\cdot\frac{\|u\|_X}{\|u\|_{t'}}$$
$$\le C\cdot|M(t)-M(t')|$$

and

$$\frac{\|u\|_t}{\|u\|_{t'}} \leq e^{|CM(t)-CM(t')|}.$$

Since $m \in BV([0,T], L^\infty(\Omega))$, $t \mapsto CM(t)$ has also bounded variation. Hence, assumption (i') holds.

To show (iii), let us estimate

$$\|A(t)u - A(s)u\|_X = \|[m(t) - m(s)]\Delta u_1\|_{L^2} \leq \sup_x |m(t,x) - m(s,x)| \cdot \|u\|_Y. \tag{2.2}$$

Since m is of bounded variation, A is of bounded variation, too.

According to Corollary 1.4, there exists an evolution family for $(A(t))_{t\in[0,T]}$. So, for initial conditions $u(0,\cdot) \in D$, $u'(0,\cdot) \in H_0^1(\Omega)$ there exists a unique solution $u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ to (2.1), such that for all t with countably many exceptions it holds that $t \to u(t,\cdot)$ is continuous with values in $H_0^1 \cap H^2$ and $t \to u'(t,\cdot)$ is continuous with values in H_0^1 and differentiable with values in L^2. Moreover, if $T = +\infty$ then the solutions are bounded on $\mathbb{R}_+$.

As a second example we assume the following equation in the divergence form

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\,\frac{\partial u}{\partial x_j}(t,x)\right) + \sum_{i=1}^{n} b_i(t,x)\,\frac{\partial u}{\partial x_i}\,(t,x) + c(t,x)u(t,x). \tag{2.3}$$

Let

$$a_{ij} \in BV([0,T], W^{1,\infty}(\Omega)) \cap L^\infty([0,T], \mathrm{Lip}\,(\Omega)), \quad a_{ij} = a_{ji}$$

and

$$b_i, c \in BV([0,T], L^\infty(\Omega))$$

and there exists $\gamma > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq \gamma\|\xi\|^2, \quad \xi = (\xi_i) \in \mathbb{R}^n.$$

In Kato [Kat85], there is proved the existence of an evolution family provided the coefficients are Lipschitz continuous in time variable.

Denote

$$C(t)u := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\,\frac{\partial u}{\partial x_j}(t,x)\right), \qquad D(C(t)) = D$$

and

$$C_1(t)u = \sum_{i=1}^{n} b_i(t,x)\,\frac{\partial u}{\partial x_i}(t,x) + c(t,x)u(t,x), \qquad D(C_1(t)) = H_0^1(\Omega).$$

We want to show that

$$\tilde{A}(t) := \begin{pmatrix} 0 & I \\ C(t) + C_1(t) & 0 \end{pmatrix}, \qquad D(\tilde{A}(t)) = D \times H_0^1(\Omega)$$

(with $X = H_0^1 \times L^2$ and $Y = D \times H_0^1$) satisfies the assumptions of Corollary 1.4. First of all we show, that

$$A(t) := \begin{pmatrix} 0 & I \\ C(t) & 0 \end{pmatrix}, \qquad D(A(t)) = D \times H_0^1(\Omega)$$

satisfies (i').

We define equivalent scalar products on X by

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_t = \int_\Omega \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial u_1(x)}{\partial x_i} \frac{\partial v_1(x)}{\partial x_j} \, \mathrm{d}x + \int_\Omega u_2 v_2 \, \mathrm{d}x.$$

It is easy to show that $A(t)$ is skew-adjoint with respect to $(\cdot,\cdot)_t$. Hence, according to Stone's theorem (see, e.g., [EN00], Theorem II.3.24), $A(t) \in G(X_t, 1, 0)$. Since $\tilde{A}(t) - A(t)$ is uniformly bounded, we have $\tilde{A}(t) \in G(X_t, 1, \tilde{M})$ by Proposition 3.5 in [Kat70].

It remains to show that the mapping $t \mapsto \|\cdot\|_t$ is of bounded variation. Then assumption (i') will be verified. We have

$$\begin{aligned} |\,\|u\|_t^2 - \|u\|_{t'}^2| &\le \left| \int_\Omega \sum_{i,j=1}^n [a_{ij}(t,x) - a_{ij}(t',x)] \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j} \, \mathrm{d}x \right| \\ &\le \max_{1 \le i,j \le n} \|a_{ij}(t) - a_{ij}(t')\|_\infty \int_\Omega \sum_{i,j=1}^n \left| \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j} \right| \mathrm{d}x \\ \le n \max_{1 \le i,j \le n} \|a_{ij}(t) - a_{ij}(t')\|_\infty \int_\Omega \nabla u_1 \cdot \nabla u_1 \, \mathrm{d}x &\le n \max_{1 \le i,j \le n} \|a_{ij}(t) - a_{ij}(t')\|_\infty \|u\|_X. \end{aligned}$$

The rest follows by the same arguments as in the previous case. In fact, we have for some $K > 0$

$$|\,\|u\|_t + \|u\|_{t'}| \ge \frac{1}{K} \|u\|_X$$

by ellipticity of $(a_{ij}(t))_{i,j=1}^n$. Hence,

$$|\,\|u\|_t - \|u\|_{t'}| \le |M(t) - M(t')| \cdot \|u\|_X,$$

where

$$M(t) := C \cdot \mathrm{Var}_0^t(\max a_{ij}).$$

Finally, we obtain

$$\frac{\|u\|_t}{\|u\|_{t'}} \leq e^{|CM(t)-CM(t')|},$$

hence the assumption (i') holds.

We show that $\tilde{A}$ is of bounded variation. It holds that

$$\|\tilde{A}(t)u - \tilde{A}(s)u\|_X \leq \left\| \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left([a_{ij}(t) - a_{ij}(s)]\frac{\partial u_1}{\partial x_j}\right)\right\|_{L^2} \tag{2.4}$$

$$+\left\| \sum_{i=1}^{n} |b_i(t) - b_i(s)| \left|\frac{\partial u_1}{\partial x_i}\right|\right\|_{L^2} + \|(c(t) - c(s))u_1\|_{L^2} \leq CB(t,s)\cdot \|u\|_Y$$

where

$$B(t,s) := \max\{\max_{ij} \|a_{ij}(t) - a_{ij}(s)\|_{W^{1,\infty}}, \max_{i} \|b_i(t) - b_i(s)\|_\infty, \|c(t) - c(s)\|_\infty\}.$$

Hence, variation of $\tilde{A}$ is estimated by variation of the coefficients a_{ij}, b_i, and c.

Corollary 1.4 now yields existence of an evolution family for (2.3) and solutions in the same space as in the previous example.

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Abstract

We present a new generation theorem for evolution families which is suitable for hyperbolic equations of first and second order with coefficients of bounded variation in time.

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