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Integral closure and normality of some classes of Veronese-type ideals (**)

1 - Introduction

Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K. The monomial ideals of R are ideals generated by monomials and they have been intensively studied. Some problems arise when we would study good properties of monomial ideals, such that integral closure, normality, and the same properties for some algebras related to them [**1**], [**10**]. The most important of such algebras is the Rees algebra $Rees(I) = \bigoplus_{i \geq 0} I^i t^i$ ([**1**], §1.5, §4.5). An important result says that if I is normal, then $Rees(I)$ is normal ([**10**], 3.3.18).

In this paper we are interested to study the integral closure and the normality of special classes of monomial ideals. In [**9**] the varieties of Veronese-type are studied. We introduce the monomial ideals of Veronese bi-type in the polynomial ring over a field in two sets of variables.

The paper is organized as follows. In the section 1 we consider a special class of ideals of Veronese-type $I_{q,2}$ that are monomial ideals of degree q generated by the set $\{X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} \,|\, \sum_{j=1}^{n} a_{i_j} = q, \quad 0 \leq a_{i_j} \leq 2\}$. We study the integral closure of $I_{q,2}$. These ideals of Veronese-type can arise from the edges and the walks of a graph with loops. A graph G on vertex set $V = \{x_1, \ldots, x_n\}$ has loops if it is not requiring $x_i \neq x_j$ for all edges $\{x_i, x_j\}$ of G. A graph G with loops is called complete if a pair $\{x_i, x_j\}$ is an edge of G for all $x_i, x_j \in V$.

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If G is a complete graph with loops, then the edge ideal $I(G)$ is integrally closed [**6**]. Now we find a geometric description of the integral closure of $I(G)$ in the sense of [**10**] (§7.3). Moreover we prove that also the generalized ideal $I_q(G)$ is integrally closed because it is an ideal of Veronese-type $I_{q,2}$.

In the second part of this paper, starting from the special class of Veronese-type ideals $I_{q,2}$, we introduce a class of Veronese bi-type ideals in the polynomial ring in two sets of variables $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$. They are monomial ideals generated in the same degree q: $L_{q,2} = \sum_{r+s=q} I_{r,2}J_{s,2}$, where $r, s \geq 1$, $I_{r,2}$ is generated on degree r in the variables $X_1, \ldots, X_n$ and $J_{s,2}$ is generated on degree s in the variables $Y_1, \ldots, Y_m$. We prove that $L_{q,2}$ is integrally closed. These ideals can be associated to bipartite graphs with loops, called quasi-bipartite graphs [**6**]. A graph G with loops is quasi-bipartite if its vertex set V can be partitioned into disjoint subsets V_1 and V_2, any edge joins a vertex of V_1 with a vertex of V_2 and there exists some vertex of V with a loop. A quasi-bipartite graph G is strong if all the vertices of V_1 are joined to all vertices of V_2 and for each vertex of V there is a loop.

We prove that the edge ideal of a strong quasi-bipartite graph is not integrally closed and we give an expression for its integral closure. Moreover we show that the generalized ideal $I_q(G)$ associated to a strong quasi-bipartite graph G is integrally closed for $q \geq 3$ because it is an ideal of Veronese bi-type.

In section 3 we study the normality of these ideals obtaining the same informations for the generalized ideals associated to complete graphs with loops and to quasi-bipartite graphs. In [**3**] it is illustrate a criterion to show that the ideals of Veronese-type are normal. Now we apply similar technics to those used in [**3**] and [8] to prove the normality of $L_{q,2}$.

2 - Integral closure of Veronese-type ideals $I_{q,2}$

Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K, I be a monomial ideal of R.

The *integral closure* of I is the set of all elements of R which are integral over I. The integral closure of a monomial ideal is again a monomial ideal. In [**10**] it is given the following description for the integral closure of I:

$$\overline{I} = (f \mid \ f \text{ is monomial in } R \text{ and } \ f^i \in I^i, \text{for some } i \geq 1).$$

In general $I \subseteq \overline{I}$. If $I = \overline{I}$, I is said to be *integrally closed* or *complete*.
If all the powers I^k are integrally closed I is said to be *normal*.

The *Veronese ideal* of degree q is the ideal I_q of $R = K[X_1, \ldots, X_n]$ which is generated by all the monomials in the variables $X_1, \dots, X_n$ of degree q: $I_q = (X_1, \dots, X_n)^q$.

The special class of *ideals of Veronese-type* of degree q is given by the monomial ideals $I_{q,2}$ generated by the set:

$$\{X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} | \sum_{j=1}^{n} a_{i_j} = q, \quad 0 \le a_{i_j} \le 2\}.$$

We have $I_{q,2} \subseteq I_q$ and $I_{q,2} = I_q$ for $q = 1, 2$.

Example 2.1. $R = K[X_1, X_2, X_3]$
$I_{3,2} = (X_1^2X_2, X_1^2X_3, X_1X_2^2, X_2^2X_3, X_1X_3^2, X_2X_3^2, X_1X_2X_3) \subset I_3$.

It is known that I_q is a normal ideal ([**10**]). Now we study the combinatoric of the integral closure of the Veronese ideals I_q.

Let I be a monomial ideal of $R = K[X_1, \dots, X_n]$ generated by the monomials $\underline{X}^{a_1}, \dots, \underline{X}^{a_r}$, with $\underline{X}^{a_i} = X_1^{a_{i_1}} \cdots X_n^{a_{i_n}}$. Each $\underline{X}^{a_i}$ is associated to $a_i = (a_{i_1}, \dots, a_{i_n}) \in \mathbb{N}^n$.

We put $\mathcal{A} = \{a_1, a_2, \dots, a_r\} \subset \mathbb{N}^n$ and we define the set

$$d\mathcal{A} = \{a_{j_1} + a_{j_2} + \cdots + a_{j_d} | 1 \le j_1 \le \cdots \le j_d \le r\}.$$

Example 2.2. *Let* $R = K[X_1, X_2]$, $I = (X_1X_2^2, X_1^2X_2)$
$a_1 = (1, 2), \quad a_2 = (2, 1), \quad A = \{(1, 2), (2, 1)\}$
$2A = \{(a_1 + a_1), (a_2 + a_2), (a_1 + a_2)\} = \{(2, 4), (4, 2), (3, 3)\}$.

Let $d\mathcal{A} = \{a'_1, a'_2, \dots, a'_T\}$, where $a'_j = a_{j_1} + a_{j_2} + \cdots + a_{j_d}$, with $1 \le j_1 \le \cdots \le j_d \le r$. In $d\mathcal{A}$ there are $T = \binom{d+r-1}{d}$ elements. By definition

$$\operatorname{conv}(d\mathcal{A}) = \{\sum_{i=1}^{T} \lambda_i a'_i | \sum_{i=1}^{T} \lambda_i = 1, \lambda_i \in \mathbb{Q}_+\}$$

is the *convex hull* of $d\mathcal{A}$.

In [**10**] it is given a geometric description of the integral closure of a monomial ideal using the convex hull of the set $\mathcal{A}$ associated to the ideal. Now we generalize this result to the integral closure of the power of a monomial ideal using the convex hull of $d\mathcal{A}$ in order to give a geometric description of $\overline{I_q}$.

Proposition 2.1. *Let* $R = K[X_1, \dots, X_n]$ *be the polynomial ring,* I *be an ideal of* R *generated by the monomials* $\underline{X}^{a_1}, \dots, \underline{X}^{a_r}$. *For all* $d \ge 1$ *we have:*

$$\overline{I^d} = (\{\underline{X}^{\lceil \alpha \rceil} | \alpha \in \operatorname{conv}(d\mathcal{A})\}),$$

where $\ulcorner\alpha\urcorner$ *is the upper right corner of* α *whose entries are given by:*

$$\ulcorner\alpha_i\urcorner = \begin{cases} \alpha_i & \text{if } \alpha_i \in \mathbb{N} \\ \llcorner\alpha_i\lrcorner + 1 & \text{if } \alpha_i \notin \mathbb{N} \end{cases}$$

with $\llcorner\alpha_i\lrcorner$ *the integral part of* α_i *and* $\mathcal{A} = \{a_1, \ldots, a_r\}$.

Proof. Set $H = \{\underline{X}^{\ulcorner\alpha\urcorner} | \alpha \in \operatorname{conv}(d\mathcal{A})\}$ and prove that $\overline{I^d} = (H)$. Let $\underline{X}^{\ulcorner\alpha\urcorner} \in H$, $\alpha = \sum_{i=1}^{T} \lambda_i a_i$, with $a_i \in d\mathcal{A}$, $\sum_{i=1}^{T} \lambda_i = 1$, $\lambda_i \in \mathbb{Q}_+$ and $T = \binom{d+r-1}{d}$ be the number of the elements of $d\mathcal{A}$. As $\ulcorner\alpha\urcorner \geq \alpha$ with respect to the order on $\mathbb{Q}^n$, $\exists\, \delta \in \mathbb{Q}^n_+$ such that $\ulcorner\alpha\urcorner = \delta + \alpha$, $\exists\ p > 0$ such that $p\delta \in \mathbb{N}^n$ and $p\lambda_i \in \mathbb{N}$ for all $i = 1, \ldots, T$. Then $\underline{X}^{p\ulcorner\alpha\urcorner} = \underline{X}^{p\delta}\underline{X}^{p\alpha} = \underline{X}^{p\delta}(\underline{X}^{a_1})^{p\lambda_1} \cdots (\underline{X}^{a_T})^{p\lambda_T}$ is an element of $(\overline{I^d})^p$. By definition of integral closure it follows that $\underline{X}^{\ulcorner\alpha\urcorner} \in \overline{I^d}$. Hence $(H) \subseteq \overline{I^d}$. Conversely, we have $\overline{I^d} \subseteq (H)$ (see [**10**], 7.3.4).

Proposition 2.2. *Let* $R = K[X_1, \ldots, X_n]$ *be the polynomial ring over a field* K *and* I_q *be the Veronese ideal of degree* q. *Then:*

$$\overline{I_q} = (\{\underline{X}^{\ulcorner\alpha\urcorner} | \alpha \in \operatorname{conv}(q\mathcal{A})\}),$$

where $\mathcal{A} = \{e_1, \ldots, e_n\}$ *and* e_i *is the i-th unit vector of* $\mathbb{R}^n$.

Proof. $I_q = (X_1, \ldots, X_n)^q$ and its generators are associated to the set $q\mathcal{A}$, where $\mathcal{A} = \{e_1, \ldots, e_n\}$ and e_i is the i-th unit vector of $\mathbb{R}^n$. I_q is a normal ideal, then $\overline{I_q} = I_q = (X_1, \ldots, X_n)^q$. Hence by Proposition 2.1 it follows that $\overline{I_q} = (\{\underline{X}^{\ulcorner\alpha\urcorner} | \alpha \in \operatorname{conv}(q\mathcal{A})\})$.

For ideals of Veronese-type we give the following result.

Theorem 2.1. *Let* $R = K[X_1, \ldots, X_n]$ *be the polynomial ring over a field* K. *The ideal of Veronese-type* $I_{q,2}$ *is integrally closed.*

Proof. Let $\underline{X}^{a_1}, \ldots, \underline{X}^{a_r}$ be the generators of $I_{q,2}$ where $\underline{X}^{a_i} = X_1^{a_{i_1}} \cdots X_n^{a_{i_n}}$ with $\sum_{j=1}^{n} a_{i_j} = q$, $0 \leq a_{i_j} \leq 2$ and $a_i \in \mathbb{N}^n$ for $i = 1, \ldots, r$.

By the geometric description of the integral closure of a monomial ideal given in [**10**] (7.3.4), one has:

$$\overline{I_{q,2}} = (\{\underline{X}^{\ulcorner\alpha\urcorner} | \alpha \in \operatorname{conv}(a_1, \ldots, a_r)\}).$$

Let f be a generator of $\overline{I_{q,2}}$, $f = \underline{X}^{\ulcorner\alpha\urcorner}$ with $\alpha = \sum_{i=1}^{r} \lambda_i a_i \in \operatorname{conv}(a_1, \ldots, a_r)$, $\sum_{i=1}^{r} \lambda_i = 1$, $\lambda_i \in \mathbb{Q}_+$. It follows that $\alpha = (\sum_{i=1}^{r} \lambda_i a_{i_1}, \ldots, \sum_{i=1}^{r} \lambda_i a_{i_n}) \in \mathbb{Q}^n_+$. By definition of $I_{q,2}$, in each generator $\underline{X}^{a_i} = X_{i_1}^{a_{i_1}} \cdots X_{i_n}^{a_{i_n}}$ one has that $a_{i_j} = 0, 1, 2$. If $\lambda_i \in \mathbb{N}$ for $i = 1, \ldots, r$ then $\lambda_i = 1$ and $\lambda_j = 0 \; \forall j \neq i$, hence $\underline{X}^{\ulcorner\alpha\urcorner} = \underline{X}^{a_i}$, $i = 1, \ldots, r$. If $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ with $\sum_{i=1}^{r} \lambda_i = 1$, then one obtains a monomial $\underline{X}^{\ulcorner\alpha\urcorner}$ with $\ulcorner\alpha\urcorner \geq a_i$, that is $\alpha_j \geq a_{i_j}$ for some j, $1 \leq j \leq n$, with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $a_i = (a_{i_1}, \ldots, a_{i_n})$. It follows that $\underline{X}^{\ulcorner\alpha\urcorner}$ is divided by $\underline{X}^{a_i}$ for some $i = 1, \ldots, r$. Therefore $\overline{I_{q,2}}$ is generated by $\underline{X}^{a_i}$, for all $1 \leq i \leq r$ and $\underline{X}^{\ulcorner\alpha\urcorner}$ with $\ulcorner\alpha\urcorner \geq a_i$. Hence the minimal system of generators of $\overline{I_{q,2}}$ is $\underline{X}^{a_1}, \ldots, \underline{X}^{a_r}$. It is proved that $\overline{I_{q,2}} = I_{q,2}$.

As an application we observe that the ideals of Veronese-type can be associated to graphs with loops.

A graph G consists of a finite set $V = \{x_1, \ldots, x_n\}$ of vertices and a collection $E(G)$ of subsets of V, that consists of pairs $\{x_i, x_j\}$, for some $x_i, x_j \in V$.

A graph G has loops if it is not requiring $x_i \neq x_j$ for all edges $\{x_i, x_j\}$ of G. Then the edge $\{x_i, x_i\}$ is said a loop of G.
A graph G with loops is called *complete* if each pair $\{x_i, x_j\}$ is an edge of G for all $x_i, x_j \in V$.

Let G be a graph with loops on vertices $x_1, \ldots, x_n$ and $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K, with one variable X_i for each vertex x_i.

Definition 2.1. *The edge ideal $I(G)$ associated to a graph G with loops is the ideal of R generated by the monomials of degree two, X_iX_j, on the variables $X_1, \ldots, X_n$, such that $\{x_i, x_j\} \in E(G)$ for $1 \leq i \leq j \leq n$:*

$$I(G) = (\{X_iX_j | \{x_i, x_j\} \in E(G)\}).$$

Remark 2.1. *Let G be a complete graph with loops on vertices $x_1, \ldots, x_n$. The edge ideal $I(G)$ is the ideal of R generated by all the monomials of degree two, X_iX_j for all $1 \leq i \leq j \leq n$*

$$I(G) = (X_1^2, \ldots, X_n^2, X_1X_2, \ldots, X_1X_n, \ldots, X_{n-1}X_n) = (X_1, \ldots, X_n)^2 = (I_1)^2,$$

where I_1 is the monomial ideal of R generated by all the variables $X_1, \ldots, X_n$. In this case $I(G)$ is the Veronese ideal I_2.

In [**6**] it is proved that the edge ideal of a complete graph with loops is normal. Now we give a geometric description of the integral closure of the edge ideal of a complete graph with loops as in [**10**] (7.3.4).

Proposition 2.3. *Let G be a complete graph with loops on vertices $x_1, \ldots, x_n$*

and $I(G)$ be the edge ideal. Then

$$\overline{I(G)} = (\{\underline{X}^{\ulcorner \alpha \urcorner} | \alpha \in \operatorname{conv}(2\mathcal{A})\}),$$

where $\mathcal{A} = \{e_1, \ldots, e_n\}$ with e_i the i-th unit vector of $\mathbb{R}^n$.

Proof. The result follows by Proposition 2.2 because $I(G) = I_2$.

Definition 2.2. *Let G be a graph with loops. A walk of length q is an alternating sequence of vertices and edges*

$$w = \{x_{i_0}, l_{i_1}, x_{i_1}, l_{i_2}, \ldots, x_{i_{q-1}}, l_{i_q}, x_{i_q}\},$$

where $l_{i_j} = \{x_{i_{j-1}}, x_{i_j}\}$ is the edge joining $x_{i_{j-1}}$ and x_{i_j}, or l_{i_j} is the loop $\{x_{i_j}, x_{i_j}\}$, $1 \leq i_1 \leq i_2 \leq \ldots \leq i_q \leq n$.

Example 2.3. *Let G be a graph with loops on vertices x_1, x_2, x_3 and edge set $E(G) = \{\{x_1, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_3\}\}$. A walk of length 3 is*

$$w = \{x_1, l_1, x_1, l_2, x_2, l_3, x_3\}$$

where $l_1 = \{x_1, x_1\}$ is the loop on x_1, $l_2 = \{x_1, x_2\}$ is the edge joining x_1 and x_2, $l_3 = \{x_2, x_3\}$ is the edge joining x_2 and x_3.

Remark 2.2. *In a walk $w = \{x_{i_0}, l_{i_1}, x_{i_1}, l_{i_2}, \ldots, x_{i_{q-1}}, l_{i_q}, x_{i_q}\}$ two vertices coincide only if the edge joining them is a loop. Otherwise the vertices are distinct. For example, $w = \{x_1, l_1, x_1, l_2, x_2, l_3, x_1\}$ is not a walk of length 3 in a graph G.*

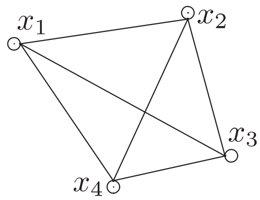
Definition 2.3. *Let G be a graph with loops on vertices $x_1, \ldots, x_n$. The generalized graph ideal, denoted by $I_q(G)$, is the ideal of $R = K[X_1, \ldots, X_n]$ generated by the monomials $X_1^{a_{i_1}} \cdots X_n^{a_{i_n}}$ of degree q such that $\{x_i, x_{i+1}\}$ is an edge of G or x_i has a loop for $i, i+1 \in \{1, \ldots, n\}$.*

Remark 2.3. *The generalized ideal $I_q(G)$ is generated by the monomials of degree q of R corresponding to the walks of length $q-1$ in G:*

$$I_q(G) = (\{X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} | \sum_{j=1}^{n} a_{i_j} = q, \quad 0 \leq a_{i_j} \leq 2\}) = I_{q,2}.$$

$I_q(G)$ is an ideal of Veronese-type. The variables in each generator of $I_q(G)$ have at most degree 2. In fact, in the monomial $X_1^{a_{i_1}} \cdots X_n^{a_{i_n}}$ one has $a_{i_j} = 2$ if G has a loop in x_j or $a_{i_j} = a_{i_{j+1}} = 1$ if $\{x_j, x_{j+1}\}$ is an edge of G.

Example 2.4. *Let G be the complete graph with loops on 4 vertices*



$I_5(G) = ({X_1}^2X_2X_3X_4, {X_1}^2{X_2}^2X_3, {X_1}^2{X_2}^2X_4, {X_1}^2X_2{X_3}^2, {X_1}^2{X_3}^2X_4,$
${X_1}^2X_2{X_4}^2, {X_1}^2X_3{X_4}^2, X_1{X_2}^2X_3X_4, {X_2}^2{X_3}^2X_4, X_1{X_2}^2{X_4}^2, {X_2}^2X_3{X_4}^2,$
$X_1X_2{X_3}^2X_4, X_1{X_3}^2{X_4}^2, X_2{X_3}^2{X_4}^2, X_1X_2X_3{X_4}^2)$.

Proposition 2.4. *Let G be a complete graph with loops on vertices $x_1, \ldots, x_n$. The generalized ideal $I_q(G)$ is integrally closed.*

Proof. It follows from Theorem 2.1 because $I_q(G) = I_{q,2}$.

3 - Integral closure of Veronese bi-type ideals $L_{q,2}$

Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K in two sets of variables. Put $a_i = (a_{i_1}, \ldots, a_{i_n}) \in \mathbb{N}^n$, $b_i = (b_{i_1}, \ldots, b_{i_m}) \in \mathbb{N}^m$, $i = 1, \ldots, r$ and

$$(a_i, b_i) = (a_{i_1}, \ldots, a_{i_n}, b_{i_1}, \ldots, b_{i_m}) \in \mathbb{N}^{n+m}.$$

Let I be a monomial ideal of R generated by the monomials $\underline{X}^{a_1}\underline{Y}^{b_1}, \ldots, \underline{X}^{a_r}\underline{Y}^{b_r}$, where $\underline{X}^{a_i}\underline{Y}^{b_i}$ stands for $X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} Y_1^{b_{i_1}} \cdots Y_m^{b_{i_m}}$ for $i = 1, \ldots, r$. The integral closure of I is the following:

$$\overline{I} = (\underline{X}^{\ulcorner \alpha \urcorner}\underline{Y}^{\ulcorner \beta \urcorner} | (\alpha, \beta) \in \mathrm{conv}\,((a_1, b_1), \ldots, (a_r, b_r))),$$

where $\mathrm{conv}\,((a_1, b_1), \ldots, (a_r, b_r)) = \{\sum_{i=1}^r \lambda_i(a_i, b_i) | \sum_{i=1}^r \lambda_i = 1, \lambda_i \in \mathbb{Q}_+\}$ is the *convex hull* of (a_i, b_i), for $i = 1, \ldots, r$.

In [4] a new monomial ideal of R is defined starting from I:

$$\overline{\overline{I}} = (\{\underline{X}^{\ulcorner \alpha \urcorner}\underline{Y}^{\ulcorner \beta \urcorner} | \alpha \in \mathrm{conv}\,(a_1, \ldots, a_r), \beta \in \mathrm{conv}\,(b_1, \ldots, b_r)\}),$$

with $\alpha \in \mathbb{Q}_+^n$ and $\beta \in \mathbb{Q}_+^m$.
In general, we have the inclusion $\overline{I} \subseteq \overline{\overline{I}}$ ([4], Prop.1).

We define the special class of *ideals of Veronese bi-type* of degree q the monomial ideals of R

$$L_{q,2} = \sum_{r+s=q} I_{r,2} J_{s,2}, \quad r, s \geq 1,$$

where $I_{r,2}$ is the special class of ideals of Veronese-type of degree r in the variables $X_1, \dots, X_n$ and $J_{s,2}$ is the special class ideals of Veronese-type of degree s in the variables $Y_1, \dots, Y_m$.

Example 3.1. *Let $R = K[X_1, X_2; Y_1, Y_2]$ be the polynomial ring.*
$L_{4,2} = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2,$
$X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2,$
$X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

Now we study the integral closure of this class of Veronese bi-type ideals.

Theorem 3.1. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K. The ideal of Veronese bi-type $L_{q,2}$ is integrally closed.*

Proof. Let $\underline{X}^{a_1}\underline{Y}^{b_1}, \dots, \underline{X}^{a_r}\underline{Y}^{b_r}$ be the generators of $L_{q,2}$, with $\underline{X}^{a_i}\underline{Y}^{b_i} = X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} Y_1^{b_{i_1}} \cdots Y_m^{a_{i_m}}$ with $\sum_{j=1}^n a_{i_j} + \sum_{j=1}^m b_{i_j} = q$, $0 \leq a_{i_j} \leq 2$ and $0 \leq b_{i_j} \leq 2$ for $i = 1, \dots, r$, and $q \geq 2$.

By the geometric description of the integral closure of a monomial ideal given in [**10**] (7.3.4), one has:

$$\overline{L_{q,2}} = (\{\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} | (\alpha, \beta) \in \operatorname{conv}((a_1, b_1), \dots, (a_r, b_r))\}),$$

where $\operatorname{conv}((a_1, b_1), \dots, (a_r, b_r)) = \{\sum_{i=1}^r \lambda_i(a_i, b_i) | \sum_{i=1}^r \lambda_i = 1, \lambda_i \in \mathbb{Q}_+\}$.

Let f be a generator of $\overline{L_{q,2}}$, $f = \underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$. $(\alpha, \beta) = (\sum_{i=1}^r \lambda_i a_{i_1}, \dots, \sum_{i=1}^r \lambda_i a_{i_n}, \sum_{i=1}^r \lambda_i b_{i_1}, \dots, \sum_{i=1}^r \lambda_i b_{i_m}) \in \mathbb{Q}_+^{n+m}$. By definition of $L_{q,2}$ in each generator $\underline{X}^{a_i}\underline{Y}^{b_i} = X_{i_1}^{a_{i_1}} \cdots X_{i_t}^{a_{i_t}} Y_{i_1}^{b_{i_1}} \cdots Y_{i_k}^{b_{i_k}}$ one has that $a_{i_j}, b_{i_j} = 0, 1, 2$ such that $\sum_{j=1}^n a_{i_j} + \sum_{j=1}^m b_{i_j} = q$. If $\lambda_i \in \mathbb{N}$ with $\sum_{i=1}^r \lambda_i = 1$ then $\lambda_i = 1$ and $\lambda_j = 0$ $\forall j \neq i$, hence $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} = \underline{X}^{a_i}\underline{Y}^{b_i}$ for $1 \leq i \leq r$. If $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ with $\sum_{i=1}^r \lambda_i = 1$, then one obtains a monomial $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ with $\ulcorner\alpha\urcorner \geq a_i$ and $\ulcorner\beta\urcorner \geq b_i$, that is $\alpha_i \geq a_{i_j}$ and $\beta_i \geq b_{i_j}$. It follows that the monomial $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ is divided by $\underline{X}^{a_i}\underline{Y}^{b_i}$ for some $i = 1, \dots, r$. Therefore $L_{q,2}$ is generated by $\underline{X}^{a_i}\underline{Y}^{b_i}$, $\forall 1 \leq i \leq r$ and by $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ with $\ulcorner\alpha\urcorner \geq a_i$ and $\ulcorner\beta\urcorner \geq b_i$. Then the minimal system of generators of $\overline{L_{q,2}}$ is $\underline{X}^{a_1}\underline{Y}^{b_1}, \dots, \underline{X}^{a_r}\underline{Y}^{b_r}$. Hence $\overline{L_{q,2}} = L_{q,2}$.

For $q = 2$ the inclusion $\overline{L_{q,2}} \subseteq \overline{\overline{L_{q,2}}}$ is true as equality.

Proposition 3.1. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K and $L_{2,2} \subset R$. Then*

$$\overline{L_{2,2}} = \overline{\overline{L_{2,2}}}.$$

Proof. $L_{2,2} = I_1 J_1$. By [4] (Proposition 2) the thesis follows.

For $q = 3$ it is possible to give the description of the integral closure of $L_{q,2}$ in terms of ideals of mixed products [8].

Proposition 3.2. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K and $L_{3,2} \subset R$. Then*

$$\overline{L_{3,2}} = \overline{\overline{I_1 J_2}} + \overline{\overline{I_2 J_1}}.$$

Proof. $L_{3,2}$ is integrally closed, then $\overline{L_{3,2}} = L_{3,2} = I_1 J_2 + I_2 J_1$. So we prove that $\overline{\overline{I_1 J_2}} + \overline{\overline{I_2 J_1}} = I_1 J_2 + I_2 J_1$. Let f be a generator of $\overline{\overline{I_1 J_2}}$, with $\overline{\overline{I_1 J_2}} = (\underline{X}^{\ulcorner \alpha \urcorner} \underline{Y}^{\ulcorner \beta \urcorner} | \alpha \in \operatorname{conv}(a_1, \dots, a_r), \beta \in \operatorname{conv}(b_1, \dots, b_r))$, where $(a_1, b_1), \dots, (a_r, b_r)$ are the exponent vectors of the generators of $I_1 J_2$. Let $f = \underline{X}^{\ulcorner \alpha \urcorner} \underline{Y}^{\ulcorner \beta \urcorner}$, by hypotheses $\alpha = \sum_{i=1}^{r} \lambda_i a_i$, with $\sum_{i=1}^{r} \lambda_i = 1$ and $\beta = \sum_{i=1}^{r} \mu_i b_i$, with $\sum_{i=1}^{r} \mu_i = 1$. It follows: if $\lambda_i \in \mathbb{N}$ then $\underline{X}^{\ulcorner \alpha \urcorner} = X_i$, $\forall\, 1 \le i \le n$ and if $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ then $\underline{X}^{\ulcorner \alpha \urcorner} = X_1 \cdots X_n$. In similar way if $\mu_j \in \mathbb{N}$, then $\underline{Y}^{\ulcorner \beta \urcorner} = Y_j Y_k$ $\forall 1 \le j \le k \le m$ and if $\mu_{i_j} \in \mathbb{Q}_+ \setminus \mathbb{N}$ then $\underline{Y}^{\ulcorner \beta \urcorner} = Y_1 \cdots Y_m$ or $\underline{Y}^{\ulcorner \beta \urcorner} = Y_j (Y_1 \cdots Y_m)$ for all $1 \le j \le m$. Therefore $\overline{\overline{I_1 J_2}}$ is generated by the products of the monomials $\underline{X}^{\ulcorner \alpha \urcorner}$ and $\underline{Y}^{\ulcorner \beta \urcorner}$ as defined before. It follows that the minimal system of generators of $\overline{\overline{I_1 J_2}}$ is $\{X_i Y_j Y_k | 1 \le i \le n, 1 \le j \le k \le m\}$. Hence $\overline{\overline{I_1 J_2}} = I_1 J_2$. In the same way we obtain that $\overline{\overline{I_2 J_1}}$ is generated by $\{X_i X_j Y_k |\ \ 1 \le i \le j \le n,\ \ 1 \le k \le m\}$. Hence $\overline{\overline{I_2 J_1}} = I_2 J_1$. It follows the thesis.

In general we give the following description of the integral closure of $L_{q,2}$.

Proposition 3.3. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K and $L_{q,2} \subset R$ with $q > 3$. Then*

$$\overline{L_{q,2}} = \sum_{r+s=q} \overline{\overline{I_{r,2} J_{s,2}}}, \quad r, s \ge 1.$$

Proof. $L_{q,2}$ is integrally closed, then $\overline{L_{q,2}} = L_{q,2} = \sum_{r+s=q} I_{r,2}J_{s,2}$. So we prove that in general $\overline{I_{r,2}J_{s,2}} = I_{r,2}J_{s,2}$ for $r, s \geq 1$. Let $H = \overline{I_{r,2}J_{s,2}} = (\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} | \alpha \in \mathrm{conv}\,(a_1, \ldots, a_r), \beta \in \mathrm{conv}\,(b_1, \ldots, b_r))$, where $(a_1, b_1), \ldots, (a_r, b_r)$ are the exponent vectors of the generators of $I_{r,2}J_{s,2}$. Let $f = \underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ be a generator of H, by hypotheses $\alpha = \sum_{i=1}^{r} \lambda_i a_i$, with $\sum_{i=1}^{r} \lambda_i = 1$ and $\beta = \sum_{i=1}^{r} \mu_i b_i$, with $\sum_{i=1}^{r} \mu_i = 1$. It follows: if $\lambda_i \in \mathbb{N}$ then $\underline{X}^{\ulcorner\alpha\urcorner} = \underline{X}^{a_i}$, $\forall\ 1 \leq i \leq r$ and if $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ one obtains a monomial $\underline{X}^{\ulcorner\alpha\urcorner}$ with $\ulcorner\alpha\urcorner \geq a_i$. In similar way if $\mu_j \in \mathbb{N}$, then $\underline{Y}^{\ulcorner\beta\urcorner} = \underline{Y}^{b_j}$ $\forall 1 \leq j \leq r$ and if $\mu_{i_j} \in \mathbb{Q}_+ \setminus \mathbb{N}$ one obtains a monomial $\underline{Y}^{\ulcorner\beta\urcorner}$ with $\ulcorner\beta\urcorner \geq b_j$. Therefore H is generated by the products of the monomials $\underline{X}^{\ulcorner\alpha\urcorner}$ and $\underline{Y}^{\ulcorner\beta\urcorner}$ as defined before. Hence the minimal system of generators of H is $\{\underline{X}^{a_i}\underline{Y}^{b_i} | 1 \leq i \leq r\}$. It follows $H = \sum_{r+s=q} I_{r,2}J_{s,2}$. Hence the thesis.

This class of ideals of Veronese bi-type in two sets of variables is associated to bipartite graphs with loops.

A graph G with loops is *quasi-bipartite* [**6**] if its vertex set V can be partitioned into disjoint subsets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_m\}$, and any edge joins a vertex of V_1 with a vertex of V_2 and there exists some vertex of V with a loop.

A graph G with loops is a *strong quasi-bipartite* if all the vertices of V_1 are joined to all the vertices of V_2 and for each vertex of V there is a loop.

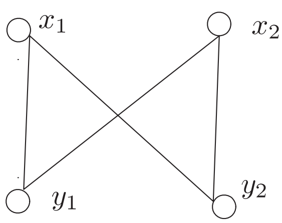
Let G be a graph with loops on vertices $x_1, \ldots, x_n; y_1, \ldots, y_m$.

Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be a polynomial ring over a field K, with one variable X_i for each vertex x_i and Y_j for y_j, $i = 1, \ldots, n, j = 1, \ldots, m$.

Definition 3.1. *The edge ideal $I(G)$ associated to a quasi-bipartite graph G is the ideal of R generated by the monomials of degree two corresponding to the edges and loops of G.*

Remark 3.1. *Let G be a strong quasi-bipartite graph and $I(G)$ the edge ideal. In general $I(G)$ is not integrally closed.*

For example we consider the bipartite graph on 4 vertices:



$I(G) = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2, {X_1}^2, {X_2}^2, {Y_1}^2, {Y_2}^2) \subset K[X_1, X_2, Y_1, Y_2]$
$\overline{I(G)} = ({X_1}^2, {X_2}^2, X_1X_2, X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2, Y_1Y_2, {Y_1}^2, {Y_2}^2)$, [**2**].
$I(G) \neq \overline{I(G)}$. *We observe that* $\overline{I(G)} = I_2^2 + I_1J_1 + J_2^2$, *where* $I_1J_1 = L_{2,2}$.

The structure of the integral closure of $I(G)$ is given in the following result.

Theorem 3.2. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K and G be a strong quasi-bipartite graph G. Then:*

$$\overline{I(G)} = I_1^2 + I_1J_1 + J_1^2.$$

Proof. Let $\underline{X}^{a_1}\underline{Y}^{b_1}, \ldots, \underline{X}^{a_r}\underline{Y}^{b_r}$ be the generators of $I(G)$, where $\underline{X}^{a_i}\underline{Y}^{b_i}$ is a monomial of degree 2 of type X_i^2, Y_j^2 or X_iY_j.

By the geometric description of the integral closure of a monomial ideal given in [**10**] (7.3.4), one has:

$$\overline{I(G)} = (\{\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} | (\alpha, \beta) \in \operatorname{conv}((a_1, b_1), \ldots, (a_r, b_r))\}),$$

where $\operatorname{conv}((a_1, b_1), \ldots, (a_r, b_r)) = \{\sum_{i=1}^r \lambda_i(a_i, b_i) | \sum_{i=1}^r \lambda_i = 1, \lambda_i \in \mathbb{Q}_+\}$.

Let f be a generator of $\overline{I(G)}$, $f = \underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$. $(\alpha, \beta) = (\sum_{i=1}^r \lambda_i a_{i_1}, \ldots, \sum_{i=1}^r \lambda_i a_{i_n}, \sum_{i=1}^r \lambda_i b_{i_1}, \ldots, \sum_{i=1}^r \lambda_i b_{i_m}) \in \mathbb{Q}_+^{n+m}$, with $a_i = (a_{i_1}, \ldots, a_{i_n}) \in \mathbb{N}^n$ and $b_i = (b_{i_1}, \ldots, b_{i_m}) \in \mathbb{N}^m$ for $i = 1, \ldots, r$. By definition of $I(G)$ in each generator $\underline{X}^{a_i}\underline{Y}^{b_i}$ one has that $a_{i_j}, b_{i_j} = 0, 1, 2$. Then $\sum_{i=1}^r \lambda_i a_{i_k} = \lambda_{i_{j_1}} + \cdots + \lambda_{i_{j_m}} + 2\lambda_{i_k}, 1 \leq i_{j_1} < i_{j_2} < \cdots < i_{j_m} \leq r$ and $i_k \notin \{i_{j_1}, i_{j_2}, \cdots, i_{j_m}\}$, and $\sum_{i=1}^r \lambda_i b_{i_t} = \lambda_{i_{l_1}} + \cdots + \lambda_{i_{l_n}} + 2\lambda_{i_s}, 1 \leq i_{l_1} < i_{l_2} < \cdots < i_{l_n} \leq r$ and $i_s \notin \{i_{l_1}, i_{l_2}, \cdots, i_{l_n}\}$.

Hence if $\lambda_i \in \mathbb{N}$ with $\sum_i \lambda_i = 1$ one obtains $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} = \underline{X}^{a_i}\underline{Y}^{b_i}, \forall 1 \leq i \leq r$, that is $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} = X_i^2, \underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} = Y_j^2$ or $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} = X_iY_j$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Instead if $\lambda_{i_k} = \dfrac{1}{2}$ with $\sum_i \lambda_i = 1$, one obtains the monomials $\underline{X}^{\ulcorner\alpha\urcorner}Y^{\ulcorner\beta\urcorner}$ equal to X_iX_j for all $1 \leq i < j \leq n$. In the same way if $\lambda_{i_t} = \dfrac{1}{2}$ with $\sum_i \lambda_i = 1$, one obtains the monomials $\underline{X}^{\ulcorner\alpha\urcorner}Y^{\ulcorner\beta\urcorner}$ equal to Y_iY_j for all $1 \leq i < j \leq m$.

Otherwise if $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ one obtains a monomial $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ with $\ulcorner\alpha\urcorner \geq a_i$ and $\ulcorner\beta\urcorner \geq b_i$, that is $\alpha_i \geq a_{i_j}$ and $\beta_i \geq b_{i_j}$. It follows that the monomial $\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ is divided by $\underline{X}^{a_i}\underline{Y}^{b_i}$ for some $i = 1, \ldots, r$. Therefore the minimal system of generators of $\overline{I(G)}$ is $\{X_i^2, X_iX_k, Y_j^2, Y_jY_l, X_iY_j\}$ for all $1 \leq i \leq n$, $1 \leq k \leq n$, $i \neq k$ and $1 \leq j \leq m$, $1 \leq l \leq m, j \neq l$. Hence $\overline{I(G)} = I_1^2 + I_1J_1 + J_1^2$.

Remark 3.2. *By definition $L_{2,2} = I_1J_1$, it follows that $L_{2,2} \subset \overline{I(G)}$.*

Proposition 3.4. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K and $I(G)$ be the edge ideal of a strong quasi-bipartite graph. Then $\overline{I(G)} \subset \overline{\overline{I(G)}}$.*

Proof. Let $I(G) = (\{X_i^2, X_iY_j, Y_j^2 | \ 1 \leq i \leq n, \ 1 \leq j \leq m\})$ and r be the number of the generators of $I(G)$. For each generator $\underline{X}^{a_i}\underline{Y}^{b_i}$ of $I(G)$ we consider

$(a_i, b_i) \in \mathbb{N}^{n+m}$ with $(a_{i_1}, \ldots, a_{i_n}) \in \mathbb{N}^n$, $(b_{i_1}, \ldots, b_{i_m}) \in \mathbb{N}^m$. More precisely there are m exponent vectors a_i equal to $e_i = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$ for $i = 1, \ldots, n$ and n equal to $2e_i$ for $i = 1, \ldots, n$, where e_i is the i-th unit vector of $\mathbb{R}^n$. Analogously there are n exponent vectors b_j equal to $e_j = (0, \ldots, \underbrace{1}_{j}, \ldots, 0)$ for $j = 1, \ldots, m$ and m equal to $2e_j$ for $j = 1, \ldots, m$.

By definition $\overline{\overline{I(G)}} = (\underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner} | \alpha \in \operatorname{conv}(a_1, \ldots, a_r), \beta \in \operatorname{conv}(b_1, \ldots, b_r))$ and let $f = \underline{X}^{\ulcorner\alpha\urcorner}\underline{Y}^{\ulcorner\beta\urcorner}$ be a generator of $\overline{\overline{I(G)}}$.

By hypotheses $\alpha = \sum_{i=1}^{r} \lambda_i a_{i_j} \in \operatorname{conv}(a_1, \ldots, a_r)$, with $\sum_{i=1}^{r} \lambda_i = 1$, $1 \le j \le m$. Hence $\alpha = (\sum_{i=1}^{r} \lambda_i a_{i_1}, \ldots, \sum_{i=1}^{r} \lambda_i a_{i_n})$. Then $\sum_{i=1}^{r} \lambda_i a_{i_k} = \lambda_{i_{j_1}} + \cdots + \lambda_{i_{j_m}} + 2\lambda_{i_k}$, $1 \le i_{j_1} < i_{j_2} < \cdots < i_{j_m} \le r$ and $i_k \notin \{i_{j_1}, i_{j_2}, \cdots, i_{j_m}\}$. It follows: if $\lambda_i \in \mathbb{N}$ with $\sum_i^r \lambda_i = 1$ we obtain $\underline{X}^{\ulcorner\alpha\urcorner} = X_i$ or $\underline{X}^{\ulcorner\alpha\urcorner} = X_i^2$ $\forall 1 \le i \le n$ or $\underline{X}^{\ulcorner\alpha\urcorner} = 1$ and if $\lambda_i \in \mathbb{Q}_+ \setminus \mathbb{N}$ $\underline{X}^{\ulcorner\alpha\urcorner} = X_1 \cdots X_n$.

In the same way $\beta = \sum_{i=1}^{r} \mu_i b_{i_j}$, with $\sum_{i=1}^{r} \mu_i = 1$, then: if $\mu_i \in \mathbb{N}$, $\underline{Y}^{\ulcorner\beta\urcorner} = Y_j$ or $\underline{Y}^{\ulcorner\beta\urcorner} = Y_i^2$ $\forall 1 \le j \le m$ or $\underline{Y}^{\ulcorner\beta\urcorner} = 1$ and if $\mu_i \in \mathbb{Q}_+ \setminus \mathbb{N}$, $\underline{Y}^{\ulcorner\beta\urcorner} = Y_1 \cdots Y_m$.

Therefore $\overline{\overline{I(G)}}$ is generated by all the products of the monomials $\underline{X}^{\ulcorner\alpha\urcorner}$ and $\underline{Y}^{\ulcorner\beta\urcorner}$ before defined. Hence $\overline{\overline{I(G)}} = R$. It follows the thesis.

Definition 3.2. *Let G be a quasi-bipartite graph on t vertices. A walk of length q in G is an alternating sequence* $w = \{v_{i_0}, l_{i_1}, v_{i_1}, l_{i_2}, \ldots, v_{i_{q-1}}, l_{i_q}, v_{i_q}\}$, *where* v_{i_j} *is a vertex of G and* $l_{i_j} = \{v_{i_{j-1}}, v_{i_j}\}$ *is the edge joining* $v_{i_{j-1}}$ *and* v_{i_j} *or a loop if* $v_{i_{j-1}} = v_{i_j}$, $1 \le i_1 \le i_2 \le \ldots \le i_q \le t$.

Example 3.2. *Let G be a strong quasi-bipartite graph on vertices* $\{x_1, x_2; y_1, y_2\}$. *A walk of length* 2 *is*

$$w = \{x_1, l_1, x_1, l_2, y_1\}$$

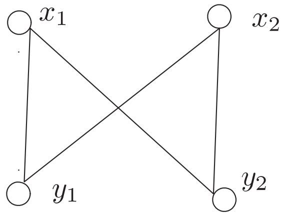
where $l_1 = \{x_1, x_1\}$ *is the loop on* x_1 *and* $l_2 = \{x_1, y_1\}$ *is the edge joining* x_1 *and* y_1. *(A walk w in G can not have the edges* $\{x_i, x_j\}$, *with* $i \ne j$ *and* $\{y_s, y_t\}$ *with* $s \ne t$, *because G is bipartite).*

Definition 3.3. *Let G be a quasi-bipartite graph. The generalized ideal* $I_q(G)$ *associated to G is the ideal of R generated by the monomials of degree q corresponding to the walks of length* $q-1$.

Remark 3.3. *Let G be a strong quasi-bipartite graph. The generalized ideal* $I_q(G)$ *is generated by all the monomials of degree* $q \ge 3$ *corresponding to the walks of length* $q-1$ *and the variables in each generator of* $I_q(G)$ *have at most degree* 2 (*see* Remark 2.3).

Therefore $I_q(G) = L_{q,2} = \sum_{r+s=q} I_{r,2}J_{s,2}$ *for* $q \geq 3$.

Example 3.3. *Let* $R = K[X_1, X_2; Y_1, Y_2]$ *be a polynomial ring over a field* K *and* G *be the strong quasi-bipartite graph on vertices* x_1, x_2, y_1, y_2*:*



$I_3(G) = I_1J_2 + I_2J_1 = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, X_1X_2Y_1,$
$X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$.
$I_4(G) = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2,$
$X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2,$
$X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

Proposition 3.5. *Let* $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ *be the polynomial ring over a field* K *and* G *be a strong quasi-bipartite graph* G*. The generalized ideal* $I_q(G)$ *is integrally closed for* $q \geq 3$.

Proof. It follows from Theorem 3.1 because $I_q(G) = \sum_{r+s=q} I_{r,2}J_{s,2}$ for $q \geq 3$.

Remark 3.4. *Let* G *be a strong quasi-bipartite graph* G*.* $I_2(G) = I(G)$ *is not integrally closed* (Theorem 3.2).

4 - Normality of Veronese by-type ideals

We study the normality of these classes of ideals of Veronese-type and Veronese bi-type using similar technics to those introduced in [8].

Theorem 4.1. *Let* $R = K[X_1, \ldots, X_n]$ *be the polynomial ring over a field* K*. The ideal of Veronese-type* $I_{q,2}$ *is normal.*

Proof. We must prove that $I^p_{q,2}$ is integrally closed for all $p \geq 1$. We proceed by induction on p, for all $p \geq 1$. If $p = 1$, $I_{q,2}$ is integrally closed by Theorem 2.1.

If $p > 1$ we assume that $I^t_{q,2}$ is integrally closed for $t < p$. The integral closure of $I^p_{q,2}$ is a monomial ideal given by ([**10**], 7.3.3):

$$\overline{I^p_{q,2}} = (\{f \in R | \, f \text{ monomial and } f^m \in I^{mp}_{q,2} \text{ for some } m \geq 1\}).$$

Assume $M = \overline{I^p_{q,2}}/I^p_{q,2} \neq (0)$. If $\wp \in Ass_R(M) \setminus \{m\}$, where $m = (X_1, \dots, X_n)$ and $X_1 \notin \wp$ for instance, then $(I^p_{q,2})_\wp \subseteq (I^p_{q-1,2})_\wp$. By induction hypothesis we obtain $M_\wp = (0)$ which is impossible. Hence we have $Ass_R(M) = \{m\}$.

Let f be a monomial such that $f \in \overline{I^p_{q,2}}$ and $f \notin I^p_{q,2}$. We observe that by the description of $\overline{I^p_{q,2}}$ it follows that any monomial in $\overline{I^p_{q,2}}$ has degree at least qp. We have $f \in \overline{I^p_{q,2}} \subset \overline{I^{p-1}_{q,2}} = I^{p-1}_{q,2}$ by induction hypothesis on p, so we can write:

$$f = f_1 \cdots f_{p-1} u,$$

where $f_1, \dots, f_{p-1}$ are monomials of degree q in $I_{q,2}$ and u is a monomial with $\deg(u) \geq q$. We can write, after permutation of variables, $u = X_1^{b_1} \cdots X_\lambda^{b_\lambda}$, $b_i \geq 1$ for all i, $|\mathrm{supp}(u)| \leq p-1$.

If $X_1 \notin \mathrm{supp}(f_i)$ for some i, then there is $X_{j_1} \in \mathrm{supp}(f_i) \setminus \mathrm{supp}(u)$. So we can write:

$$f = f_1 \cdots f_{i-1} \left(\frac{f_i}{X_{j_1}} X_1 \right) f_{i+1} \cdots f_{p-1} X_1^{b_1 - 1} X_2^{b_2} \cdots X_\lambda^{b_\lambda} X_{j_1}.$$

We have the following cases by the same technic used in [**10**] (7.4.5).

I) The number of f_i's not containing X_1 is greater or equal than b_1, then we write

$$f = z_1 \cdots z_{p-1} X_2^{b_2} \cdots X_\lambda^{b_\lambda} X_{j_1} \cdots X_{j_{b_1}}$$

where z_i are monomials in $I_{q,2} \setminus mI_{q,2}$, the variables $X_2, \dots, X_\lambda, X_{j_1}, \dots, X_{j_{b_1}}$ are distinct and $\lambda + b_1 \leq t$.

II) The number r of f_i that don't contain X_1 is less than b_1, then we write

$$f = z_1 \cdots z_{p-1} X_1^{b_1 - r} X_2^{b_2} \cdots X_\lambda^{b_\lambda} X_{j_1} \cdots X_{j_r}$$

where z_i are monomials in $I_{q,2} \setminus mI_{q,2}$ with $X_1 \in \mathrm{supp}(z_i)$ for all i and $X_1, \dots, X_\lambda$, $X_{j_1}, \dots, X_{j_r}$ are distinct variables.

To complete the proof we apply the same considerations to the variables $X_2, \dots, X_\lambda$ in order to obtain

$$f = y_1 \cdots y_{p-1} X_{i_1}^{a_1} \cdots X_{i_r}^{a_r} X_{j_1} \cdots X_{j_s},$$

where y_i are monomials in $I_{q,2} \setminus mI_{q,2}$, $a_i \geq 1$ for all i, the variables $X_{i_1}, \dots, X_{i_r}$, $X_{j_1}, \dots, X_{j_s}$ are distinct and $X_{i_1}, \dots, X_{i_r} \in \mathrm{supp}(y_i)$ for all $i = 1, \dots, p-1$.

We observe that $2r + s \leq q - 1$ otherwise $f \in I^p_{q,2}$ (because for $2r + s > q - 1$ the monomial $h = X_{i_1}^{a_1} \cdots X_{i_r}^{a_r} X_{j_1} \cdots X_{j_s}$ is a multiple of a monomial that generates $I_{q,2}$, that is $h \in I_{q,2}$ and it follows $f \in I^p_{q,2}$).

Now we substitute $X_{i_1} = \ldots = X_{i_r} = 1$ in f and we denote g the monomial obtained after this computation

$$g = {y'}_1 \cdots y'_{p-1} X_{j_1} \cdots X_{j_s},$$

where y_i' is obtained by y_i substituting $X_{i_1} = \ldots = X_{i_r} = 1$. Then $\deg(y_i') = \deg(y_i) - (\deg_{X_{i_1}}(y_i) + \cdots + \deg_{X_{i_r}}(y_i)) = q - d_i$, where $\deg_{X_{i_j}}(y_i)$ is the degree of the variable X_{i_j} in the monomials y_i. We obtain $\deg(g) = \sum_{i=1}^{p-1}(q - d_i) + s = (p-1)(q - \sum_{i=1}^{p-1} d_i) + s$. By the structure of $I_{q,2}$ we observe that the degree of each variable X_i in its generators has degree 1 or 2, then it follows $d_i = k_i + 2(r - k_i) = 2r - k_i$ with $1 \le k_i \le r$. Then $\deg(g) = \sum_{i=1}^{p-1}(q - 2r + k_i) + s = (p-1)(q - 2r + \sum_{i=1}^{p-1} k_i) + s$. On the other hand, since $f \in \overline{I^p_{q,2}}$, then $\deg(f) \ge pq$. It follows $\deg(g) \ge p(\sum_{i=1}^{p-1}(q - d_i)) = p(q - 2r + \sum_{i=1}^{p-1} k_i)$. Hence $(p-1)(q - 2r + \sum_{i=1}^{p-1} k_i) + s \ge p(q - 2r + \sum_{i=1}^{p-1} k_i) \;\Rightarrow\; s - q + 2r - \sum_{i=1}^{p-1} k_i \ge 0 \Rightarrow s + 2r \ge q + \sum_{i=1}^{p-1} k_i$ for $1 \le k_i \le r$, that is an absurd because $2r + s \le q - 1$. It follows that $f \in I^p_{q,2}$. Hence $\overline{I^p_{q,2}} = I^p_{q,2}$.

Theorem 4.2. *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K. The ideal of Veronese bi-type $L_{q,2}$ is normal.*

Proof. We use induction on q, for all $q \ge 2$. For $q = 2$: $L_{2,2} = I_1 J_1$ is normal [8].

For $q > 2$: we assume that $L_{q,2}$ is normal for $t < q$. We must prove that $L^p_{q,2}$ is integrally closed for all $p \ge 1$. If $p = 1$, $L_{q,2}$ is integrally closed by Theorem 3.1.

If $p > 1$ we set $M = \overline{L^p_{q,2}}/L^p_{q,2}$ and suppose $M \ne (0)$. Then we take an associated prime ideal $\wp$ of M. Since $M \hookrightarrow R/L^p_{q,2}$, an associated prime ideal of M is an associated of $R/L^p_{q,2}$, this implies that $\wp$ is a face ideal (since every associated prime is a face ideal see [**10**] 5.1.3)). We suppose that $\wp \ne m$ where m is the maximal ideal $m = R_+$. If a variable $X_i \notin \wp$ then $(L^p_{q,2})_\wp \subseteq (L^p_{q-1,2})_\wp$. We have

$$M_\wp = (\overline{L^p_{q,2}}/L^p_{q,2})_\wp = (\overline{L^p_{q,2}})_\wp/(L^p_{q,2})_\wp \subseteq (\overline{L_{q-1,2}{}^p})_\wp/(L^p_{q-1,2})_\wp = (0),$$

because $L^p_{q-1,2}$ is integrally closed by induction hypothesis on q [**5**]. This is a contradiction, because $M \ne (0)$ and $\wp$ is in the support of M. Hence the only associated prime of M is the maximal ideal m. Then there exists a monomial $f \in \overline{L^p_{q,2}} \setminus L^p_{q,2}$ such that $(L^p_{q,2} : f) = m$.

The support of f contains one of the variables Y_i: if $f = X_1^{a_1} \cdots X_n^{a_n}$, then from $f \in \overline{L^p_{q,2}}$ it follows $f^i \in L^{pi}_{q,2}$ for some $i \ge 1$ and this is impossible by the structure of the generators of $L_{q,2}$. Let $Y_1 \in \operatorname{supp}(f)$ such that $\deg_{Y_1}(f) \ge \deg_{Y_i}(f)$ for all $i > 1$. Then we can write $Y_1 f = h\omega_1 \cdots \omega_p$, where $\omega_1 \ldots \omega_p$ are monomials of $L_{q,2}$ and h is a monomial of R such that $Y_1 \notin \operatorname{supp}(h)$ because $f \notin L^p_{q,2}$. We have the following cases by the same technic used in [**10**] (7.5.8).

I) We assume that Y_j divides h for some $j \ne 1$. Let $d = \deg_{Y_1}(f)$. As Y_1^{d+1} divides $Y_1 f$ then Y_1^{d+1} divides $\omega_1 \cdots \omega_p$. Assume that $Y_1 \in \operatorname{supp}(\omega_i)$ for $i = 1, \ldots, d+1$ and note that $Y_j \in \text{supp}(\omega_i)$ for all $i = 1, \ldots, d+1$; in fact if $Y_j \notin \text{supp}(\omega_i)$ the equality

$$Y_1 f = \omega_1 \cdots (\omega_i Y_j / Y_1) \cdots \omega_{d+1} \cdots \omega_p (Y_1 h / Y_j),$$

implies that $f \in L^p_{q,2}$. So Y_j^{d+1} divides f that contradicts the choice of Y_1.

II) Assume that $h = X_1^{a_1} \cdots X_n^{a_n}$ and X_j divides h for some fixed j.

Suppose that there exists a monomial ω_l of the form

$$\omega_l = (X_{i_1} \cdots X_{i_s})(Y_1 Y_{j_2} \cdots Y_{j_t})$$

with $1 \le i_1 \le \ldots \le i_s \le n, 1 \le j_2 \le \ldots \le j_t \le m$ and $Y_1 \in \text{supp}(\omega_l)$. If $Y_1 \notin \text{supp}(\omega_l)$ and $X_j \in \text{supp}(\omega_l)$, then we can write

$$Y_1 f = \omega_1 \cdots \omega_{l-1} (X_{i_1} \cdots X_{i_k} X_j)(Y_{j_2} \cdots Y_{j_t}) \omega_{l+1} \cdots \omega_p (Y_1 h / X_j),$$

it follows $f \in L^p_{q,2}$, that is a contradiction again.

Then there exists a monomial ω_q of the form

$$\omega_q = (X_{s_1} \cdots X_{s_k})(Y_{t_1} \cdots Y_{t_r})$$

with $1 \le s_1 \le \ldots \le s_k \le n,\ 1 \le t_1 \le \ldots \le t_r \le m$ and $X_j \notin \text{supp}(\omega_q)$. $\{X_{t_1}, \ldots, X_{t_r}\} \not\subseteq \{Y_{j_2}, \ldots, Y_{j_r}\}$ and let $Y_{t_1} \notin \{Y_{j_2}, \ldots, Y_{j_r}\}$.
From the equality

$$Y_1 f = h \omega_l \omega_q \prod_{i \ne l,q} \omega_i = (Y_1 h / X_j)(Y_{t_1} \omega_l / Y_1)(X_j \omega_q / Y_{t_1}) \prod_{i \ne l,q} \omega_i,$$

it follows $f \in L^p_{q,2}$. This is contradiction as in the case I).

We conclude that there is not a monomial $f \in \overline{L^p_{q,2}}$ and $f \notin L^p_{q,2}$. Hence $L^p_{q,2}$ is integrally closed for all $p > 1$. It follows the thesis.

Corollary 4.1. *Let G be a complete graph with loops on vertices $x_1, \ldots, x_n$. The generalized ideal $I_q(G)$ is normal.*

Proof. It follows from Theorem 4.1.

Corollary 4.2. *Let G be a strong quasi-bipartite graph. The generalized ideal $I_q(G)$ is normal for $q \ge 3$.*

Proof. It follows from Theorem 4.2.

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Abstract

The combinatoric of the integral closure and the normality of special classes of Veronese-type ideals in the polynomial ring over a field in one set and two sets of variables are studied. Moreover we associate these monomial ideals to complete graphs and complete bipartite graphs with loops.

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