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On half-space problems for the linearized discrete Boltzmann equation (**)

Contents

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1 - Introduction

The planar stationary Boltzmann equation [**18**], [**19**] with boundary conditions reads

$$\begin{cases} \xi^1 \dfrac{dF}{dx} = Q(F,F),\ F = F(x,\xi) \\ F(0,\xi) = F_0(\xi) \text{ for } \xi^1 > 0 \\ F \to M_\infty \text{ as } x \to \infty \end{cases} \tag{1}$$

where $x \in \mathbb{R}_+$, $\xi = (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3$, $M_\infty = \dfrac{\rho_\infty}{(2\pi T_\infty)^{3/2}} e^{-|\xi - \boldsymbol{u}_\infty|^2/(2T_\infty)}$ and the collision integral $Q(F,F)$ is quadratic in F (for more details see [**18**],[**19**]). If the distribution F is close to a Maxwellian distribution $M = \dfrac{\rho}{(2\pi T)^{3/2}} e^{-|\xi - \boldsymbol{u}|^2/(2T)}$, with ρ, $\boldsymbol{u} = (u^1, u^2, u^3)$, and T constant, then the non-linear equation (1) can be approximated by the linear equation

$$\begin{cases} \xi^1 \dfrac{df}{dx} + Lf = 0, f = f(x,\xi) \\ f(0,\xi) = f_0(\xi) \text{ for } \xi^1 > 0 \end{cases} \tag{2}$$

where $Lf = -2M^{-1/2} Q(M, M^{1/2} f)$ and $F = M + M^{1/2} f$, and where we assume that f fulfills the boundary condition $f \to (a + \boldsymbol{b} \cdot \xi + d|\xi|^2) M^{1/2}$ as $x \to \infty$, for constant $a, d \in \mathbb{R}$ and $\boldsymbol{b} \in \mathbb{R}^3$, at infinity. This problem has been extensively studied in the literature, see [**5**], [**17**], [**21**], [**25**], and the review paper [**6**].

Eq. (2) can by a shift in the velocity space be rewritten as

$$\begin{cases} \left(\xi^1 + u^1\right) \dfrac{df}{dx} + L_0 f = 0 \\ f(0,\xi) = f_0(\xi) \text{ for } \xi^1 + u^1 > 0 \end{cases} \tag{3}$$

where $L_0 f = -2M_0^{-1/2} Q\left(M_0, M_0^{1/2} f\right)$ and $M_0 = \dfrac{\rho}{(2\pi T)^{3/2}} e^{-|\xi|^2/(2T)}$ (cf. [**21**]).

We can consider more general boundary conditions at $x = 0$ (at the wall) in

Eq. (1) [**18**], [**32**], [**33**]:

$$F(0,\xi) = g_0(\xi) + \int\limits_{\xi_*^1<0} K(\xi,\xi_*)F(0,\xi_*)\, d\xi_* \quad \text{for } \xi^1 > 0, \tag{4}$$

where (i) $g_0(\xi) \geq 0$ for $\xi^1 > 0$; and (ii) the kernel $K(\xi,\xi_*)$, fulfills $K(\xi,\xi_*) \geq 0$ for $\xi^1 > 0$ and $\xi_*^1 < 0$, if the boundary is at rest.

For a non-condensable gas (i.e. with no mass flux of the gas across the wall) we can put $g_0(\xi) \equiv 0$. A particular case is the boundary conditions introduced by Maxwell in [**27**, Appendix],

$$F(0,\xi) = (1-a)F(0,\xi_-) + \frac{a\sigma_w}{(2\pi T_w)^{3/2}}\, e^{-|\xi|^2/(2T_w)} \quad \text{for } \xi^1 > 0, \text{ with}$$

$$\sigma_w = -\sqrt{\frac{2\pi}{T_w}} \int\limits_{\xi^1<0} \xi^1 F(0,\xi)\, d\xi \quad \text{and} \quad \xi_- = \left(-\,\xi^1, \xi^2, \xi^3\right),$$

where T_w is the temperature of the wall and a, with $0 \leq a \leq 1$, is the accommodation coefficient. The case $a = 1$ is called diffuse reflection, and $a = 0$ specular reflection. The Maxwell boundary conditions can be obtained by taking

$$K(\xi,\xi_*) = (1-a)\delta(\xi_* - \xi + 2\xi^1\boldsymbol{e}_1) - \frac{a}{2\pi T_w^2}\,\xi_*^1 e^{-|\xi|^2/(2T_w)},$$

with $\boldsymbol{e}_1 = (1,0,0)$, in Eq. (4).

In this paper we study the corresponding problem for the general discrete velocity model [**7**], [**16**], [**23**], [**31**]. Discrete velocity models (DVMs) of the Boltzmann equation are models, where the velocity is discretized, i.e. the velocity is assumed to be able to take only a finite (or in general a discrete) number of different values. It is a well-known fact that the Boltzmann equation can be approximated by DVMs [**13**], [**22**], [**28**], [**29**], and that these approximations can be used for numerical methods. The study of DVMs can also give a better conceptual understanding and new ideas, which can be applied to the Boltzmann equation.

In the planar stationary case, the general DVM reduces to a system of ordinary differential equations. One can consider the Boltzmann equation (for such problems) as a limiting case of the system of ODEs, when the number of discrete velocities tends to infinity [**10**]. The DVM is considered as its finite-dimensional approximation, which in principle can explain many "qualitative" features of the solution of the Boltzmann equation. We continue here the study of DVMs in the directions formulated in [**10**], [**11**]. Especially, the results in [**11**] (see Section 2.2 below) on the dimensions of the stable, unstable and center manifolds of the singular points (Maxwellians for DVMs) are important tools in these studies. We apply these results to typical half-space problems of rarefied gas dynamics. Some first steps in these studies were done, for DVMs with inflow boundary conditions, in [**11**]. A classification of well-posed half-space problems for linearized DVMs (with general boundary conditions) is made. We want to stress that the structure of the solutions is independent of the number of velocities.

The same results as in this paper can also be obtained for DVMs for mixtures [**8**] (see [**1**] for the Boltzmann equation for binary mixtures). We have also, based on similar ideas, obtained results for the weakly non-linear half-space problem for the discrete Boltzmann equation, which will be presented in a future paper. Existence of weak shock wave solutions for the discrete Boltzmann equation has also been proved based on the same ideas [**9**].

This paper is organized as follows: in Section 2, we introduce the planar stationary discrete Boltzmann equation and review some of its properties. We make an expansion around an equilibrium Maxwellian, and note, using the Shoshitalshvili Theorem, that in the non-degenerate case the linear and weakly non-linear systems are topologically equivalent. We also review, in Theorem 2.1, the results in [**11**] on the dimensions of the stable, unstable and center manifolds of the system of ODEs. A proof of Theorem 2.1 is appended in Appendix A.

The half-space problem and the main results on existence and uniqueness are stated in Section 3 (Theorem 3.1). The boundary conditions at the "wall" are discussed in more detail in Section 4. The results of [**11**] (Theorem 2.1) are used to investigate the number of additional conditions needed to obtain well-posedness of the homogeneous and inhomogeneous problems in Section 5 and Section 6 respectively, and thereby to prove Theorem 3.1 in Section 3. Explicit solutions of the half-space problems, the homogeneous and the inhomogeneous, are also obtained in Section 5 and Section 6 respectively. The asymptotic flow, in the homogeneous case with inflow boundary conditions, is discussed briefly in Section 7. The more general case when we allow velocities inducing a singular "velocity-matrix" (that is, if we allow velocities that have zero as first component) is studied in Section 8. Calculations of the above mentioned dimensions are carried out explicitly for axially symmetric DVMs (in the "shock wave context"), when we have expanded around a non-drifting Maxwellian in Section 9. The results are in accordance with the results for the "continuous" Boltzmann equation [**21**]. A special type of axially symmetric models [**2**], [**3**], [**4**] is also studied in Section 9 and our results are applied for a plane 12-velocity model. In Section 10, the half-space problems, stated in Section 3, in the homogeneous case with inflow boundary conditions, are solved exactly, for a (simplified) linearized discrete kinetic model of BGK-type. We also use the solution of the discrete problem to find an exact solution of a half-space problem, for a corresponding continuous linearized model.

2 - Linearized discrete Boltzmann equation

The planar stationary system for the discrete Boltzmann equation (DBE) reads

$$\xi_i^1 \frac{dF_i}{dx} = Q_i(F,F), \;\; x \in \mathbb{R}_+, \;\;\; i = 1, \dots, n, \tag{5}$$

where $\mathrm{V} = \{\xi_1, \dots, \xi_n\} \subset \mathbb{R}^d$ is a finite set of velocities, $F_i = F_i(x) = F(x, \xi_i)$, and $F = F(x^1, \xi)$ represents the microscopic density of particles with velocity $\xi = (\xi^1, \dots, \xi^d)$ at position $\boldsymbol{x} = (x^1, \dots, x^d) \in \mathbb{R}^d$. We also assume (except in Section 8) that

$$\xi_i^1 \neq 0, \quad \text{for } i = 1, \dots, n.$$

For a function $g = g(\xi)$ (possibly depending on more variables than ξ), we will identify g with its restrictions to the points $\xi \in \mathrm{V}$, i.e.

$$g = (g_1, \dots, g_n), \;\text{ with } g_i = g(\xi_i).$$

Consistently, we say that g is non-negative (positive), $g \geq 0$ ($g > 0$), if and only if $g_i \geq 0$ ($g_i > 0$) for all $1 \leq i \leq n$.

Then Eq. (5) can be rewritten as

$$B\frac{df}{dx} = Q(f,f), \;\text{ with } x \in \mathbb{R}_+ \text{ and } B = \mathrm{diag}(\xi_1^1, \dots, \xi_n^1). \tag{6}$$

Below we review some properties of the discrete Boltzmann equation.

The collision operators $Q_i(F,F)$ in (5) are given by the bilinear expressions

$$Q_i(F,G) = \frac{1}{2}\sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl}\left(F_k G_l + G_k F_l - F_i G_j - G_i F_j\right), \tag{7}$$

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0, \tag{8}$$

with equality unless the conservation laws

$$\xi_i + \xi_j = \xi_k + \xi_l \quad \text{and} \quad |\xi_i|^2 + |\xi_j|^2 = |\xi_k|^2 + |\xi_l|^2 \tag{9}$$

are satisfied (preservation of momentum and energy).

Remark 2.1. *Our main results, presented in Section* 3, *do not depend on the preservation of energy* (*even if we indeed use it in some of our applications*), *i.e., Eq.* (9) *could be replaced by*

$$\xi_i + \xi_j = \xi_k + \xi_l \;,$$

without affecting our main results. In fact, our main results do not depend on what set of collision invariants (*cf. Eqs.* (11)) *we have.*

The collision operator $Q = (Q_1, \dots, Q_n)$ is symmetric in its arguments and from the relations (8)

$$\langle H, Q(F,G)\rangle = \frac{1}{8}\sum_{i,j,k,l=1}^{n} \Gamma_{ij}^{kl}\left(H_i + H_j - H_k - H_l\right)\left(F_k G_l + G_k F_l - F_i G_j - G_i F_j\right). \tag{10}$$

A function $\phi = \phi(\xi)$ is a collision invariant, if and only if

$$\phi_i + \phi_j = \phi_k + \phi_l, \tag{11}$$

for all indices such that $\Gamma_{ij}^{kl} \neq 0$. We have the trivial collision invariants (also called the physical collision invariants) $\phi_0 = 1$, $\phi_1 = \xi^1, \dots, \phi_d = \xi^d$, $\phi_{d+1} = |\xi|^2$ (including all linear combinations of these). By the relation (10)

$$\langle \phi, Q(F,F)\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{n} \Gamma_{ij}^{kl}\left(\phi_i + \phi_j - \phi_k - \phi_l\right)\left(F_k F_l - F_i F_j\right), \tag{12}$$

which is zero, independently of our choice of non-negative function F, if and only if ϕ is a collision invariant. Here and below, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^n$.

We consider below (even if this restriction is not necessary in our general context) only normal DVMs. That is, DVMs without spurious (or non-physical) collision invariants, i.e. any collision invariant is of the form

$$\phi = a + \boldsymbol{b} \cdot \xi + c|\xi|^2 \tag{13}$$

for some constant $a, c \in \mathbb{R}$ and $\boldsymbol{b} \in \mathbb{R}^d$ (methods of their construction are described in [**12**], [**14**], [**15**]). In this case the equation

$$\langle \phi, Q(F,F)\rangle = 0,$$

has the general solution (13).

A Maxwellian distribution (or just a Maxwellian) is a function $M = M(\xi)$, such that

$$Q(M,M) = 0 \quad \text{and} \quad M > 0.$$

All Maxwellian distributions are of the form

$$M = e^{\phi} = Ae^{\boldsymbol{b}\cdot\xi + c|\xi|^2}, \quad \text{with } A = e^a > 0, \tag{14}$$

where ϕ is a collision invariant (13). To obtain this, we can use the relation

$$(z - y)\log\frac{y}{z} \leq 0, \tag{15}$$

with equality if and only if $y = z$, which is valid for all $y, z \in \mathbb{R}_+$. Assuming that a function F is positive, we let $\phi = \log F$ in Eq. (12), and obtain, by the relation (15), that

$$\langle \log F, Q(F,F)\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{n} \Gamma_{ij}^{kl}\left(F_k F_l - F_i F_j\right)\log\frac{F_i F_j}{F_k F_l} \leq 0, \tag{16}$$

with equality if and only if

$$F_k F_l = F_i F_j, \tag{17}$$

for all indices such that $\Gamma_{ij}^{kl} \neq 0$. Hence, there is equality in Eq. (16) if and only if F is a Maxwellian. Taking the logarithm of Eqs. (17), F is a Maxwellian if and only if $\phi = \log F$ is a collision invariant.

2.1 - *Linearization*

Given a Maxwellian M we denote

$$F = M + M^{1/2} f, \tag{18}$$

in Eq. (5), and obtain

$$\xi_i^1 \frac{df_i}{dx} = -(Lf)_i + S_i(f,f), \quad i = 1, \dots, n,$$

where L is the linearized collision operator ($n \times n$ matrix) given by

$$Lf = -2M^{-1/2}Q(M, M^{1/2}f), \tag{19}$$

and S is the quadratic part given by

$$S(f,g) = M^{-1/2}Q(M^{1/2}f, M^{1/2}g). \tag{20}$$

In more explicit forms, the operators (19) and (20) read

$$(Lf)_i = -\sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl} M_j^{1/2}\left(M_k^{1/2} f_l + M_l^{1/2} f_k - M_i^{1/2} f_j - M_j^{1/2} f_i\right), \tag{21}$$

and

$$S_i(f,f) = \sum_{j,k,l=1}^{n} \Gamma_{ij}^{kl} M_j^{1/2}\left(f_k f_l - f_i f_j\right),$$

for $i=1,\dots,n$. By Eqs. (21) and the relations (8), we obtain the equality

$$\langle g, Lf\rangle = \frac{1}{4}\sum_{i,j,k,l=1}^{n} \Gamma_{ij}^{kl}\left(M_k^{1/2}f_l + M_l^{1/2}f_k - M_i^{1/2}f_j - M_j^{1/2}f_i\right)$$
$$\times\left(M_k^{1/2}g_l + M_l^{1/2}g_k - M_i^{1/2}g_j - M_j^{1/2}g_i\right).$$

Hence, the matrix L is symmetric and semi-positive, i.e.

$$\langle g, Lf\rangle = \langle Lg, f\rangle \text{ and } \langle f, Lf\rangle \geq 0,$$

for all functions $g=g(\xi)$ and $f=f(\xi)$. Furthermore, $\langle f, Lf\rangle = 0$ if and only if

$$M_k^{1/2}f_l + M_l^{1/2}f_k = M_i^{1/2}f_j + M_j^{1/2}f_i \tag{22}$$

for all indices satisfying $\Gamma_{ij}^{kl}\neq 0$. We denote $f=M^{1/2}\phi$ in Eq. (22), and obtain Eq. (11), by the relations $M_iM_j = M_kM_l \neq 0$. Hence, since L is semi-positive,

$$Lf = 0 \text{ if and only if } f = M^{1/2}\phi, \tag{23}$$

where ϕ is a collision invariant (13). In consequence,

$$\left\langle S(f,f), M^{1/2}\phi\right\rangle = \langle Q(F,F),\phi\rangle + \left\langle F, LM^{1/2}\phi\right\rangle = 0 \tag{24}$$

for all collision invariants ϕ.

The system (6) transforms into

$$B\frac{df}{dx} + Lf = S(f,f). \tag{25}$$

The diagonal matrix B (6) (under our assumptions) has no zero diagonal elements and is non-singular. We denote $f|_{x=0} = f_0$. Then we can rewrite Eq. (25) as

$$f(x) = e^{-xB^{-1}L}f_0 + \int_0^x e^{(\sigma - x)B^{-1}L}[S(f,f)](\sigma)\,d\sigma.$$

Moreover, the nonlinear term is not very important for the topological classification of the stationary point $f=0$ of the system (25), at least not when the matrix $B^{-1}L$ is diagonalizable (the non-degenerate case $l=0$, see Section 2.2). In fact, there is an invertible matrix P, such that

$$P^{-1}B^{-1}LP = J + N,$$

where J is a diagonal matrix

$$J = \operatorname{diag}(\underbrace{0,\dots,0}_{p+l},\lambda_1,\dots,\lambda_q),$$

$$\text{with } n = p+l+q,\ \lambda_1,\dots,\lambda_{m^+} > 0 \text{ and } \lambda_{m^++1},\dots,\lambda_q < 0,$$

$N^2 = 0$ and $P^T BP$ is a diagonal matrix with non-zero diagonal elements.

We denote

$$g = P^{-1}f, \quad \text{with } g = (g_0, g_+, g_-), \ \ g_0 \in \mathbb{R}^{p+l}, \ \ g_+ \in \mathbb{R}^{m^+} \text{ and } g_- \in \mathbb{R}^{m^-},$$

where $m^- = q - m^+$, in the system (25), and obtain the system

$$\frac{dg}{dx} = -(J+N)g + P^{-1}B^{-1}S(Pg,Pg)$$

or, equivalently,

$$\begin{cases} \dfrac{dg_0}{dx} = \widetilde{N}g_0 + G_0(g_0,g_+,g_-) \\ \dfrac{dg_+}{dx} = -\widetilde{J}_0 g_+ + G_+(g_0,g_+,g_-) \\ \dfrac{dg_-}{dx} = \widetilde{J}_1 g_- + G_-(g_0,g_+,g_-) \end{cases} \tag{26}$$

with $(G_0, G_+, G_-) = P^{-1}B^{-1}S(Pg,Pg)$, $\widetilde{N}^2 = 0$, $\widetilde{J}_0 = \operatorname{diag}(\lambda_1,\dots,\lambda_{m^+})$ and $\widetilde{J}_1 = -\operatorname{diag}(\lambda_{m^++1},\dots,\lambda_q)$. The system (26) is, by the Shoshitalshvili Theorem [30, pag. 161], in a neighborhood of the equilibrium point $f = 0$, topologically equivalent to the system

$$\begin{cases} \dfrac{dg_0}{dt} = \widetilde{N}g_0 + G_0(g_0,k_1(g_0),k_2(g_0)) \\ \dfrac{dg_+}{dt} = -\widetilde{J}_0 g_+ \\ \dfrac{dg_-}{dt} = \widetilde{J}_1 g_- \end{cases}$$

for some C^1-functions $k_1(g_0)$ and $k_2(g_0)$. If $B^{-1}L$ is diagonalizable (the non-degenerate case $l = 0$, see Section 2.2), then the system (25) is topologically equivalent (in a neighborhood of the stationary point $f = 0$) to the corresponding linearized system

$$B\frac{df}{dx} + Lf = 0 \tag{27}$$

since $G_0 \equiv 0$ according to Eqs. (23), (24).

This might indicate some similarities for the weakly non-linear half-space problem with the linearized case treated in this paper. In fact, if we assume that we have made the expansion (18) around a Maxwellian M, such that $F \to M$ as $x \to \infty$, then similar results, to the ones in part (i) of Theorem 3.1, can be obtained for the weakly non-linear half-space problem [8] (cf. the results by Ukai et al. for the continuous Boltzmann equation with inflow boundary conditions [34]). This results will be discussed in more details in a future paper.

2.2 - *Characteristic numbers for the linearized discrete Boltzmann equation*

We denote by $n^{\pm}$, where $n^{+} + n^{-} = n$, and $m^{\pm}$, with $m^{+} + m^{-} = q$, the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices B and $B^{-1}L$ respectively, and by m^0 the number of zero eigenvalues of $B^{-1}L$. Moreover, we denote by k^+, k^-, and l the numbers of positive, negative, and zero eigenvalues of the $p \times p$ matrix K ($p = d + 2$ for normal DVMs), with entries $k_{ij} = \langle y_i, y_j \rangle_B = \langle y_i, By_j \rangle$, such that $\{y_1, \dots, y_p\}$ is a basis of the null-space of L, i.e. in our case $\text{span}\,(y_1, \dots, y_p) = N(L) = \text{span}\big(M^{1/2}, M^{1/2}\xi^1, \dots, M^{1/2}\xi^d, M^{1/2}|\xi|^2\big)$. Here and below, we denote $\langle \cdot, \cdot \rangle_B = \langle \cdot, B\cdot \rangle$ and by $N(L)$ the null-space of L.

In applications, the number p of collision invariants is usually relatively small compared to n (note that formally $n = \infty$ for the continuous Boltzmann equation whenas $p \leq 5$). Also, the matrix B is diagonal and therefore all its eigenvalues are known. This explains the importance of the following result by Bobylev and Bernhoff [**11**].

Theorem 2.1. *The numbers of positive, negative and zero eigenvalues of $B^{-1}L$ are given by*

$$\begin{cases} m^+ = n^+ - k^+ - l \\ m^- = n^- - k^- - l\,. \\ m^0 = p + l \end{cases} \tag{28}$$

For the sake of completeness and due to the importance of this theorem, we have included the proof in Appendix A.

Remark 2.2. *In* [**11**] *Theorem* 2.1 *is proved for any real symmetric matrices L and B, such that L is semi-positive and B is invertible, i.e. such that*

$$\langle h, h \rangle_L = \langle h, Lh \rangle \geq 0 \quad \textit{for all } h \in \mathbb{R}^n, \quad \det B \neq 0 \quad \textit{and} \quad \dim\,(N(L)) = p \geq 1.$$

In the proof of Theorem 2.1 a basis

$$\{u_1, \dots, u_q, y_1, \dots, y_k, z_1, \dots, z_l, w_1, \dots, w_l\} \tag{29}$$

of $\mathbb{R}^n$, such that

$$y_i, z_r \in N(L), \quad B^{-1}Lw_r = z_r \text{ and } B^{-1}Lu_a = \lambda_a u_a, \tag{30}$$

and

$$\begin{aligned}
&\langle u_a, u_\beta\rangle_B = \lambda_a \delta_{a\beta}, \quad \text{with } \lambda_1, \dots, \lambda_{m^+} > 0 \text{ and } \lambda_{m^++1}, \dots, \lambda_q < 0,\\
&\langle y_i, y_j\rangle_B = \gamma_i \delta_{ij}, \quad \text{with } \gamma_1, \dots, \gamma_{k^+} > 0 \text{ and } \gamma_{k^++1}, \dots, \gamma_k < 0,\\
&\langle u_a, z_r\rangle_B = \langle u_a, w_r\rangle_B = \langle u_a, y_i\rangle_B = \langle w_r, y_i\rangle_B = \langle z_r, y_i\rangle_B = 0,\\
&\langle w_r, w_s\rangle_B = \langle z_r, z_s\rangle_B = 0 \text{ and } \langle w_r, z_s\rangle_B = \delta_{rs},
\end{aligned} \tag{31}$$

is constructed.

3 - Statement of the problem and main results

We consider the inhomogeneous (or homogeneous if $g = 0$) linearized problem

$$B\frac{df}{dx} + Lf = g, \tag{32}$$

where $g = g(x) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$, with one of the boundary conditions

(O) the solution tends to zero at infinity, i.e.

$$f(x) \to 0 \text{ as } x \to \infty;$$

(P) the solution is bounded, i.e.

$$|f(x)| < \infty \text{ for all } x \in \mathbb{R}_+;$$

(Q) the solution can be slowly increasing, i.e.

$$|f(x)|\, e^{-\varepsilon x} \to 0 \text{ as } x \to \infty, \text{ for all } \varepsilon > 0;$$

at infinity.

In the case of boundary conditions (O) at infinity we additionally assume that

$$g(x) \in N(L)^\perp \text{ for all } x \in \mathbb{R}_+. \tag{33}$$

Remark 3.1. *The boundary conditions* (O) *correspond to the case when we have made the expansion* (18) *around a Maxwellian M, such that* $F \to M$ *as* $x \to \infty$. *The boundary conditions* (P) *and* (Q) *are the boundary conditions in the Milne and Kramers problem respectively.*

We can (without loss of generality) assume that

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \tag{34}$$

where

$$\begin{aligned} &B_+ = \operatorname{diag}(\xi_1^1, \dots, \xi_{n^+}^1) \ \text{ and } \ B_- = -\operatorname{diag}(\xi_{n^++1}^1, \dots, \xi_n^1), \ \text{ with} \\ &\xi_1^1, \dots, \xi_{n^+}^1 > 0 \ \text{ and } \ \xi_{n^++1}^1, \dots, \xi_n^1 < 0. \end{aligned} \tag{35}$$

We also define the projections $R_+ : \mathbb{R}^n \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^n \to \mathbb{R}^{n^-}$, $n^- = n - n^+$, by

$$R_+ s = s^+ = (s_1, \dots, s_{n^+}) \ \text{ and } \ R_- s = s^- = (s_{n^++1}, \dots, s_n)$$

for $s = (s_1, \dots, s_n)$.

At $x = 0$ we assume the general boundary conditions (cf. Eqs. (44) below)

$$f^+(0) = Cf^-(0) + h_0, \tag{36}$$

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$. We introduce the operator $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^{n^+}$, given by

$$\mathcal{C} = R_+ - CR_- .$$

In order to be able to obtain existence and uniqueness of solutions of the linearized half-space problems we will assume that the matrix C fulfills the condition

$$\begin{aligned} &\dim \mathcal{C}U_+ = m^+, \\ &\text{with } U_+ = \operatorname{span}(u \,|\, Lu = \lambda Bu, \ \text{with } \lambda > 0) = \operatorname{span}(u_1, \dots, u_{m^+}), \end{aligned} \tag{37}$$

as we consider boundary conditions (O) at infinity, the condition

$$\dim \mathcal{C}X_+ = n^+, \quad \text{with } X_+ = \operatorname{span}(u_1, \dots, u_{m^+}, y_1, \dots, y_{k^+}, z_1, \dots, z_l), \tag{38}$$

as we consider boundary conditions (P) at infinity, and the condition (38) or the condition

$$\begin{aligned} &\dim \mathcal{C}\widetilde{X}_+ = n^+, \\ &\text{with } \widetilde{X}_+ = \operatorname{span}(u_1, \dots, u_{m^+}, y_1, \dots, y_{k^+}, z_1 + w_1, \dots, z_l + w_l), \end{aligned} \tag{39}$$

as we consider boundary conditions (Q) at infinity.

Remark 3.2. *For the continuous Boltzmann equation* (*with* $d = 3$)*, if we have made the expansion* (18) *around a non-drifting Maxwellian*

$$M = \frac{\rho_\infty}{(2\pi T_\infty)^{3/2}} e^{-|\xi|^2/2T_\infty},$$

$k^+ = 1$, $l = 3$ and the collision invariants y_1, y_2, z_1, z_2, z_3 can be chosen as [**21**]

$$y_1 = \left(\frac{\xi^1}{\sqrt{2T_\infty}} + \frac{|\xi|^2}{\sqrt{30}T_\infty}\right)M^{1/2}, \quad y_2 = \left(-\frac{\xi^1}{\sqrt{2T_\infty}} + \frac{|\xi|^2}{\sqrt{30}T_\infty}\right)M^{1/2},$$

$$z_1 = \left(\sqrt{\frac{5}{2}} - \frac{|\xi|^2}{\sqrt{10}T_\infty}\right)M^{1/2}, \qquad z_2 = \frac{\xi^2}{\sqrt{T_\infty}}M^{1/2} \text{ and } z_3 = \frac{\xi^3}{\sqrt{T_\infty}}M^{1/2}.$$

Moreover,

$$w_j = L^{-1}\xi^1 z_j,$$

in the continuous case, and the continuous analogue of equation $Lu = \lambda Bu$ is

$$Lg = \lambda\xi^1 g, \ \ g = g(\xi), \tag{40}$$

(*see* [**18**] *for a discussion on the eigenvalue problem* (40)). *We also want to point out that, in the continuous case, the boundary conditions* (*before the expansion* (18)), *that correspond to conditions* (36), *are given by Eqs.* (4).

Theorem 3.1. (i) *Assume that the conditions* (33) *and* (37) *are fulfilled and that*

$$h_0, \mathcal{C}e^{xB^{-1}L}B^{-1}g(x) \in \mathcal{C}U_+ \quad \textit{for all } x \in \mathbb{R}_+. \tag{41}$$

Then the system (32) *with the boundary conditions* (O) *and* (36) *has a unique solution.*

(ii) *Assume that the condition* (38) *is fulfilled. Then the system* (32) *with the boundary conditions* (P) *and* (36) *has a unique solution with the asymptotic flow*

$$f_A(x) = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \eta_j z_j,$$

if the k^- parameters $\mu_{k^++1}, \ldots, \mu_k$ are prescribed.

(iii) *Assume that the condition* (38) *is fulfilled. Then the system* (32) *with the boundary conditions* (Q) *and* (36) *has a unique solution with the asymptotic flow*

$$f_A(x) = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left(\left(\eta_j - xa_j\right) z_j + a_j w_j\right), \tag{42}$$

if the $k^- + l$ parameters $\mu_{k^++1}, \ldots, \mu_k$ and $a_1, \ldots, a_l$ are prescribed.

(iv) *Assume that the condition* (39) *is fulfilled. Then the system* (32) *with the boundary conditions* (Q) *and* (36) *has a unique solution with the asymptotic flow* (42) *if the $k^- + l$ parameters $\mu_{k^++1}, \ldots, \mu_k$ and $\vartheta_1, \ldots, \vartheta_l$, $\vartheta_i = \eta_i - a_i$, are prescribed.*

Especially, for the homogeneous system (27) where $g = 0$, the condition (41) is reduced to

$$h_0 \in \mathcal{C}U_+ \ . \tag{43}$$

Theorem 3.1 is proved below in Section 6.

The asymptotic flow for the homogeneous case with inflow boundary conditions (i.e. with $C = 0$, see also Corollary 3.1 below) is discussed in Section 7.

Remark 3.3. *If the condition* (37) *is fulfilled, then the condition* (43) *implies that we have* $k^+ + l$ *conditions on* h_0.

Remark 3.4. *If the conditions*

$$\mathcal{C}U_- \subseteq \mathcal{C}U_+ \ ,$$

with

$$\begin{aligned} U_- &= span(\{u |\ Lu = \lambda Bu, \ with \ \lambda < 0\} \cup \{z_1, \dots, z_l\}) \\ &= span\big(u_{m^+ + 1}, \dots, u_q, z_1, \dots, z_l\big), \end{aligned}$$

and (33) *are fulfilled, then*

$$\mathcal{C}e^{xB^{-1}L}B^{-1}g(x) \in \mathcal{C}U_+ \quad for \ all \ x \in \mathbb{R}_+.$$

Lemma 3.1. *Let* B_+ *and* B_- *be the matrices defined in Eqs.* (35). *Then*

i) the condition (38) *is fulfilled, if*

$$C^T B_+ C < B_- \ \ on \ \ R_- X_+ \, ;$$

ii) the conditions (37) *and* (39) *are fulfilled, if*

$$C^T B_+ C \leq B_- \ \ on \ \ R_- U_+ \ \ and \ \ R_- \widetilde{X}_+, \ \ respectively.$$

Proof. Let $u \in X_+$ and $C^T B_+ C < B_-$ on $R_- X_+$. Then

$$\langle u, u \rangle_B \geq 0.$$

Furthermore, if $u \neq 0$ and $\mathcal{C}u = 0$, then

$$\langle u, u \rangle_B = \langle Cu^-, Cu^- \rangle_{B_+} - \langle u^-, u^- \rangle_{B_-} = \big\langle (C^T B_+ C - B_-) u^-, u^- \big\rangle < 0.$$

Hence, if $\mathcal{C}u = 0$, then $u = 0$. That is, $\dim \mathcal{C}X_+ = \dim X_+ = n^+$, and part *i)* of the lemma is proved.

Part *ii)* of the lemma is proved in a similar way.

Corollary 3.1. *If* $C = 0$, *then the conditions* (37)-(39) *are fulfilled. In particular,* $\{u_1^+, \dots, u_{m^+}^+, y_1^+, \dots, y_{k^+}^+, z_1^+, \dots, z_l^+\}$ *is a basis of* $\mathbb{R}^{n^+}$.

Corollary 3.2. *Assume that* $n^+ = n^-$ *and that we have a set of velocities* V, *such that*

$$\xi_{i+n^+} = (-\xi_i^1, \dots, \xi_i^d),\ \xi_i^1 > 0,$$

and that we have made the expansion (18) *around a non-drifting Maxwellian* M, *i.e. with* $\boldsymbol{b} = \boldsymbol{0}$ *in Eq.* (14). *Then conditions* (37) *and* (39) *are fulfilled for the Maxwell-type boundary conditions*

$$F^+(0) = C_M F^-(0), \quad \text{with } C_M = (1-a)I + aC_{0d}\,, \quad 0 \le a \le 1,$$

where I *is the identity matrix and* C_{0d} *is the* $n^+ \times n^+$ *matrix, with the elements* $c_{d,ij} = \dfrac{\xi_j^1 M_i^{1/2} M_j^{1/2}}{\langle B_+ M^+, 1\rangle}$. *Furthermore, if* $a \neq 0$, *then also condition* (38) *is fulfilled.*

Corollary 3.2 is proved in Section 4.

Remark 3.5. *All our results can be extended in a natural way, to yield also for singular matrices* B, *if*

$$N(L) \cap N(B) = \{0\},$$

by Lemma 8.1 *in Section* 8.

4 - Boundary conditions

If $M = Ae^{\boldsymbol{b}\cdot\xi + c|\xi|^2} \in \mathbb{R}^n$ is the Maxwellian we have made the expansion (18) around, i.e.,

$$F(x) = M + M^{1/2} f(x),$$

then the general boundary conditions (cf. boundary conditions (4) in the continuous case)

$$F^+(0) = C_0 F^-(0) + a_0,$$

where C_0 is a given $n^+ \times n^-$ matrix and $a_0 \in \mathbb{R}^{n^+}$, at $x = 0$, lead to the following C and h_0 in Eq. (36),

$$C = M_+^{-1/2} C_0 M_-^{1/2} \quad \text{and} \quad h_0 = M_+^{-1/2}(C_0 M^- - M^+ + a_0), \tag{44}$$

with

$$M_+^{-1/2} = \mathrm{diag}(M_1^{-1/2}, \dots, M_{n^+}^{-1/2}) \quad \text{and} \quad M_-^{1/2} = \mathrm{diag}(M_{n^++1}^{1/2}, \dots, M_n^{1/2}).$$

EXAMPLE 4.1 – *If we assume inflow boundary conditions, i.e.* $C_0 = 0$*, as is the case when we have complete condensation, then* $C = 0$ *and* $h_0 = M_+^{-1/2}(a_0 - M^+)$.

EXAMPLE 4.2 – *Let* $n^- = n^+$. *The discrete version of the Maxwell-type boundary conditions reads*

$$F^+(0) = C_0F^-(0), \quad \text{with } C_0 = (1-a)I + aC_{0d}, \quad 0 \le a \le 1,$$

where I *is the identity matrix and* C_{0d} *is the* $n^+ \times n^+$ *matrix, with the elements* $c_{0d,ij} = \dfrac{\xi^1_{n^++j} M_{0i}}{\langle B_- M_0^-, 1\rangle}$ *for some Maxwellian* M_0 **[24]**. *The cases* $a = 0$ *and* $a = 1$ *correspond to specular and diffuse reflection, respectively.*

After the expansion (18)*, the Maxwell-type boundary conditions reads*

$$\begin{aligned} &f^+(0) = C_M f^-(0) + h_0, \quad \textit{with } C_M = (1-a)M_+^{-1/2}M_-^{1/2} + aC_d, \quad 0 \le a \le 1, \\ &h_0 = M_+^{-1/2}\Big((1-a)M^- + a\frac{\langle B_- M^-, 1\rangle}{\langle B_- M_0^-, 1\rangle} M_0^+ - M^+\Big), \end{aligned} \tag{45}$$

where C_d *is the* $n^+ \times n^+$ *matrix, with the elements*

$$c_{d,ij} = \frac{\xi^1_{n^++j} M_i^{-1/2} M_{n^++j}^{1/2} M_{0i}}{\langle B_- M_0^-, 1\rangle}.$$

We obtain

$$\begin{aligned} \langle C_d^T B_+ C_d u, u\rangle &= \frac{\langle B_- \sqrt{M^-}, u\rangle^2 \langle B_+ M_0^+, M_0^+ M_+^{-1}\rangle}{\langle B_- M_0^-, 1\rangle^2} \\ &\le \frac{\langle B_- M^-, 1\rangle \langle B_+ M_0^+, M_0^+ M_+^{-1}\rangle}{\langle B_- M_0^-, 1\rangle^2} \langle B_- u, u\rangle, \end{aligned}$$

with equality if and only if $u \in \mathrm{span}\,(\sqrt{\mathrm{M}^-})$.

Moreover, if $M = M_0$*, then*

$$c_{d,ij} = \frac{\xi^1_{n^++j} M_i^{1/2} M_{n^++j}^{1/2}}{\langle B_- M^-, 1\rangle}, \quad h_0 = (1-a)M_+^{-1/2}(M^- - M^+),$$

and

$$\left\langle C_d^T B_+ C_d u, u\right\rangle \leq \frac{\langle B_+ M^+, 1\rangle}{\langle B_- M^-, 1\rangle}\langle B_- u, u\rangle,$$

with equality if and only if $u \in \operatorname{span}(\sqrt{M^-})$.

We now consider the Maxwell-type boundary conditions (45).

If we assume that

$$\xi_{i+n^+} = (-\xi_i^1, \dots, \xi_i^d),\ \xi_i^1 > 0, \quad \text{for } i = 1, \dots, n^+,$$

then $M^- = e^{-2b\xi^1} M^+$ and therefore, $M_+^{-1/2} M_-^{1/2} = \operatorname{diag}(e^{-b\xi_1^1}, \dots, e^{-b\xi_{n^+}^1})$, where b is the first component of $\boldsymbol{b}$ in Eqs. (14). Note also that $B_- = B_+$.

We assume that we have made the expansion (18) around the Maxwellian M_0, i.e. with $M = M_0$ in Eq. (45) . Then $h_0 = (1-a)(e^{-2b\xi^1} - 1)M_+^{1/2}$ and

$$\left\langle C_d^T B_+ C_d u, u\right\rangle \leq \frac{\left\langle B_- M^-, e^{2bB_-}\right\rangle}{\langle B_- M^-, 1\rangle}\langle B_- u, u\rangle \leq \langle B_- u, u\rangle, \tag{46}$$

if $b \leq 0$, with equality for $u \neq 0$, in the last inequality, if and only if $b = 0$, and, in the first inequality, if and only if $u \in \operatorname{span}(\sqrt{M^-})$.

If we additionally assume that $M = M_0$ is a non-drifting Maxwellian (i.e. with $\boldsymbol{b} = \mathbf{0}$ in Eqs. (14)), then the Maxwell-type boundary conditions reads

$$f^+(0) = C_M f^-(0), \quad \text{with } C_M = (1-a)I + aC_d \ (0 \leq a \leq 1),$$

where C_d is the $n^+ \times n^+$ matrix, with the elements $c_{d,ij} = \dfrac{\xi_j^1 M_i^{1/2} M_j^{1/2}}{\langle B_+ M^+, 1\rangle}$. By Eq. (46) and the inequality

$$\langle B_+ C_d u, u\rangle = \frac{\left\langle B_+ \sqrt{M^+}, u\right\rangle^2}{\left\langle B_+ M^+, 1\right\rangle} \leq \langle B_+ u, u\rangle,$$

which can be obtained by the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} &\left\langle C_M^T B_+ C_M u, u\right\rangle \\ &= (1-a)^2 \langle B_+ u, u\rangle + 2\left(a - a^2\right)\langle B_+ C_d u, u\rangle + a^2 \left\langle C_d^T B_+ C_d u, u\right\rangle \\ &\leq \langle B_- u, u\rangle. \end{aligned} \tag{47}$$

We have equality in Eq. (47) if and only if $a = 0$ or $u \in \operatorname{span}(\sqrt{M^-})$. But, if $R_+ g = C_M R_- g$, and $R_- g \in \operatorname{span}(\sqrt{M^-})$, then $g \in \operatorname{span}(\sqrt{M})$, and hence, if $g \neq 0$, then g neither belong to X_+ nor $\widetilde{X}_+$. By Lemma 3.1, conditions (37) and (39) are fulfilled for $C = C_M$, and moreover, if $a \neq 0$, then also condition (38) is fulfilled. This proves Corollary 3.2.

Remark 4.1. *In the general case of Remark* 2.2 *we fix an orthonormal basis*

$$\{e_1, \ldots, e_n\}, \quad \textit{with } \langle e_i, e_j \rangle = \delta_{ij},$$

of $\mathbb{R}^n$, *such that*

$$Be_i = b_i e_i,$$

where

$$b_1, \ldots, b_{n^+} > 0 \quad \textit{and} \quad b_{n^++1}, \ldots, b_n < 0,$$

and define $R_+ : \mathbb{R}^n \to \mathbb{R}^{n^+}$ *and* $R_- : \mathbb{R}^n \to \mathbb{R}^{n^-}$, *by*

$$R_+ s = s^+ = (s_1, \ldots, s_{n^+}) \quad \textit{and} \quad R_- s = s^- = (s_{n^++1}, \ldots, s_n)$$

for $s = \sum_{i=1}^{n} s_i e_i$. *We also introduce the matrices* B_+ *and* B_-, *defined by*

$$B_+ = diag\,(b_1, \ldots, b_{n^+}) \quad \textit{and} \quad B_- = -\ diag(b_{n^++1}, \ldots, b_n).$$

5 - Homogeneous problem

We start by treating the homogenous case, even if it can be obtained from the inhomogeneous case just by letting $g = 0$.

We consider the linearized homogeneous problem (27) with the boundary conditions (36) and one of the boundary conditions (O), (P) and (Q) at infinity.

The Jordan normal form of $B^{-1}L$ (with respect to the basis (29)-(31)) is

$$\begin{pmatrix} \lambda_1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & \lambda_q & & & & & & & & \\ & & & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & 0 & 1 & & & \\ & & & & & & 0 & 0 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & 0 & 0 \end{pmatrix},$$

where there are l blocks of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the Jordan normal form of $e^{-xB^{-1}L}$ is

$$\begin{pmatrix} e^{-x\lambda_1} & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & e^{-x\lambda_q} & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & 1 & -x & & & & \\ & & & & & & 0 & 1 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & 1 & -x \\ & & & & & & & & & 0 & 1 \end{pmatrix},$$

and hence, for any $f_0 \in \mathbb{R}^n$, we obtain

$$e^{-xB^{-1}L} f_0 = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left((\eta_j - x a_j) z_j + a_j w_j \right) + \sum_{r=1}^{q} \beta_r e^{-\lambda_r x} u_r, \tag{48}$$

where

$$\mu_i = \frac{\langle f_0, y_i \rangle_B}{\langle y_i, y_i \rangle_B}, \ \beta_r = \frac{\langle f_0, u_r \rangle_B}{\lambda_r}, \quad a_j = \langle f_0, z_j \rangle_B \quad \text{and} \quad \eta_j = \langle f_0, w_j \rangle_B.$$

Accordingly, the solution of the system (27) can be written on the form (48)

$$f(x) = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left((\eta_j - x a_j) z_j + a_j w_j \right) + \sum_{r=1}^{q} \beta_r e^{-\lambda_r x} u_r,$$

where $\mu_1, \dots, \mu_k, \beta_1, \dots, \beta_q, \eta_1, \dots, \eta_l, a_1, \dots, a_l$ are arbitrary constants.

The solutions of the system (27) with the boundary conditions

(O) are

$$f(x) = \sum_{r=1}^{m^+} \beta_r e^{-\lambda_r x} u_r,$$

which form an $(n^+ - k^+ - l)$-parameter subspace of $\mathbb{R}^n$;

(P) are

$$f(x) = \sum_{r=1}^{m^+} \beta_r e^{-\lambda_r x} u_r + \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \eta_j z_j,$$

which form an $(n^+ + k^-)$-parameter subspace of $\mathbb{R}^n$;

(Q) are

$$f(x) = \sum_{r=1}^{m^+} \beta_r e^{-\lambda_r x} u_r + \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left(\eta_j z_j + a_j (w_j - x z_j) \right),$$

which form an $(n^+ + k^- + l)$-parameter subspace of $\mathbb{R}^n$.

First we assume that the boundary conditions (O) and (36) are fulfilled. Then we obtain a linear system

$$\sum_{r=1}^{m^+} \beta_r \mathcal{C} u_r = h_0, \tag{49}$$

for the unknown parameters $\beta_1, \dots, \beta_{m^+}$. The system (49) has a solution for all h_0 satisfying the conditions

$$h_0 \in \mathcal{C}U_+, \text{with}$$

$$U_+ = \text{span}\,(u \mid Lu = \lambda Bu, \text{ with } \lambda > 0) = \text{span}\,(u_1, \dots, u_{m^+}),$$

and a unique solution if and only if, additionally, condition (37) is fulfilled, i.e. if and only if $\dim \mathcal{C}U_+ = m^+$.

Now we pass on to the case when the boundary conditions (Q) and (36) are fulfilled. We denote

$$f_\infty = \lim_{x\to\infty}\left(f(x) - x\,\frac{df(x)}{dx}\right) = \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left(\eta_j z_j + a_j w_j\right).$$

and prescribe the $k^- + l$ parameters

$$\mu_i = \frac{\langle f_\infty, y_i\rangle_B}{\langle y_i, y_i\rangle_B}, \quad i = k^+ + 1, \dots, k, \tag{50}$$

and

$$a_j = \left\langle f_\infty, z_j \right\rangle_B, \quad j = 1, \dots, l. \tag{51}$$

We obtain the linear system

$$\sum_{r=1}^{m^+} \beta_r \mathcal{C} u_r + \sum_{i=1}^{k^+} \mu_i \mathcal{C} y_i + \sum_{j=1}^{l} \eta_j \mathcal{C} z_j = h_0 - \sum_{i=k^+ +1}^{k} \mu_i \mathcal{C} y_i - \sum_{j=1}^{l} a_j \mathcal{C} w_j \tag{52}$$

with $\mathcal{C} = R_+ - CR_-$, for the unknown parameters $\mu_1, \dots, \mu_{k^+}, \beta_1, \dots, \beta_{m^+}, \eta_1, \dots, \eta_l$, with $m^+ + k^+ + l = n^+$. The system (52) has a unique solution, if the condition (38) is fulfilled.

The result for the case with the boundary conditions (Q), can be used for the case with the boundary conditions (P), if we prescribe $a_1 = \dots = a_l = 0$ in Eqs. (51). That is, the problem (27), (36) with the boundary conditions (P), has a unique solution for all h_0, if the condition (38) is fulfilled and the parameters (50) are prescribed.

We still assume that the boundary conditions (Q) and (36) are fulfilled and prescribe the parameters (50), but instead of prescribing the parameters (51), we prescribe

$$\vartheta_j = \eta_j - a_j = \langle f_\infty, w_j - z_j \rangle_B, \quad j = 1, \ldots, l.$$

Then we obtain the linear system

$$\begin{aligned}\sum_{r=1}^{m^+} \beta_r \mathcal{C} u_r + \sum_{i=1}^{k^+} \mu_i \mathcal{C} y_i + \sum_{j=1}^{l} \varpi_j \mathcal{C} \frac{z_j + w_j}{2} \\ = h_0 - \sum_{i=k^+ +1}^{k} \mu_i \mathcal{C} y_i - \sum_{j=1}^{l} \vartheta_j \mathcal{C} \frac{z_j - w_j}{2}\end{aligned} \tag{53}$$

with $\mathcal{C} = R_+ - CR_-$, for the unknown parameters $\mu_1, \ldots, \mu_{k^+}$, $\beta_1, \ldots, \beta_{m^+}$, $\varpi_1, \ldots, \varpi_l$, $\varpi_j = \eta_j + a_j$, with $m^+ + k^+ + l = n^+$. The system (53) has a unique solution, if the condition (39) is fulfilled.

Remark 5.1. *The problem* (27), (36) *with the boundary conditions* (Q) *has a solution, with* $k^- + l$ *free parameters, for all* h_0, *if*

$$\begin{aligned}&\dim \mathcal{C} X_1 = n^+,\\ &X_1 = span(u \mid LB^{-1}Lu = 0 \vee Lu = \lambda Bu, \ with \ \lambda > 0)\\ &\quad = span(u_1, \ldots, u_{m^+}, y_1, \ldots, y_k, z_1, \ldots, z_l, w_1, \ldots, w_l),\end{aligned}$$

and the problem (27), (36) *with the boundary conditions* (P) *has a solution, with* k^- *free parameters, for all* h_0, *if*

$$\begin{aligned}&\dim \mathcal{C} X_2 = n^+,\\ &X_2 = span(u \mid Lu = \lambda Bu, \ with \ \lambda \geq 0)\\ &\quad = span(u_1, \ldots, u_{m^+}, y_1, \ldots, y_k, z_1, \ldots, z_l).\end{aligned}$$

6 - Inhomogeneous problem

We now consider the inhomogeneous linearized problem (32), where $g = g(x) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ with the boundary conditions (36).

The formal solution of the system (32) reads

$$f(x) = \sum_{i=1}^{k} \mu_i(x) y_i + \sum_{j=1}^{l} \big(\eta_j(x) z_j + a_j(x)(w_j - x z_j)\big) + \sum_{r=1}^{q} \beta_r(x) u_r,$$

where

$$\mu_i(x) = \mu_i(0) + \int_0^x \widetilde{\mu}_i(\sigma)\,d\sigma,\ a_j(x) = a_j(0) + \int_0^x \widetilde{a}_j(\sigma)\,d\sigma,$$

$$\eta_j(x) = \eta_j(0) + \int_0^x \widetilde{\eta}_j(\sigma)\,d\sigma \text{ and } \beta_r(x) = \beta_r(0)e^{-\lambda_r x} + \int_0^x e^{(\sigma-x)\lambda_r}\widetilde{\beta}_r(\sigma)\,d\sigma,$$

with (in the notations of (29)-(31))

$$\widetilde{\mu}_i(x) = \frac{\langle g(x), y_i\rangle}{\langle y_i, y_i\rangle_B},\quad \widetilde{a}_j(x) = \langle g(x), z_j\rangle,\quad \widetilde{\eta}_j(x) = \langle g(x), w_j\rangle$$
$$\text{and } \widetilde{\beta}_r(x) = \frac{\langle g(x), u_r\rangle}{\lambda_r}. \tag{54}$$

First we assume that the boundary conditions (O) are fulfilled, and that $g(x) \in N(L)^\perp$ for all $x \in \mathbb{R}_+$. The system (32) with the boundary conditions (O), has the general solution

$$f(x) = \sum_{j=1}^{l} \eta_j(x)z_j + \sum_{r=1}^{q} \beta_r(x)u_r,$$

where

$$\begin{cases} \eta_j(x) = -\int_x^\infty \widetilde{\eta}_j(\sigma)\,d\sigma \quad \text{for } j = 1, \dots, l \\ \beta_r(x) = \beta_r(0)e^{-\lambda_r x} + \int_0^x e^{(\sigma-x)\lambda_r}\widetilde{\beta}_r(\sigma)\,d\sigma \ \text{ for } r = 1, \dots, m^+ \\ \beta_r(x) = -\int_x^\infty e^{(\sigma-x)\lambda_r}\widetilde{\beta}_r(\sigma)\,d\sigma \ \text{ for } r = m^+ + 1, \dots, q \end{cases}$$

with $\widetilde{\eta}_1(x), \dots, \widetilde{\eta}_l(x)$ and $\widetilde{\beta}_1(x), \dots, \widetilde{\beta}_q(x)$ given by Eqs. (54).

If we take the boundary conditions (36) in consideration, we obtain the linear system

$$\mathcal{C}\sum_{r=1}^{m^+} \beta_{r,0}u_r = h_0 + \mathcal{C}\int_0^\infty \sum_{r=m^++1}^{q} e^{\sigma\lambda_r}\widetilde{\beta}_r(\sigma)u_r + \sum_{j=1}^{l} \widetilde{\eta}_j(\sigma)z_j\,d\sigma, \tag{55}$$

with $\mathcal{C} = R_+ - CR_-$, for the unknown parameters $\beta_{1,0}, \dots, \beta_{m^+,0}$ ($\beta_{r,0} = \beta_r(0)$). The system (55) has a solution if the conditions

$$h_0, \mathcal{C}e^{xB^{-1}L}B^{-1}g(x) \in \mathcal{C}U_+ \quad \text{for all } x \in \mathbb{R}_+,$$

$$\text{where} \quad U_+ = \text{span}(u \mid Lu = \lambda Bu, \text{ with } \lambda > 0) = \text{span}(u_1, \ldots, u_{m^+}),$$

are fulfilled and a unique solution if and only if, additionally, condition (37) is fulfilled, i.e. if and only if $\dim \mathcal{C}U_+ = m^+$.

This proves part (*i*) of Theorem 3.1.

Now we assume that the boundary conditions (Q) are fulfilled. The solution of the system (32) with the boundary conditions (Q) reads

$$f(x) = \sum_{i=1}^{k} \mu_i(x)y_i + \sum_{j=1}^{l} \big(\eta_j(x)z_j + a_j(x)(w_j - xz_j)\big) + \sum_{r=1}^{q} \beta_r(x)u_r,$$

where

$$\begin{cases} \mu_i(x) = \mu_i(0) + \int\limits_0^x \widetilde{\mu}_i(\sigma)\, d\sigma \ \text{ for } i = 1, \ldots, k \\ a_j(x) = a_j(0) + \int\limits_0^x \widetilde{a}_j(\sigma)\, d\sigma \quad \text{for } j = 1, \ldots, l \\ \eta_j(x) = \eta_j(0) + \int\limits_0^x \widetilde{\eta}_j(\sigma)\, d\sigma \quad \text{for } j = 1, \ldots, l \\ \beta_r(x) = \beta_r(0)e^{-\lambda_r x} + \int\limits_0^x e^{(\sigma - x)\lambda_r}\widetilde{\beta}_r(\sigma)\, d\sigma \quad \text{for } r = 1, \ldots, m^+ \\ \beta_r(x) = -\int\limits_x^\infty e^{(\sigma - x)\lambda_r}\widetilde{\beta}_r(\sigma)\, d\sigma \ \text{ for } r = m^+ + 1, \ldots, q \end{cases}$$

with $\widetilde{\mu}_1(x), \ldots, \widetilde{\mu}_k(x), \widetilde{a}_1(x), ..., \widetilde{a}_l(x), \widetilde{\eta}_1(x), \ldots, \widetilde{\eta}_l(x)$, and $\widetilde{\beta}_1(x), \ldots, \widetilde{\beta}_q(x)$ given by Eqs. (54).

We denote

$$f_\infty = \lim_{x \to \infty} (f(x) - x\frac{df(x)}{dx}) = \sum_{i=1}^{k} \mu_i^\infty y_i + \sum_{j=1}^{l} (\eta_j^\infty z_j + a_j^\infty w_j),$$

where

$$\mu_i^\infty = \mu_i(0) + \int_0^\infty \widetilde{\mu}_i(\sigma)\, d\sigma, \quad a_j^\infty = a_j(0) + \int_0^\infty \widetilde{a}_j(\sigma)\, d\sigma \quad \text{and}$$

$$\eta_j^\infty = \eta_j(0) + \int_0^\infty \widetilde{\eta}_j(\sigma)\, d\sigma.$$

Furthermore, we prescribe the $k^- + l$ parameters

$$\mu_i^\infty = \frac{\langle f_\infty, y_i \rangle_B}{\langle y_i, y_i \rangle_B}, \quad i = k^+ + 1, \dots, k, \tag{56}$$

and

$$a_j^\infty = \langle f_\infty, z_j \rangle_B , \quad j = 1, \dots, l, \tag{57}$$

and assume that the boundary conditions (36) are fulfilled. We remind the notation $\mathcal{C} = R_+ - CR_-$ and obtain the linear system

$$\begin{aligned} &\sum_{r=1}^{m^+} \beta_{r,0} \mathcal{C} u_r + \sum_{i=1}^{k^+} \mu_{i,0} \mathcal{C} y_i + \sum_{j=1}^{l} \eta_{j,0} \mathcal{C} z_j \\ &= h_0 + \int_0^\infty \sum_{r=m^++1}^{q} e^{\sigma \lambda_r} \widetilde{\beta}_r(\sigma) \mathcal{C} u_r \, d\sigma - \sum_{j=1}^{l} a_j(0) \mathcal{C} w_j - \sum_{i=k^++1}^{k} \mu_i(0) \mathcal{C} y_i \end{aligned} \tag{58}$$

for the unknown parameters $\mu_{1,0}, \dots, \mu_{k^+,0}, \beta_{1,0}, \dots, \beta_{m^+,0}$, and $\eta_{1,0}, \dots, \eta_{l,0}$ ($\mu_{i,0} = \mu_i(0)$, $\beta_{r,0} = \beta_r(0)$, $\eta_{j,0} = \eta_j(0)$), with $m^+ + k^+ + l = n^+$.

The system (58) has a unique solution for all h_0 , if condition (38) is fulfilled.

We now have proved part (iii) of Theorem 3.1.

The result can be used, if we prescribe $a_j^\infty = 0$ for $j = 1, \dots, l$, to conclude that the problem (32),(36) with the boundary conditions (P), has a unique solution for all h_0, if the condition (38) is fulfilled and the parameters (56) are prescribed. This proves part (ii) of Theorem 3.1.

We still assume that the boundary conditions (Q) and (36) are fulfilled and prescribe the parameters (56), but instead of prescribing the parameters (57), we prescribe

$$\vartheta_j^\infty = \eta_j^\infty - a_j^\infty = \langle f_\infty, w_j - z_j \rangle_B, \quad j = 1, \dots, l.$$

Then we obtain the linear system

$$\begin{aligned} &\sum_{r=1}^{m^+} \beta_{r,0} \mathcal{C} u_r + \sum_{i=1}^{k^+} \mu_{i,0} \mathcal{C} y_i + \sum_{j=1}^{l} \varpi_j \mathcal{C} \frac{z_j + w_j}{2} \\ &= h_0 + \int_0^\infty \sum_{r=m^++1}^{q} e^{\sigma \lambda_r} \widetilde{\beta}_r(\sigma) \mathcal{C} u_r \, d\sigma - \sum_{i=k^++1}^{k} \mu_i(0) \, \mathcal{C} y_i - \sum_{j=1}^{l} \vartheta_j(0) \, \mathcal{C} \, \frac{z_j - w_j}{2} \end{aligned} \tag{59}$$

for the unknown parameters $\mu_{1,0}, \dots, \mu_{k^+,0}$, $\beta_{1,0}, \dots, \beta_{m^+,0}$, $\varpi_1, \dots, \varpi_l$, ($\mu_{i,0} = \mu_i(0)$, $\beta_{r,0} = \beta_r(0)$, $\varpi_j = \eta_j(0) + a_j(0)$), with $m^+ + k^+ + l = n^+$.

The system (59) has a unique solution, if the condition (39) is fulfilled. This proves part (iv) of Theorem 3.1, and Theorem 3.1 is proved.

7 - Asymptotic flow

We consider the problem

$$\begin{cases} B\dfrac{df}{dx} + Lf = 0 \\ f^+(0) = f_0 \end{cases}, \tag{60}$$

where $f_0 \in \mathbb{R}^{n^+}$.

If we consider the problem (60) together with the boundary conditions (Q), (50) and (51) at infinity, then the asymptotic flow is given by

$$\begin{aligned} f_A(x) &= \sum_{i=1}^{k} \mu_i y_i + \sum_{j=1}^{l} \left[\eta_j z_j + a_j\left(w_j - x z_j\right)\right] \\ &= \sum_{i=1}^{k^+} \mu_i y_i + \sum_{j=1}^{l} \eta_j z_j + \sum_{i=1}^{k^-} \theta_i \omega_i + \sum_{j=1}^{l} a_j\left(w_j - x z_j\right), \end{aligned}$$

where $\theta_i = \mu_{k^+ + i}$ and $\omega_i = y_{k^+ + i}$, for $i = 1, \dots, k^-$. The coefficients $\theta_1, \dots, \theta_{k^-}$, and $a_1, \dots, a_l$, are all prescribed, but what can we say about the coefficients $\mu_1, \dots, \mu_{k^+}$, and $\eta_1, \dots, \eta_l$?

The boundary conditions $f^+(0) = f_0$ lead to the system

$$f_0 = \sum_{i=1}^{k^+} \mu_i y_i^+ + \sum_{j=1}^{l} \eta_j z_j^+ + \sum_{r=1}^{m^+} \beta_r u_r^+ + \sum_{i=1}^{k^-} \theta_i \omega_i^+ + \sum_{j=1}^{l} a_j w_j^+$$

or, equivalently,

$$\widetilde{f}_0 = \sum_{i=1}^{k^+} \mu_i y_i^+ + \sum_{j=1}^{l} \eta_j z_j^+ + \sum_{r=1}^{m^+} \beta_r u_r^+, \tag{61}$$

with

$$\widetilde{f}_0 = f_0 - \sum_{i=1}^{k^-} \theta_i \omega_i^+ - \sum_{j=1}^{l} a_j w_j^+. \tag{62}$$

Eqs. (61), (62) have a unique solution for unknown $\mu_1, \dots, \mu_{k^+}$, $\eta_1, \dots, \eta_l$, and $\beta_1, \dots, \beta_{m^+}$, by Corollary 3.1.

We can assume that $\theta_1 = \dots = \theta_{k^-} = a_1 = \dots = a_l = 0$ (otherwise we just substitute $\widetilde{f} = f - \sum\limits_{i=1}^{k^-} \theta_i \omega_i - \sum\limits_{j=1}^{l} a_j w_j$). Let

$$\mathcal{X} = \operatorname{span}(u_1, \dots, u_{m^+}) = \operatorname{span}(u \mid Lu = \lambda Bu, \text{ with } \lambda > 0) \text{ and}$$

$$\mathcal{Y} = \operatorname{span}(y_1, \dots, y_{k^+}, z_1, \dots, z_l).$$

Then, by Corollary 3.1, $R_+\mathcal{X} \oplus R_+\mathcal{Y} = \mathbb{R}^{n^+}$ and hence, $f_0 = u^+ + y^+$ for some $u \in \mathcal{X}$ and $y \in \mathcal{Y}$. The asymptotic flow is $y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix}$, where $y \in \mathcal{Y}$. Furthermore, $f(0) = u + y$, where $u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in \mathcal{X}$ (u and y are uniquely determined, for a given f_0). The problem is (at least for large n) that we normally need to know $R_+\mathcal{X}$, to be able to find y. If, as in Section 10 , $\{y_1^+, \dots, y_{k^+}^+, z_1^+, \dots, z_l^+\}$ is orthogonal to the subspace

$$\mathcal{U} = \text{span}\big(B_+u_1^+, \dots, B_+u_{m^+}^+\big)$$

of $\mathbb{R}^{n^+}$, then we can obtain the asymptotic flow, without knowing $R_+\mathcal{X}$.

8 - Extension to singular operators B

To study the case when the operator B is singular (i.e. the case when $\xi_i^1 = 0$ for some i) we assume (cf. [**26**]) that

$$N(L) \cap N(B) = \{0\}, \tag{63}$$

and introduce the orthogonal projections

$$P_0 : \mathbb{R}^n \to N(B) \;\text{ and }\; P_1 : \mathbb{R}^n \to \text{Im}(B).$$

The assumption (63) ensures that the operator P_0LP_0 is non-singular on $N(B)$.

The system (32) (with $g(x) = 0$ in the homogeneous case) is equivalent with the system

$$\begin{cases} P_0LP_0f + P_0LP_1f = P_0g(x) \\ P_1BP_1\dfrac{df}{dx} + P_1LP_1f + P_1LP_0f = P_1g(x) \end{cases},$$

or, equivalently,

$$\begin{cases} P_0f = -(P_0LP_0)^{-1}P_0LP_1f + (P_0LP_0)^{-1}P_0g(x) \\ \widetilde{B}\,\dfrac{dP_1f}{dx} + \widetilde{L}P_1f = \widetilde{g}(x) \end{cases},$$

where

$$\widetilde{L} = P_1L(I - P_0(P_0LP_0)^{-1}P_0L)P_1 \;,\; \widetilde{B} = P_1BP_1 \;\text{ and}$$

$$\widetilde{g}(x) = P_1(I - LP_0(P_0LP_0)^{-1}P_0)g(x).$$

The restrictions, $\widetilde{L}_{\text{Im}}$ and $\widetilde{B}_{\text{Im}}$, of $\widetilde{L}$ and $\widetilde{B}$ to $\text{Im}(B)$, are linear operators $\big(\widetilde{n} \times \widetilde{n}$ matrices, with $\widetilde{n} = n - \dim(N(B))\big)$ on $\text{Im}(B)$.

Lemma 8.1. *Let $g(x) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ and assume that*

$$N(L) \cap N(B) = \{0\}.$$

Then the linear operators $\widetilde{L}_{\mathrm{Im}}$ and $\widetilde{B}_{\mathrm{Im}}$ on $\mathrm{Im}(B)$ have the following properties: $\widetilde{L}_{\mathrm{Im}}$ and $\widetilde{B}_{\mathrm{Im}}$ are real symmetric operators, $\widetilde{L}_{\mathrm{Im}}$ is semi-positive, $\widetilde{B}_{\mathrm{Im}}$ is non-singular, $\dim(N(\widetilde{L}_{\mathrm{Im}})) = p$ and the numbers k^+, k^- and l are the same for the system

$$\widetilde{B}_{\mathrm{Im}} \frac{dP_1 f}{dx} + \widetilde{L}_{\mathrm{Im}} P_1 f = \widetilde{g}(x), \tag{64}$$

as for the original system (32). Furthermore, $\widetilde{g}(x) \in L^1(\mathbb{R}_+, \mathrm{Im}(B))$.

Proof. It is obvious that the operators $\widetilde{L}$ and $\widetilde{B}$ are real and symmetric and that $\widetilde{B}$ is non-singular on $\mathrm{Im}(B)$. Hence, this is true also for the restrictions to $\mathrm{Im}(B)$. The linear operator $\widetilde{L}_{\mathrm{Im}}$ is semi-positive, since

$$\begin{aligned} 0 &= \left\langle (P_0L - P_0LP_0(P_0LP_0)^{-1}P_0L)P_1h, (P_0LP_0)^{-1}P_0LP_1h \right\rangle \\ &= \left\langle L(I - P_0(P_0LP_0)^{-1}P_0L)P_1h, P_0(P_0LP_0)^{-1}P_0LP_1h \right\rangle \end{aligned}$$

if $h \in \mathrm{Im}(B)$, and hence,

$$\begin{aligned} \left\langle \widetilde{L}_{\mathrm{Im}}h, h \right\rangle &= \left\langle \widetilde{L}h, h \right\rangle = \left\langle L(I - P_0(P_0LP_0)^{-1}P_0L)P_1h, P_1h \right\rangle \\ &= \left\langle L(I - P_0(P_0LP_0)^{-1}P_0L)P_1h, (I - P_0(P_0LP_0)^{-1}P_0L)P_1h \right\rangle \geq 0 \end{aligned}$$

for all $h \in \mathrm{Im}(B)$. By assumption (63), $\dim(N(\widetilde{L})) = \dim(N(L)) = p$ and $N(\widetilde{L}) \subseteq P_1N(L)$, since

$$\begin{aligned} \widetilde{L}P_1h &= P_1L(I - P_0(P_0LP_0)^{-1}P_0L)P_1h \\ &= P_1L(P_1 + P_0(P_0LP_0)^{-1}P_0LP_0)h = P_1L(P_1 + P_0)h = 0 \end{aligned}$$

if $Lh = L(P_0 + P_1)h = 0$, for $h \in \mathbb{R}^n$. Hence,

$$N(\widetilde{L}_{\mathrm{Im}}) = N(\widetilde{L}) = P_1N(L).$$

Furthermore, the numbers k^+, k^- and l are the same for the system (64) as for the original system, since

$$\left\langle \widetilde{B}_{\mathrm{Im}}P_1h, P_1h \right\rangle = \langle BP_1h, P_1h \rangle = \langle Bh, h \rangle$$

for all $h \in \mathbb{R}^n$. Clearly, $\widetilde{g}(x) \in L^1(\mathbb{R}_+, \mathrm{Im}(B))$, if $g(x) \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ and the lemma is proved.

9 - Axially symmetric DVMs

In this section we consider only such symmetric sets of velocities V, such that

$$\text{if } \xi_i = (\xi_i^1, \ldots, \xi_i^d) \in \mathrm{V}, \text{ then } (\pm\,\xi_i^1, \ldots, \pm\,\xi_i^d) \in \mathrm{V} \tag{65}$$

for any combinations of signs. We can, without loss of generality, assume that

$$(\xi_{i+N}^1, \xi_{i+N}^2, \ldots, \xi_{i+N}^d) = (-\,\xi_i^1, \xi_i^2, \ldots, \xi_i^d) \text{ and } \xi_i^1 > 0,$$

for $i = 1, \ldots, N$, with $n = 2N$.

EXAMPLE 9.1 – *The plane* 12*-velocity model* [**12**]*, with velocities* $(\pm 1, \pm 1)$*,* $(\pm 1, \pm 3)$ *and* $(\pm 3, \pm 1)$*, and the infinitely many* (*obvious, from the constructions in* [**12**] – *"with three corners of a square in the model, add the fourth"*) *symmetric normal extensions of this model are examples of* (*normal*) *such DVMs. These extensions include the plane square models, with* (*all combinations of*) *coordinates from the set of all odd integers with absolute values less or equal than a maximal odd integer* (*these models are called Nicodin p-th squares in* [**20**]*, but are also, at least implicitly, constructed in* [**12**]).

EXAMPLE 9.2 – *In three dimensions the* 32*-velocity model, with velocities* $(\pm 1, \pm 1, \pm 1)$*,* $(\pm 1, \pm 1, \pm 3)$*,* $(\pm 1, \pm 3, \pm 1)$ *and* $(\pm 3, \pm 1, \pm 1)$*, and the infinitely many* (*obvious*) *symmetric normal extensions of this model are examples of such* (*normal*) *DVMs. These extensions include the cubic models, with* (*all combinations of*) *coordinates from the set of all odd integers with absolute values less or equal than a maximal odd integer. The* 32*-velocity model can be obtained by normal extensions, with the starting point in the* 9*-velocity asymmetric normal model with velocities* $(\pm 1, \pm 1, \pm 1)$ *and* $(3, -1, 1)$*. The* 24*-velocity models, with velocities* $(\pm 1, \pm 1, \pm 1)$*,* $(\pm 1, \pm 3, \pm 1)$ *and* $(\pm 3, \pm 1, \pm 1)$*, and* $(\pm 1, \pm 1, \pm 1)$*,* $(\pm 1, \pm 1, \pm 3)$ *and* $(\pm 3, \pm 1, \pm 1)$*, respectively, are DVMs with fewer velocities* (*earlier in the "evolution"*)*, that can be constructed from the same asymmetric model.*

9.1 - *Explicit calculation of the characteristic numbers*

We assume now that (i) we have a symmetric set (65) of velocities; (ii) our DVM is normal; (iii) we have made the expansion (18) around a non-drifting Maxwellian M, i.e. with $\boldsymbol{b} = \mathbf{0}$ in Eq. (14); and (iv)

$$B = \mathrm{diag}(\xi_1^1, \ldots, \xi_N^1, -\xi_1^1, \ldots, -\xi_N^1), \quad \text{with } \xi_1^1, \ldots, \xi_N^1 > 0.$$

Here we study, instead of Eq. (27), the equation

$$(B+uI)\frac{df}{dx}+Lf=0, \tag{66}$$

(cf. Eq. (3)). Note, however, that Eqs. (27), (66) are never equivalent for non-zero u, as Eqs. (2), (3) are in the continuous case, for DVMs with a finite number of velocities.

The linearized collision operator L has the null-space

$$N(L)=\operatorname{span}(\phi_1,\dots,\phi_{d+2}),$$

where

$$\begin{cases} \phi_1 &= M^{1/2}\cdot(1,\dots,1) \\ \phi_2 &= M^{1/2}\cdot(\xi_1^1,\dots,\xi_N^1,-\xi_1^1,\dots,-\xi_N^1) \\ \phi_3 &= M^{1/2}\cdot(|\xi_1|^2,\dots,|\xi_N|^2,|\xi_1|^2,\dots,|\xi_N|^2) \\ \phi_{i+2} &= M^{1/2}\cdot(\xi_1^i,\dots,\xi_N^i,-\xi_1^i,\dots,-\xi_N^i),\ \ i=2,\dots,d \end{cases}. \tag{67}$$

Then

$$K=\begin{pmatrix} u\chi_1 & \chi_2 & u\chi_3 & & & \\ \chi_2 & u\chi_2 & \chi_4 & & & \\ u\chi_3 & \chi_4 & u\chi_5 & & & \\ & & & u\chi_6 & & \\ & & & & \ddots & \\ & & & & & u\chi_{d+4} \end{pmatrix},$$

where $K=(\langle\phi_i,\phi_j\rangle_{B+uI})$, $\chi_1=\langle\phi_1,\phi_1\rangle$, $\chi_2=\langle\phi_2,\phi_2\rangle$, $\chi_3=\langle\phi_1,\phi_3\rangle$, $\chi_4=\langle\phi_2,\phi_3\rangle_B$, $\chi_5=\langle\phi_3,\phi_3\rangle$ and $\chi_{i+4}=\langle\phi_i,\phi_i\rangle$, $i=2,\dots,d$. Hence,

$$\det(K)=\chi_6\cdots\chi_{d+4}u^d(u^2(\chi_1\chi_2\chi_5-\chi_2\chi_3^2)-(\chi_1\chi_4^2+\chi_2^2\chi_5-2\chi_2\chi_3\chi_4)),$$

and the degenerate values of u, i.e. the values of u for which $l\geq 1$, are

$$u_0=0 \text{ and } u_\pm=\pm\sqrt{\frac{\chi_1\chi_4^2+\chi_2^2\chi_5-2\chi_2\chi_3\chi_4}{\chi_2(\chi_1\chi_5-\chi_3^2)}}\ . \tag{68}$$

Note that, by the Cauchy-Schwarz inequality, $\chi_1\chi_5-\chi_3^2>0$, and hence,

$$\chi_1\chi_4^2+\chi_2^2\chi_5-2\chi_2\chi_3\chi_4=\frac{(\chi_1\chi_4-\chi_2\chi_3)^2+\chi_2^2(\chi_1\chi_5-\chi_3^2)}{\chi_1}>0\ .$$

Moreover, by denoting

$$\begin{cases} \varphi_1 = \sqrt{\chi_2}(\chi_2\chi_5 - \chi_3\chi_4)\phi_1 + \sqrt{\Upsilon}\phi_2 + \sqrt{\chi_2}(\chi_1\chi_4 - \chi_2\chi_3)\phi_3 \\ \varphi_2 = \sqrt{\chi_2}(\chi_2\chi_5 - \chi_3\chi_4)\phi_1 - \sqrt{\Upsilon}\phi_2 + \sqrt{\chi_2}(\chi_1\chi_4 - \chi_2\chi_3)\phi_3 \, , \\ \varphi_3 = \chi_4\phi_1 - \chi_2\phi_3 \end{cases}$$

with $\Upsilon = (\chi_1\chi_4^2 + \chi_2^2\chi_5 - 2\chi_2\chi_3\chi_4)(\chi_1\chi_5 - \chi_3^2)$, we obtain

$$\widetilde{K} = \frac{\Upsilon}{\chi_1\chi_5 - \chi_3^2}\begin{pmatrix} u2\chi_2(\chi_1\chi_5 - \chi_3^2) - 2\sqrt{\chi_2\Upsilon} & 0 & 0 \\ 0 & u2\chi_2(\chi_1\chi_5 - \chi_3^2) + 2\sqrt{\chi_2\Upsilon} & 0 \\ 0 & 0 & u \end{pmatrix},$$

where $\widetilde{K} = (\langle\varphi_i, \varphi_j\rangle_{B+uI})$. We obtain the following table for the values of k^+, k^- and l:

	$u < u_-$	$u = u_-$	$u_- < u < 0$	$u = 0$	$0 < u < u_+$	$u = u_+$	$u_+ < u$
k^+	0	0	1	1	$d+1$	$d+1$	$d+2$
k^-	$d+2$	$d+1$	$d+1$	1	1	0	0
l	0	1	0	d	0	1	0

EXAMPLE 9.3 – *The degenerate values of* u *for the* 12*-velocity model in Example* 9.1 *are*

$$u = 0 \text{ and } u = \pm\sqrt{\frac{1+50s^2}{1+10s^2}}\,,$$

and for the 32*-velocity model in Example* 9.2 *the degenerate values of* u *are*

$$u = 0 \text{ and } u = \pm\sqrt{\frac{3+121s^2}{3+33s^2}}\,,$$

Remark 9.1. *For the continuous Boltzmann equation* (*with* $d = 3$) *the numbers* $\chi_1, \dots, \chi_5$ *are given by*

$$\chi_1 = \rho,\ \ \chi_2 = \rho T,\ \ \ \chi_3 = 3\rho T,\ \ \ \chi_4 = 5\rho T^2 \ \ \textit{and}\ \ \chi_5 = 15\rho T^2,$$

(*where* ρ *and* T *denote the density and the temperature respectively*), *if we have made the expansion* (18) *around a non-drifting Maxwellian*

$$M = \frac{\rho}{(2\pi T)^{3/2}} e^{-|\xi|^2/2T}.$$

Therefore, for the Boltzmann equation (*with* $d = 3$) *the degenerate values* (68) *are* (cf. [**21**])

$$u_0 = 0 \text{ and } u_\pm = \pm\sqrt{\frac{5T}{3}}\,.$$

9.2 - *Characteristic numbers and Jordan normal form in a special case*

Now, additionally to assumptions (i)-(iv) above, we assume that (v) the coefficients Γ_{ij}^{kl} in Eq. (7) satisfy the additional symmetric conditions

$$\Gamma_{ij}^{kl} = \Gamma_{\pi(i)\pi(j)}^{\pi(k)\pi(l)},$$

where $\pi(i) = \begin{cases} i+N, & \text{if } 1 \leq i \leq N \\ i-N, & \text{if } N+1 \leq i \leq 2N \end{cases}$. We also let $u = u_0 = 0$, i.e., we return to study Eq. (27).

Then $L = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix}$, where L_1 and L_2 are two $N \times N$ matrices (cf. [**2**], [**3**], [**4**]). We choose

$$\begin{cases} \varphi_1 = \phi_2 + \phi_3 \\ \varphi_2 = \phi_2 - \phi_3 \\ \varphi_3 = \chi_4\phi_1 - \chi_2\phi_3 \end{cases}$$

where $\chi_2 = \langle \phi_2, \phi_2 \rangle$, $\chi_4 = \langle \phi_2, \phi_3 \rangle_B$ and ϕ_1, ϕ_2, ϕ_3 are given in Eq. (67). Then

$$K = 2\chi_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $K = (\left\langle \varphi_i, \varphi_j \right\rangle_B)$. Hence, $k^+ = k^- = 1$ and $l = d$, since $\phi_4, \ldots, \phi_{d+2}$ are all orthogonal, with respect to the scalar product $\langle \cdot, \cdot \rangle_B$, to ϕ_1, ϕ_2 and ϕ_3. Since $B^{-1}L\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \lambda \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$, with $u^+, u^- \in \mathbb{R}^N$, implies that $B^{-1}L\begin{pmatrix} u^- \\ u^+ \end{pmatrix} = -\lambda \begin{pmatrix} u^- \\ u^+ \end{pmatrix}$, we obtain (cf. [**2**], [**3**], [**4**]) that the non-negative eigenvalues of $B^{-1}L$ are $\pm\lambda_1, \ldots, \pm\lambda_{N-l-1}$, where $\lambda_i > 0$, with corresponding eigenvectors $u_1^\pm, \ldots, u_{N-l-1}^\pm$, where $R_+u_i^- = R_-u_i^+$ and $R_-u_i^- = R_+u_i^+$.

Therefore, the Jordan normal form of $B^{-1}L$ for $d = 2$ is (the number of blocks $\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$ is equal to the dimension d, that is, in this case 2)

$$\begin{pmatrix} \lambda_1 & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & \lambda_{m^+} & & & & & & & & & \\ & & & -\lambda_1 & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & -\lambda_{m^+} & & & & & & \\ & & & & & & 0 & & & & & \\ & & & & & & & 0 & & & & \\ & & & & & & & & 0 & 1 & & \\ & & & & & & & & 0 & 0 & & \\ & & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 & 0 \end{pmatrix}.$$

9.3 - *Plane* 12*-velocity model*

For $d=2$ the equations (5) admit a class of solutions satisfying

$$F_i = F_{i'} \quad \text{if} \quad \xi_i^1 = \xi_{i'}^1 \quad \text{and} \quad |\xi_i|^2 = |\xi_{i'}|^2. \tag{69}$$

This reduces the number n of equations (5) to the number $2N < n$ of different combinations $(\xi_i^1, |\xi_i|^2)$ in the velocity set. However, the structure of the collision terms (7) (in slightly different notations) remains unchanged. We can, without loss of generality, assume that

$$(\xi_{i+N}^1, |\xi_{i+N}|^2) = (-\xi_i^1, |\xi_i|^2) \text{ and } \xi_i^1 > 0$$

for $i = 1, \dots, N$. Then, the Maxwellians are of the form

$$M_i = Ae^{b\xi_i^1 + c|\xi_i|^2} = M_{i+N}e^{2b\xi_i^1}, \quad i = 1, \dots, N, \tag{70}$$

for some constant $A, b, c \in \mathbb{R}$, with $A > 0$.

For the 12-velocity model in Example 9.1, the system (5) reduces by reduction (69) to a system of the form

$$\begin{cases} \dfrac{dF_1}{dx} = \sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3 \\ \dfrac{dF_2}{dx} = \sigma_1 q_1 - \sigma_2 q_2 + \sigma_4 q_4 \\ 3\dfrac{dF_3}{dx} = -(\sigma_1 q_1 + \sigma_4 q_4) \\ -\dfrac{dF_4}{dx} = -(\sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3) \\ -\dfrac{dF_5}{dx} = \sigma_2 q_2 - \sigma_3 q_3 + \sigma_4 q_4 \\ -3\dfrac{dF_6}{dx} = \sigma_3 q_3 - \sigma_4 q_4 \end{cases},$$

where $q_1 = F_3F_4 - F_1F_2$, $q_2 = F_2F_4 - F_1F_5$, $q_3 = F_4F_5 - F_1F_6$, $q_4 = F_3F_6 - F_2F_5$ and $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \geq 0$.

The Maxwellians are of the form

$$M = K(r^2, s^2r^2, s^2, r^4, s^2r^4, s^2r^6), \tag{71}$$

where $r = \sqrt{\dfrac{M_4}{M_1}} = e^{-b}$, $s = \sqrt{\dfrac{M_2}{M_1}} = e^{4c}$, and $K = Ae^{3b+2c}$, b, c and A are constant, with $A > 0$. The null-space of L is given by

$$N(L) = \operatorname{span}(\phi_1, \phi_2, \phi_3),$$

where

$$\begin{cases} \phi_1 = M^{1/2} \cdot (1,1,1,1,1,1) = K^{1/2}(r, sr, s, r^2, sr^2, sr^3) \\ \phi_2 = M^{1/2} \cdot (1,1,3,-1,-1,-3) = K^{1/2}\left(r, sr, 3s, -r^2, -sr^2, -3sr^3\right), \\ \phi_3 = M^{1/2} \cdot (2,10,10,2,10,10) = 2K^{1/2}\left(r, 5sr, 5s, r^2, 5sr^2, 5sr^3\right) \end{cases}$$

and

$$B = \mathrm{diag}\,(1,1,3,-1,-1,-3).$$

Remark 9.2. *The degenerate values of b, i.e. the values of b for which $l \geq 1$, are*

$$b = 0 \ \ \textit{and} \ \ b = \pm \frac{\ln\left(S + \sqrt{S^2 - 1}\right)}{2},$$

$$\textit{with} \ \ S = \frac{1}{4s^2}\left(1 + 4s^2 + \sqrt{1 + \frac{40}{3}s^2(1 + 10s^2)}\right).$$

We assume below that we have linearized around a non-drifting Maxwellian, i.e. with $r = 1$ in Eq. (71),

$$M = K(1, s^2, s^2, 1, s^2, s^2),$$

where $s = e^{4c}$ and $K = Ae^{2c}$, c and A are constant, with $A > 0$. Then

$$\begin{cases} \phi_1 = K^{1/2}(1, s, s, 1, s, s) \\ \phi_2 = K^{1/2}(1, s, 3s, -1, -s, -3s) \\ \phi_3 = 2K^{1/2}(1, 5s, 5s, 1, 5s, 5s) \end{cases}$$

and

$$(\langle \phi_i, \phi_j \rangle_B) = 2K \begin{pmatrix} 0 & \chi_2 & 0 \\ \chi_2 & 0 & \chi_3 \\ 0 & \chi_3 & 0 \end{pmatrix},$$

where $\chi_1 = 1 + 2s^2$, $\chi_2 = 1 + 10s^2$ and $\chi_3 = 2(1 + 50s^2)$.

A typical choice of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ (cf. [**16**], [**23**]) is

$$\begin{cases} \sigma_1 = \sigma_3 = 2S \\ \sigma_2 = S(\sqrt{2} + \sqrt{5}). \\ \sigma_4 = S\sqrt{10} \end{cases}$$

Therefore, we assume below that $\sigma_1 = \sigma_3, \sigma_2, \sigma_4 > 0$. Then

$$L = \left(\begin{array}{ccc} (2\sigma_1+\sigma_2)s^2 & (\sigma_1-\sigma_2)s & -\sigma_1 s \\ (\sigma_1-\sigma_2)s & \sigma_1+\sigma_2+\sigma_4 s^2 & -\left(\sigma_1+\sigma_4 s^2\right) \\ -\sigma_1 s & -\left(\sigma_1+\sigma_4 s^2\right) & \sigma_1+\sigma_4 s^2 \\ -(2\sigma_1+\sigma_2)s^2 & (\sigma_2-\sigma_1)s & \sigma_1 s \\ (\sigma_2-\sigma_1)s & -\sigma_2+\sigma_4 s^2 & -\sigma_4 s^2 \\ \sigma_1 s & -\sigma_4 s^2 & \sigma_4 s^2 \end{array}\right.$$

$$\left.\begin{array}{ccc} -(2\sigma_1+\sigma_2)s^2 & (\sigma_2-\sigma_1)s & \sigma_1 s \\ (\sigma_2-\sigma_1)s & -\sigma_2+\sigma_4 s^2 & -\sigma_4 s^2 \\ \sigma_1 s & -\sigma_4 s^2 & \sigma_4 s^2 \\ (2\sigma_1+\sigma_2)s^2 & (\sigma_1-\sigma_2)s & -\sigma_1 s \\ (\sigma_1-\sigma_2)s & \sigma_1+\sigma_2+\sigma_4 s^2 & -(\sigma_1+\sigma_4 s^2) \\ -\sigma_1 s & -(\sigma_1+\sigma_4 s^2) & \sigma_1+\sigma_4 s^2 \end{array}\right),$$

and if we denote

$$\begin{cases} y_1 = (5\phi_1+\phi_2-\dfrac{\phi_3}{2})K^{-1/2} = (5, s, 3s, 3, -s, -3s) \\ y_2 = (5\phi_1-\phi_2-\dfrac{\phi_3}{2})K^{-1/2} = (3, -s, -3s, 5, s, 3s) \\ z \ = \left(\dfrac{\chi_3\phi_1-\chi_2\phi_3}{8s}\right)K^{-1/2} = (10s, -1, -1, 10s, -1, -1) \\ w \ = \left(0, -\dfrac{2}{\sigma_2}, -\dfrac{3}{\sigma_1}-\dfrac{2}{\sigma_2}, 0, \dfrac{2}{\sigma_2}, \dfrac{3}{\sigma_1}+\dfrac{2}{\sigma_2}\right) \end{cases}$$

then

$$y_1, y_2, z \in N(L), \quad Lw = Bz,$$

$$\langle y_1, y_2\rangle_B = \langle z, z\rangle_B = \langle w, w\rangle_B = \langle z, y_i\rangle_B = \langle w, y_i\rangle_B = 0 \quad \text{for } i = 1, 2,$$

$$\langle y_1, y_1\rangle_B = -\langle y_2, y_2\rangle_B = 16 \ \text{ and } \ \langle z, w\rangle_B = \frac{18}{\sigma_1}+\frac{16}{\sigma_2}.$$

Furthermore, if

$$\begin{cases} u_1 = \sqrt{2}\widetilde{u}_1 + [2\sqrt{2}\sigma_1 - \sqrt{(8\sigma_1+9\sigma_2)(\sigma_1+2\sigma_4 s^2)}]\,\widetilde{u}_2 \\ u_2 = \sqrt{2}\widetilde{u}_1 + [2\sqrt{2}\sigma_1 + \sqrt{(8\sigma_1+9\sigma_2)(\sigma_1+2\sigma_4 s^2)}]\,\widetilde{u}_2 \end{cases}, \text{ with}$$

$\widetilde{u}_1 = (3s(3\sigma_2-4\sigma_1), -3(4\sigma_1+3\sigma_2), 4\sigma_1, 3s(3\sigma_2-4\sigma_1), -9\sigma_2, 0)$ and

$\widetilde{u}_2 = (0, 3, -1, 0, -3, 1),$

then

$$Lu_i = \lambda_i Bu_i \text{ for } i = 1, 2, \quad \text{with } \lambda_1 = -\lambda_2 = \frac{\sqrt{2(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}}{3},$$

$$\langle u_1, u_2\rangle_B = \langle u_i, z\rangle_B = \langle u_i, w\rangle_B = \langle u_i, y_j\rangle_B = 0 \text{ for } i, j = 1, 2,$$

$$\text{and } \langle u_1, u_1\rangle_B = -\langle u_2, u_2\rangle_B = 12(8\sigma_1 + 9\sigma_2)\sqrt{2(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}.$$

Denote

$$u = \sqrt{2}(3s(3\sigma_2 - 4\sigma_1), -3(2\sigma_1 + 3\sigma_2), 2\sigma_1) + \sqrt{(8\sigma_1 + 9\sigma_2)(\sigma_1 + 2\sigma_4 s^2)}(0, -3, 1).$$

Then, by Theorem 3.1, the problem (32), (36), with $g = 0$ and $C = 0$ (the homogeneous case with inflow boundary conditions), and the boundary conditions

- (O) has a (unique) solution
$$f(x) = \beta e^{-\lambda_1 x} u_1,$$
if and only if $h_0 = \beta u$;
- (P) has the solution
$$f(x) = \beta e^{-\lambda_1 x} u_1 + \mu_1 y_1 + \mu_2 y_2 + \eta z$$
and hence, the asymptotic flow
$$f_A = \mu_1 y_1 + \mu_2 y_2 + \eta z,$$
where $\mu_2 = -\frac{1}{16}\langle f_A, y_2\rangle_B$ is a free parameter and
$$h_0 - \mu_2(3, -s, -3s) = \beta u + \mu_1(5, s, 3s) + \eta(10s, -1, -1);$$
- (Q) has the solution
$$f(x) = \beta e^{-\lambda_1 x} u_1 + \mu_1 y_1 + \mu_2 y_2 + \eta z + a(w - xz)$$
and hence, the asymptotic flow
$$f_A(x) = \mu_1 y_1 + \mu_2 y_2 + \eta z + a(w - xz),$$
where $\mu_2 = -\frac{1}{16}\langle f_A, y_2\rangle_B$ and $a = \frac{\sigma_1\sigma_2}{16\sigma_1 + 18\sigma_2}\langle f_A, z\rangle_B$ are two free parameters and
$$h_0 - \mu_2(3, -s, -3s) + a\left(0, \frac{2}{\sigma_2}, \frac{3}{\sigma_1} + \frac{2}{\sigma_2}\right) = \beta u + \mu_1(5, s, 3s) + \eta(10s, -1, -1).$$

10 - Exact solutions for a linearized kinetic model

10.1 - *Discrete linearized kinetic model*

Remark 10.1. *Denoting*

$$h = |B|^{1/2} f,$$

with

$$|B| = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix},$$

in Eqs. (60), *we obtain the new system*

$$\widetilde{B}\frac{dh}{dx} + \widetilde{L}h = 0,$$

where

$$\widetilde{L} = |B|^{-1/2} L |B|^{-1/2}$$

and

$$\widetilde{B} = diag\,(\underbrace{1, \ldots, 1}_{n^+}, \underbrace{-1, \ldots, -1}_{n^-}). \tag{72}$$

We notice that the matrix $\widetilde{L}$ *is again symmetric and semi-positive. That is, if the original diagonal matrix* B *is non-singular, then we can transform the system* (60) *into another system, such that the diagonal matrix has only ones and minus ones at the diagonal and the linear operator is symmetric and semi-positive.*

Remark 10.2. *We can choose an orthogonal basis*

$$\{y_1, \ldots, y_k, z_1, \ldots, z_l\}, \quad \text{with } k + l = p, \tag{73}$$

of $N(L)$*, such that*

$$\begin{gathered} \langle y_i, y_j \rangle = \delta_{ij}, \;\; \langle z_r, z_s \rangle = \delta_{rs}, \;\; \langle y_i, z_r \rangle = 0, \;\; \langle y_i, y_j \rangle_B = \gamma_i \delta_{ij}, \\ and \;\; \langle y_i, z_r \rangle_B = \langle z_r, z_s \rangle_B = 0, \;\; with \;\; \gamma_1, \ldots, \gamma_{k^+} > 0 \;\; and \;\; \gamma_{k^+ +1}, \ldots, \gamma_k < 0. \end{gathered} \tag{74}$$

If B *is on the form* (72)*, then the relations* (74) *are equivalent to the relations*

$$\left\langle y_i^{\pm}, y_j^{\pm} \right\rangle = \frac{1 \pm \gamma_i}{2}\delta_{ij}, \quad \left\langle y_i^{\pm}, z_r^{\pm} \right\rangle = 0 \quad and \quad \left\langle z_r^{\pm}, z_s^{\pm} \right\rangle = \frac{1}{2}\delta_{rs}. \tag{75}$$

We consider the system (60), with the matrix B of the form (72) and the linear operator L of the simplified form

$$L = I - \pi_L, \tag{76}$$

where $\pi_L(f) : \mathbb{R}^n \to N(L)$ is the orthogonal projection in $\mathbb{R}^n$ of f on $N(L)$ and I is the identity operator. We choose an orthogonal basis (73) of $N(L)$, such that the relations (74),(75) are fulfilled.

Remark 10.3. *If we consider the system* (60) *for an arbitrary matrix B of the form* (34)*, then instead of doing the transformation in Remark* 10.1 *of the system and then substitute* (76)

$$\widetilde{L}' = I - \pi_{\widetilde{L}},$$

for the linear operator $\widetilde{L}$*, we can directly substitute*

$$L' = |B|(I - \widetilde{\pi}_L), \tag{77}$$

where $\widetilde{\pi}_L(f) : \mathbb{R}^n \to N(L)$ *is the orthogonal, with respect to the scalar product* $\langle \cdot, \cdot \rangle_{|B|}$*, projection in* $\mathbb{R}^n$ *of* f *on* $N(L)$ *and* I *is the identity operator, for the linear operator* L *in Eq.* (60).

If $\gamma_i = -1$, then $y_i^+ = 0$, since $\langle y_i^+, y_i^+ \rangle = 0$, by (75). However, this is never the case for the Boltzmann equation, and therefore, we assume that $\gamma_i \neq -1$ for $i = 1, \ldots, k$.

The positive eigenvalues of $B^{-1}L = BL$ are

$$\lambda_{i-k^+} = -\langle y_i, y_i \rangle_B = -\gamma_i \neq 1, \;\; i = k^+ + 1, \ldots, k, \text{ and } \lambda_{k^- +1} = ... = \lambda_{m^+} = 1,$$

where $k^- = k - k^+$ and $m^+ = n^+ - k^+ - l$,

with corresponding orthogonal eigenvectors

$$u_{i-k^+} = \begin{pmatrix} u_{i-k^+}^+ \\ u_{i-k^+}^- \end{pmatrix}, \text{ where } u_{i-k^+}^+ = \frac{2y_i^+}{1+\gamma_i} \text{ and } u_{i-k^+}^- = \frac{2y_i^-}{1-\gamma_i},$$

for $i = k^+ + 1, \ldots, k$,

and

$$u_{k^- +1}, \ldots, u_{m^+} \in \mathrm{Im}(L), \text{ where } u_i^- = 0 \text{ for } i = k^- + 1, \ldots, m^+.$$

The relations

$$\langle u_i^+, y_j^+\rangle = \langle u_i, y_j\rangle_B = 0 \text{ and } \langle u_i^+, z_r^+\rangle = \langle u_i, z_r\rangle_B = 0,$$
$$i = 1, \dots, m^+,\ j = 1, \dots, k^+,\ r = 1, \dots, l.$$

are fulfilled.

We now consider Eqs. (60) with the different boundary conditions (O), (P) and (Q) at infinity.

(i) If we assume that the boundary conditions (O) at infinity are fulfilled, then there is a (unique) solution to the problem (60), if and only if

$$\langle f_0, y_i^+\rangle = \langle f_0, z_r^+\rangle = 0$$

for $i = 1, \dots, k^+$, and $r = 1, \dots, l$. In that case, the solution is

$$f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix},$$

where

$$\begin{cases} f^+ = e^{-x} f_0 + \sum\limits_{i=k^+ +1}^{k} \dfrac{2\langle f_0, y_i^+\rangle}{1+\gamma_i} (e^{\gamma_i x} - e^{-x}) y_i^+ \\ f^- = \sum\limits_{i=k^+ +1}^{k} \dfrac{2\langle f_0, y_i^+\rangle}{1-\gamma_i} e^{\gamma_i x} y_i^- \end{cases}.$$

(ii) If we assume that the boundary conditions (P) at infinity are fulfilled, then

$$f_\infty = \lim_{x\to\infty} f(x) = \sum_{i=1}^{k^+} \mu_i y_i + \sum_{j=1}^{l} \eta_j z_j + \sum_{i=k^+ +1}^{k} \theta_i y_i.$$

We prescribe

$$\theta_i = \frac{\langle f_\infty, y_i\rangle_B}{\gamma_i}, \qquad i = k^+ + 1, \dots, k,$$

and obtain

$$\begin{cases} \mu_i = \dfrac{2\langle f_0, y_i^+\rangle}{1+\gamma_i} \\ \eta_j = 2\left\langle f_0, z_j^+\right\rangle \end{cases}, \qquad i = 1, \dots, k^+, j = 1, \dots, l.$$

The solution is

$$f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix},$$

where

$$\begin{cases} f^+ = e^{-x}f_0 + \sum\limits_{i=k^++1}^{k}[\frac{2\langle f_0, y_i^+\rangle}{1+\gamma_i}(e^{\gamma_i x} - e^{-x}) + \theta_i(1-e^{\gamma_i x})]y_i^+ \\ \qquad + \sum\limits_{i=1}^{k^+}\frac{2\langle f_0, y_i^+\rangle}{1+\gamma_i}(1-e^{-x})y_i^+ + \sum\limits_{r=1}^{l} 2\langle f_0, z_r^+\rangle(1-e^{-x})z_r^+ \\ f^- = \sum\limits_{i=k^++1}^{k}\left[\frac{2\langle f_0, y_i^+\rangle}{1-\gamma_i}e^{\gamma_i x} + \theta_i\left(1 - \frac{1+\gamma_i}{1-\gamma_i}e^{\gamma_i x}\right)\right]y_i^- + \sum\limits_{i=1}^{k^+}\frac{2\langle f_0, y_i^+\rangle}{1+\gamma_i}y_i^- \\ \qquad + \sum\limits_{r=1}^{l} 2\langle f_0, z_r^+\rangle z_r^- \end{cases}.$$

(iii) We assume that the boundary conditions (Q) at infinity are fulfilled. There are vectors $w_1, \dots, w_l \in \mathrm{Im}(L)$ such that

$$B^{-1}Lw_r = z_r. \tag{78}$$

The problem (78) has the solution

$$w_r = Bz_r.$$

The relations

$$\langle w_r, w_s\rangle_B = \langle w_r, y_i\rangle_B = \langle w_r, u_j\rangle_B = 0 \text{ and } \langle w_r, z_s\rangle_B = \delta_{sr},$$

$$i = 1, \dots, k,\ j = 1, \dots, m^+, \text{ and } s, r = 1, \dots, l.$$

are fulfilled.

The asymptotic flow is

$$f_A(x) = \sum_{i=1}^{k^+}\mu_i y_i + \sum_{i=k^++1}^{k}\theta_i y_i + \sum_{j=1}^{l}[\eta_j z_j + a_j(w_j - xz_j)]$$

and

$$f_\infty = \lim_{x\to\infty}\left(f(x) - x\frac{df}{dx}\right) = \sum_{i=1}^{k^+}\mu_i y_i + \sum_{i=k^++1}^{k}\theta_i y_i + \sum_{j=1}^{l}[\eta_j z_j + a_j w_j].$$

We prescribe

$$\theta_i = \frac{\langle f_\infty, y_i\rangle_B}{\gamma_i} \quad \text{and} \quad a_j = \langle f_\infty, z_j\rangle_B, \quad i = k^+ + 1, \dots, k, \quad j = 1, \dots, l,$$

and obtain

$$\begin{cases} \mu_i = \dfrac{2\langle f_0, y_i^+\rangle}{1+\gamma_i} \\ \eta_j = 2\langle f_0, z_j^+\rangle - a_j \end{cases}, \quad i = 1, \dots, k^+, \; j = 1, \dots, l.$$

The solution is

$$f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix},$$

where

$$\text{(79)} \qquad \begin{cases} f^+ = e^{-x} f_0 + \sum\limits_{i=k^+ + 1}^{k} \left[\dfrac{2\langle f_0, y_i^+\rangle}{1+\gamma_i} (e^{\gamma_i x} - e^{-x}) + \theta_i (1 - e^{\gamma_i x}) \right] y_i^+ \\ \qquad + \sum\limits_{i=1}^{k^+} \dfrac{2\langle f_0, y_i^+\rangle}{1+\gamma_i} (1 - e^{-x}) y_i^+ + \sum\limits_{r=1}^{l} [- a_r x + 2\langle f_0, z_r^+\rangle (1 - e^{-x})] z_r^+ \\ f^- = \sum\limits_{i=k^+ + 1}^{k} \left[\dfrac{2\langle f_0, y_i^+\rangle}{1-\gamma_i} e^{\gamma_i x} + \theta_i \left(1 - \dfrac{1+\gamma_i}{1-\gamma_i} e^{\gamma_i x} \right) \right] y_i^- + \sum\limits_{i=1}^{k^+} \dfrac{2\langle f_0, y_i^+\rangle}{1+\gamma_i} y_i^- \\ \qquad + \sum\limits_{r=1}^{l} [- a_r (x+2) + 2\langle f_0, z_r^+\rangle] z_r^- \end{cases}.$$

10.2 - *Continuous linearized kinetic model*

We consider the half-space problem

$$\text{(80)} \qquad \begin{cases} \xi^1 \dfrac{dF}{dx} = Q(F,F) \\ F(0,\xi) = \rho_0 M_0 \;\text{ for } \xi^1 > 0, \\ F \underset{x\to\infty}{\to} M_\infty \end{cases} \qquad F = F(x,\xi),\; x \in \mathbb{R}_+,\; \xi \in \mathbb{R}^3,$$

for the (continuous) Boltzmann equation, where $\xi = \left(\xi^1, \xi^2, \xi^3\right)$, $M_0 = \dfrac{1}{(2\pi T_0)^{3/2}} e^{-|\xi|^2/(2T_0)}$ and $M_\infty = \dfrac{\rho_\infty}{(2\pi T_\infty)^{3/2}} e^{-|\xi - \boldsymbol{u}_\infty|^2/(2T_\infty)}$. We also assume that

$$\text{(81)} \qquad \int\limits_{\mathbb{R}^3} \xi^1 F(0,\xi)\, d\xi = 0.$$

We denote

$$F = \rho_0 M_0 + M_0^{1/2} f,$$

in Eq. (80), and obtain (neglecting the non-linear terms)

$$\begin{cases} \xi^1 \dfrac{df}{dx} + Lf = 0 \\ f(0,\xi) = 0 \ \text{ for } \xi^1 > 0 \end{cases}, \tag{82}$$

where $Lf = -2\rho_0 M_0^{-1/2} Q(M_0, M_0^{1/2} f)$. We assume that f has the asymptotic flow

$$\begin{aligned} f_A = {} & a M_0^{1/2} + \boldsymbol{b} \cdot M_0^{1/2} \xi + c M_0^{1/2} |\xi|^2 + d_1 [x M_0^{1/2} \xi^2 + L^{-1}(\xi^1 M_0^{1/2} \xi^2)] \\ & + d_2 [x M_0^{1/2} \xi^3 + L^{-1}(\xi^1 M_0^{1/2} \xi^3)] \\ & + d_3 \left[x M_0^{1/2} \left(\frac{|\xi|^2}{T} - 5 \right) + L^{-1} \left(\xi^1 M_0^{1/2} \left(\frac{|\xi|^2}{T} - 5 \right) \right) \right] \end{aligned}$$

for constant $a, c, d_1, d_2, d_3 \in \mathbb{R}$ and $\boldsymbol{b} \in \mathbb{R}^3$, at infinity. By the relation $\int_{\mathbb{R}^3} M_0 \xi^1 \, d\xi = 0$ and Eq. (81), we obtain

$$\int_{\mathbb{R}^3} f(0,\xi) M_0^{1/2} \xi^1 \, d\xi = 0. \tag{83}$$

The kernel $N(L)$ of L is given by

$$N(L) = \text{span } (M_0^{1/2}, M_0^{1/2} \xi^1, M_0^{1/2} \xi^2, M_0^{1/2} \xi^3, M_0^{1/2} |\xi|^2).$$

We introduce the scalar products

$$P(g,h) = \int_{\mathbb{R}^3} \xi^1 g h \, d\xi \quad \text{and} \quad \widetilde{P}(g,h) = \int_{\mathbb{R}^3} |\xi^1| g h \, d\xi.$$

Remark 10.4. *The substitution* (77) *corresponds, in this case, to substitute*

$$\widetilde{L} = |\xi^1| (I - \widetilde{\pi}_L), \tag{84}$$

where $\widetilde{\pi}_L(f)$ *is the orthogonal, with respect to the scalar product* $\widetilde{P}$*, projection of* f *on* $N(L)$ *and* I *is the identity operator, for the linear operator* L*. The linear operator* (84) *corresponds to the linearized kinetic model* (*with* $|\xi^1|$ *for the collision frequency* $\nu(\xi)$ *and* $N = 4$) *of BGK-type in Ref.* [**18**, p. 208].

The basis

$$\begin{cases} \phi_1 = \dfrac{3M_0^{1/2}}{4\sqrt{R}}\left[\sqrt{2}+\dfrac{\xi^1}{\sqrt{T_0}}+\dfrac{\sqrt{2}}{9}\left(\dfrac{|\xi|^2}{T_0}-5\right)\right] \\ \phi_2 = \dfrac{3M_0^{1/2}}{4\sqrt{R}}\left[\sqrt{2}-\dfrac{\xi^1}{\sqrt{T_0}}+\dfrac{\sqrt{2}}{9}\left(\dfrac{|\xi|^2}{T_0}-5\right)\right] \\ \phi_3 = \dfrac{M_0^{1/2}}{\sqrt{R}}\dfrac{\xi^2}{\sqrt{T_0}} \\ \phi_4 = \dfrac{M_0^{1/2}}{\sqrt{R}}\dfrac{\xi^3}{\sqrt{T_0}} \\ \phi_5 = \dfrac{M_0^{1/2}}{3\sqrt{R}}\left(\dfrac{|\xi|^2}{T_0}-5\right) \end{cases},$$

where $R=\sqrt{\dfrac{2T_0}{\pi}}$, of $N(L)$, is orthogonal with respect to the scalar product P and orthonormal with respect to the scalar product $\widetilde{P}$. In fact,

$$\begin{aligned} &\widetilde{P}\left(\phi_i,\phi_j\right)=\delta_{ij} \text{ and } P\left(\phi_i,\phi_j\right)=\gamma_i\delta_{ij} \text{ for } i,j=1,\dots,5, \text{ where} \\ &\gamma_1=-\gamma_2=\frac{9\sqrt{\pi}}{8} \text{ and } \gamma_3=\gamma_4=\gamma_5=0 \end{aligned} \quad . \tag{85}$$

Given a function $h=h(\xi)$, we denote

$$h^+=\begin{cases} h \text{ if } \xi^1\geq 0 \\ 0 \text{ if } \xi^1<0 \end{cases} \quad \text{and} \quad h^-=\begin{cases} 0 \text{ if } \xi^1\geq 0 \\ h \text{ if } \xi^1<0 \end{cases}.$$

The equation

$$\begin{cases} \xi^1\dfrac{d\widetilde{f}}{dx}+|\xi^1|[\widetilde{f}-\widetilde{\pi}_L(\widetilde{f})]=0 \\ \widetilde{f}(0,\xi)=g_0 \text{ for } \xi^1>0 \end{cases}, \qquad \text{with } \widetilde{f}=\widetilde{f}(x,\xi)\ ,\ x\in\mathbb{R}_+, \text{ and } \xi\in\mathbb{R}^3, \tag{86}$$

has the solution

$$\begin{aligned} \widetilde{f}(x,\xi)=&\,e^{-x}f_0^+ +\theta_2(1-e^{\gamma_2 x})\phi_2+2P\big(f_0,\phi_2^+\big)\left(\frac{e^{\gamma_2 x}-e^{-x}}{1+\gamma_2}\phi_2^+ +\frac{e^{\gamma_2 x}}{1-\gamma_2}\phi_2^-\right) \\ &-\theta_2\frac{2\gamma_2}{1-\gamma_2}e^{\gamma_2 x}\phi_2^- +\frac{2P\big(f_0,\phi_1^+\big)}{1+\gamma_1}\big(\phi_1-e^{-x}\phi_1^+\big) \\ &-\sum_{r=3}^{5}[a_{r-2}(x\phi_r+2\phi_r^-)-2P\big(h_0,\phi_r^+\big)\big(\phi_r-e^{-x}\phi_r^+\big)]. \end{aligned} \tag{87}$$

The solution (87) is the continuous form of the discrete solution (79) and it can be verified that it actually is a solution of Eq. (86), using the relations (85).

In our case, $f_0 = 0$, and hence, if we make the substitution (84), then

$$\widetilde{f}(x,\xi) = \theta_2 \left[\phi_2 - e^{\gamma_2 x} \left(\phi_2^+ + \frac{1+\gamma_2}{1-\gamma_2} \phi_2^- \right) \right] - \sum_{r=3}^{5} a_{r-2} [x\phi_r + 2\phi_r^-]$$

is a solution of Eq. (82). Since

$$\widetilde{F}(0,\xi) = -\theta_2 \frac{2\gamma_2}{1-\gamma_2} \phi_2^- - 2\sum_{r=3}^{5} a_{r-2} \phi_r^-,$$

the relations

$$P(M_0^{1/2}, \phi_2^-) = \sqrt{2R}\,\frac{16+9\sqrt{\pi}}{48}, \quad P(M_0^{1/2}, \phi_3^-) = P(M_0^{1/2}, \phi_4^-) = 0 \quad \text{and}$$

$$P(M_0^{1/2}, \phi_5^-) = -\frac{\sqrt{R}}{6},$$

and Eq. (83) imply that

$$a_3 = \frac{9\sqrt{2\pi}(16+9\sqrt{\pi})}{8(9\sqrt{\pi}-8)} \theta_2. \tag{88}$$

Therefore, the asymptotic flow is

$$\widetilde{f}_A = \theta_2 \phi_2 - \sum_{r=3}^{5} a_{r-2} [x\phi_r + 2\phi_r^-],$$

where a_3 and θ_2 fulfill Eq. (88).

Remark 10.5. *We assume above that the velocity $\boldsymbol{u}_0 = 0$. If $\boldsymbol{u}_0$ is non-zero, but orthogonal to the x-axis, then we can shift the velocity variable ξ with $\boldsymbol{u}_0$, without modifying Eq.* (82).

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A - Appendix

We remind that $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product on $\mathbb{R}^n$ and that

$$\langle \cdot, \cdot \rangle_E = \langle \cdot, E \cdot \rangle$$

for real symmetric matrices E.

If U is a subspace of $\mathbb{R}^n$, then we denote the orthogonal complement, with respect to the scalar product $\langle\cdot,\cdot\rangle_E$, by $U^{\perp_E}$, i.e.

$$U^{\perp_E}=\left\{x\in\mathbb{R}^n \middle|\, \langle x,y\rangle_E=0 \text{ for all } y\in U\right\},$$

and, in particular, if E is the identity matrix, then we denote

$$U^{\perp}=U^{\perp_E}.$$

We consider below any real symmetric matrices L and B, such that L is semi-positive and have a non-trivial null-space $N(L)$, and B is non-singular, i.e.

$$\langle x,x\rangle_L\geq 0 \text{ for all } x\in\mathbb{R}^n,\ \det B\neq 0 \text{ and } \dim\left(N(L)\right)=p\geq 1.$$

The real eigenvalue problem reads

$$B^{-1}Lx=\lambda x \text{ or, equivalently, } Lx=\lambda Bx, \tag{89}$$

with $\lambda\in\mathbb{R}$, $x\in\mathbb{R}^n$ and $x\neq 0$ (in general we have to consider $\lambda\in\mathbb{C}$ and $x\in\mathbb{C}^n$, but in our case, by Lemma A.2 below, we have a full set of real eigenvalues, and can consider only the real case).

We denote the space of real generalized eigenvectors of the matrix $B^{-1}L$ belonging to the eigenvalue $\lambda=0$ by V_0, i.e.

$$V_0=V_0\left(B^{-1}L\right)=\left\{x\in\mathbb{R}^n\middle|\,\exists\, m:\left(B^{-1}L\right)^m x=0\right\}.$$

We first show that all generalized eigenvectors of the matrix $B^{-1}L$ belonging to the eigenvalue $\lambda=0$ are in the null-space, $N\left(LB^{-1}L\right)$, of $LB^{-1}L$.

Lemma A.1

$$V_0=N\left(LB^{-1}L\right).$$

Proof. Let $k\in\mathbb{N}$. If

$$\left(B^{-1}L\right)^{k+1}x=0,$$

then

$$(L^{1/2}B^{-1}L^{1/2})^k(L^{1/2}x)=0,$$

since $N(L)=N(L^{1/2})$ and $\det B\neq 0$. But $L^{1/2}B^{-1}L^{1/2}$ is symmetric, and therefore

$$(L^{1/2}B^{-1}L^{1/2})(L^{1/2}x)=0.$$

Hence,

$$LB^{-1}Lx=0,$$

and the lemma is proved.

Remark A.1 *By* $N(L) = N(L^{1/2})$, *we obtain*

$$\dim\left(N\left(L^{1/2}B^{-1}L^{1/2}\right)\right) = \dim\left(N\left(LB^{-1}L\right)\right). \tag{90}$$

Now we create a special basis of V_0. We denote the image of the matrix L by $\mathrm{Im}(L)$, i.e.

$$\mathrm{Im}(L) = \{x \in \mathbb{R}^n \,|\, \exists y \in \mathbb{R}^n : x = Ly\}.$$

Then $\mathbb{R}^n$ is the direct sum of $N(L)$ and $\mathrm{Im}(L)$, i.e.

$$\mathbb{R}^n = N(L) \oplus \mathrm{Im}(L).$$

We can choose an orthogonal, with respect to the scalar product $\langle \cdot, \cdot \rangle_B$, basis of $N(L)$

$$\{y_1, \dots, y_k, z_1', \dots, z_l'\}, \quad \text{with } k + l = p\,, \tag{91}$$

such that

$$\begin{aligned} &\langle y_i, y_j \rangle_B = \gamma_i \delta_{ij}, \quad \text{with } \gamma_1, \dots, \gamma_{k^+} > 0 \quad \text{and} \quad \gamma_{k^++1}, \dots, \gamma_k < 0, \\ &\text{and } \langle z_r', y_i \rangle_B = \langle z_r', z_s' \rangle_B = 0. \end{aligned} \tag{92}$$

Then, for $1 \leq r \leq l$,

$$z_r' \in N(L)^{\perp_B},$$

or, equivalently,

$$Bz_r' \in N(L)^{\perp} = \mathrm{Im}(L). \tag{93}$$

By Eq. (93), there is a vector $v_r \in \mathbb{R}^n$ such that

$$Bz_r' = Lv_r \text{ or, equivalently, } z_r' = B^{-1}Lv_r\,. \tag{94}$$

On the other hand, $Lz_r' = 0$, and therefore

$$v_r \in V_0 \setminus N(L).$$

If $a_1 v_1 + ... + a_l v_l = 0$, then $a_1 z_1' + ... + a_l z_l' = 0$, and hence, $a_1 = ... = a_l = 0$. Therefore,

$$\{y_1, \dots, y_k, z_1', \dots, z_l', v_1, \dots, v_l\} \tag{95}$$

is a basis of V_0.

We will now show that, if $\lambda \neq 0$, then all generalized eigenvectors are, in fact, eigenvectors.

Lemma A.2 *The matrix $B^{-1}L$ has $q = n - p - l$ real orthogonal, with respect to the scalar product $\langle \cdot, \cdot \rangle_B$, eigenvectors*

$$u_1, \dots, u_q, \quad \text{with } \langle u_i, u_j \rangle_B = \lambda_i \delta_{ij}\,, \tag{96}$$

in $V_0^{\perp_B}$, corresponding to non-zero real eigenvalues

$$\lambda_1, \dots, \lambda_q \quad \text{where } \lambda_1, \dots, \lambda_{m^+} > 0 \quad \text{and} \quad \lambda_{m^+ +1}, \dots, \lambda_q < 0. \tag{97}$$

Proof. Multiplying Eq. (89) by $L^{1/2}$ and denoting $y = L^{1/2}x$, we obtain the new eigenvalue problem

$$Cy = \lambda y, \tag{98}$$

where the matrix $C = L^{1/2}B^{-1}L^{1/2}$ is symmetric. Therefore C has n real orthonormal eigenvectors

$$x_1, \dots, x_n, \text{ where } \langle x_i, x_j \rangle = \delta_{ij}, \tag{99}$$

which, after possible reordering, have the corresponding real eigenvalues

$$\lambda_1, \dots, \lambda_{m^+} > 0, \quad \lambda_{m^+ +1}, \dots, \lambda_q < 0 \quad \text{and} \quad \lambda_{q+1} = ... = \lambda_n = 0, \tag{100}$$

where $1 \leq q \leq n$ is still unknown. The number of zero eigenvalues is equal to $\dim \left(N(L^{1/2}B^{-1}L^{1/2}) \right)$, and hence, by Eq. (90), $\lambda = 0$ is an eigenvalue of multiplicity $p + l$ to the problem (98). Therefore,

$$q = n - p - l.$$

If y is a solution of Eq. (98), then $u = B^{-1}L^{1/2}y$ is a solution of Eq. (89) for the same λ, since

$$Lu = L^{1/2}Cy = \lambda L^{1/2}y = \lambda Bu.$$

Therefore,

$$u_i = B^{-1}L^{1/2}x_i, \quad i = 1, \dots, q, \tag{101}$$

are eigenvectors of $B^{-1}L$ that correspond to the eigenvalues $\lambda_1, \dots, \lambda_q$ in (100). The orthogonality conditions (99) imply that

$$\langle u_i, u_j \rangle_B = \left\langle L^{1/2}x_i, B^{-1}L^{1/2}x_j \right\rangle = \langle x_i, Cx_j \rangle = \lambda_i \delta_{ij}, \quad \text{with } \lambda_i \neq 0. \tag{102}$$

The vectors (101) are linearly independent ($a_1u_1 + ... + a_qu_q = 0$ implies that $a_1 = ... = a_q = 0$, by Eqs. (102)).

For all $1 \leq i \leq q$,

$$\langle u_i, y \rangle_B = \frac{1}{\lambda_i} \langle u_i, y \rangle_L = 0 \quad \text{if } y \in N(L). \tag{103}$$

On the other hand, if $w \in V_0 \setminus N(L)$, then there exists $y \in N(L)$ such that $Lw = By$, and then, by Eq. (103),

$$\langle u_i, w\rangle_B = \frac{1}{\lambda_i}\langle u_i, w\rangle_L = \frac{1}{\lambda_i}\langle u_i, y\rangle_B = 0.$$

That is, $\{u_1, \dots, u_q\} \subseteq V_0^{\perp_B}$, and the lemma is proved.

Corollary A.1 *By Lemma A.2, if $l = 0$ then all generalized eigenvectors are eigenvectors, and hence, the matrix $B^{-1}L$ is diagonalizable if and only if $l = 0$.*

Lemma A.3 *There is a basis*

$$\mathcal{B}_2 = \{y_1, \dots, y_k, z_1, \dots, z_l, w_1, \dots, w_l\} \tag{104}$$

of V_0, such that Eqs. (92) and

$$\begin{aligned} &\langle w_r, z_s\rangle_B = \delta_{rs}, \ \ \langle w_r, y_i\rangle_B = \langle z_r, y_i\rangle_B = \langle w_r, w_s\rangle_B = \langle z_r, z_s\rangle_B = 0, \\ &B^{-1}Lw_r = z_r \text{ and } z_r \in N(L), \end{aligned} \tag{105}$$

are fulfilled.

Proof. We start by substituting $v_1, \dots, v_l$ in the basis (95) by

$$v_r' = v_r - \sum_{i=1}^{k} \frac{\langle v_r, y_i\rangle_B}{\langle y_i, y_i\rangle_B} y_i, \quad r = 1, \dots, l.$$

By Eqs. (91)-(94),

$$\left\langle v_r', y_i\right\rangle_B = \left\langle z_r', y_i\right\rangle_B = \left\langle z_r', z_s'\right\rangle_B = 0, \ \ B^{-1}Lv_r' = z_r' \ \text{ and } \ z_r' \in N(L), \tag{106}$$

for $r = 1, \dots, l$, and $i = 1, \dots, k$. We denote

$$\begin{cases} w_1 = \dfrac{1}{a_1^{1/2}} v_1' - \dfrac{\beta_1}{2a_1^{3/2}} z_1' \\ z_1 \ = \dfrac{1}{a_1^{1/2}} z_1' \end{cases},$$

where

$$a_1 = \left\langle v_1', z_1'\right\rangle_B = \left\langle v_1', v_1'\right\rangle_L > 0 \quad \text{and} \quad \beta_1 = \left\langle v_1', v_1'\right\rangle_B,$$

and obtain, by Eqs. (106),

$$\begin{aligned} &\langle w_1, y_i\rangle_B = \langle z_1, y_i\rangle_B = \langle w_1, w_1\rangle_B = \langle z_1, z_1\rangle_B = \left\langle z_r', z_1\right\rangle_B = 0, \\ &\langle w_1, z_1\rangle_B = 1, \, B^{-1}Lw_1 = z_1 \ \text{ and } \ z_1 \in N(L) \end{aligned}$$

for $r = 2, \dots, l$, and $i = 1, \dots, k$.

For $j = 2, \dots, l$, after constructing $\{z_1, \dots, z_{j-1}, w_1, \dots, w_{j-1}\}$, such that

$$(107) \qquad \begin{aligned} &\langle w_r, y_i \rangle_B = \langle z_r, y_i \rangle_B = \langle w_r, w_s \rangle_B = \langle z_r, z_s \rangle_B = \langle z'_a, z_s \rangle_B = 0, \\ &\langle w_r, z_s \rangle_B = \delta_{rs}, \;\; B^{-1}Lw_r = z_r \;\text{ and }\; z_r \in N(L) \end{aligned}$$

for $i = 1, \dots, k$, $r, s = 1, \dots, j-1$, and $a = j, \dots, l$, we construct $\{z_j, w_j\}$ by the following algorithm. We start by denoting

$$\begin{cases} w'_j = v'_j - \sum\limits_{r=1}^{j-1} \left(\left\langle v'_j, w_r \right\rangle_B z_r + \left\langle v'_j, z_r \right\rangle_B w_r \right) \\ z''_j = z'_j - \sum\limits_{r=1}^{j-1} \left\langle v'_j, z_r \right\rangle_B z_r \end{cases}.$$

By Eqs. (106), (107),

$$(108) \qquad \begin{aligned} &\left\langle w'_j, z_s \right\rangle_B = \left\langle w'_j, w_s \right\rangle_B = \left\langle w'_j, y_i \right\rangle_B = \left\langle z''_j, y_i \right\rangle_B = \left\langle z''_j, z_s \right\rangle_B = \left\langle z''_j, z''_j \right\rangle_B = 0, \\ &B^{-1}Lw'_j = z''_j \;\text{ and }\; z''_j \in N(L) \end{aligned}$$

for $i = 1, \dots, k$, and $s = 1, \dots, j-1$, and moreover, for $s = 1, \dots, j-1$,

$$(109) \qquad \left\langle z''_j, w_s \right\rangle_B = \left\langle w'_j, w_s \right\rangle_L = \left\langle w'_j, z_s \right\rangle_B = 0.$$

Next, we construct $\{z_j, w_j\}$,

$$\begin{cases} w_j = \dfrac{1}{a_j^{1/2}} w'_j - \dfrac{\beta_j}{2a_j^{3/2}} z''_j \\ z_j \;= \dfrac{1}{a_j^{1/2}} z''_j \end{cases},$$

where

$$a_j = \left\langle w'_j, z''_j \right\rangle_B = \left\langle w'_j, w'_j \right\rangle_L > 0 \;\text{ and }\; \beta_j = \left\langle w'_j, w'_j \right\rangle_B.$$

Then, by Eqs. (107)-(109),

$$\begin{aligned} &\langle w_j, y_i \rangle_B = \langle z_j, y_i \rangle_B = \langle w_j, w_s \rangle_B = \langle z_j, z_s \rangle_B = \langle z'_r, z_j \rangle_B = 0, \\ &\langle w_s, z_j \rangle_B = \langle w_j, z_s \rangle_B = \delta_{sj}, \;\; B^{-1}Lw_j = z_j \;\text{ and }\; z_j \in N(L), \end{aligned}$$

for $i = 1, \dots, k$ and $s, r = 1, \dots, j$. The lemma follows.

Remark A.2 *The set $\mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1 = \{u_1, \dots, u_q\}$ consists of the vectors* (96) *and $\mathcal{B}_2$ is the basis* (104)*, is a basis of $\mathbb{R}^n$, consisting of generalized eigenvectors of the matrix $B^{-1}L$.*

Remark A.3 *We denote*

$$\boldsymbol{u} = (u_1, \dots, u_s) \ \text{ and } \ \boldsymbol{v} = (v_1, \dots, v_s), \tag{110}$$

where $v_i, u_j \in \mathbb{R}^n$ are arbitrary vectors for $i, j = 1, \dots, s$, and $s \leq n$. Let E be a symmetric $n \times n$ matrix and $\widehat{U}_1$ and $\widehat{U}_2$ two $s \times s$ matrices (hats are used for matrices of lower dimensions). Then we introduce an $s \times s$ matrix

$$\langle \boldsymbol{u} \otimes \boldsymbol{v} \rangle_E, \quad \text{with entries} \left(\langle \boldsymbol{u} \otimes \boldsymbol{v} \rangle_E \right)_{ij} = \left\langle u_i, v_j \right\rangle_E,$$

and consider the change of variables

$$\widetilde{\boldsymbol{u}} = \widehat{U}_1 \boldsymbol{u} \ \text{ and } \ \widetilde{\boldsymbol{v}} = \widehat{U}_2 \boldsymbol{v},$$

where $\boldsymbol{u}$, $\widetilde{\boldsymbol{u}}$, $\boldsymbol{v}$ and $\widetilde{\boldsymbol{v}}$ are formally understood as s-dimensional vectors (110). *It is easy to verify the following formulas*

$$\langle \widetilde{\boldsymbol{u}} \otimes \boldsymbol{v} \rangle_E = \widehat{U}_1 \langle \boldsymbol{u} \otimes \boldsymbol{v} \rangle_E \ \text{ and } \ \langle \boldsymbol{u} \otimes \widetilde{\boldsymbol{v}} \rangle_E = \langle \boldsymbol{u} \otimes \boldsymbol{v} \rangle_E \widehat{U}_2^T,$$

and hence,

$$\langle \widetilde{\boldsymbol{u}} \otimes \widetilde{\boldsymbol{v}} \rangle_E = \widehat{U}_1 \langle \boldsymbol{u} \otimes \boldsymbol{v} \rangle_E \widehat{U}_2^T.$$

Remark A.4 *Let $\phi = (\phi_1, \dots, \phi_p)$, where $N(L) = span(\phi_1, \dots, \phi_p)$. The matrix $\langle \phi \otimes \phi \rangle_B$ is symmetric and hence, there is an orthogonal matrix $\widehat{U}$ (i.e. with $\widehat{U}^{-1} = \widehat{U}^T$), such that*

$$\widehat{U} \langle \phi \otimes \phi \rangle_B \widehat{U}^{-1} = \widehat{U} \langle \phi \otimes \phi \rangle_B \widehat{U}^T = \left\langle \widetilde{\phi} \otimes \widetilde{\phi} \right\rangle_B, \ \text{where } \widetilde{\phi} = \widehat{U}\phi,$$

is a diagonal matrix, with the eigenvalues of $\langle \phi \otimes \phi \rangle_B$ on the diagonal. Hence, $\{\widetilde{\phi}_1, \dots, \widetilde{\phi}_p\}$, where $\widetilde{\phi}_i = \widehat{U}\phi_i$, is an orthogonal, with respect to the scalar product $\langle \cdot, \cdot \rangle_B$, basis of $N(L)$. By Sylvester's Theorem, for any orthogonal basis $\{\varphi_1, \dots, \varphi_p\}$ of $N(L)$, the number of positive, negative and zero diagonal elements of the matrix $\langle \widetilde{\phi} \otimes \widetilde{\phi} \rangle_B$ are the same as of the diagonal matrix $\langle \varphi \otimes \varphi \rangle_B$. Hence, the $p \times p$ matrix $K = \langle \phi \otimes \phi \rangle_B$, with entries $k_{ij} = \langle \phi_i, \phi_j \rangle_B$, has the same number of positive, negative and zero eigenvalues, independently of the choice of the basis $\{\phi_1, \dots, \phi_p\}$ of $N(L)$. That is, the matrix $K = \langle \phi \otimes \phi \rangle_B$ has k^+ positive, $k^- = k - k^+$ negative and l zero eigenvalues (with $k^+ + k^- + l = p$), independently of our choice of the basis $\{\phi_1, \dots, \phi_p\}$ of $N(L)$, since this is the case for the basis (91).

We remind that we by $n^\pm$, where $n^+ + n^- = n$, and $m^\pm$, with $m^+ + m^- = q$, denote the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices B and $B^{-1}L$ respectively, and by m^0 the number of zero eigenvalues of $B^{-1}L$.

Proof. (of Theorem 2.1). The matrix B is symmetric and non-singular, and therefore there is an orthonormal basis

$$\{e_1, \dots, e_n\}, \text{ with } \langle e_i, e_j \rangle = \delta_{ij}, \tag{111}$$

of $\mathbb{R}^n$, consisting of eigenvectors of B. After possible reordering,

$$Be_i = b_i e_i,$$

where

$$b_1, \dots, b_{n^+} > 0 \text{ and } b_{n^++1}, \dots, b_n < 0.$$

Hence,

$$\langle e_i, e_j \rangle_B = b_i \delta_{ij}, \tag{112}$$

and the basis (111) is an orthogonal, with respect to the scalar product $\langle \cdot, \cdot \rangle_B$, basis of $\mathbb{R}^n$.

We denote

$$w_r^+ = w_r + \frac{z_r}{2} \text{ and } w_r^- = w_r - \frac{z_r}{2},$$

and substitute $\{z_r, w_r\}_{1 \le r \le l}$ by $\{w_r^+, w_r^-\}_{1 \le r \le l}$ in the basis (104). By Eqs. (105),

$$\langle w_r^+, w_s^+ \rangle_B = \delta_{rs}, \ \langle w_r^-, w_s^- \rangle_B = -\delta_{rs} \text{ and } \langle w_r^+, w_s^- \rangle_B = 0. \tag{113}$$

Adding $\{u_1, \dots, u_q\}$ in (96) to our new basis of V_0, we obtain another orthogonal, with respect to the scalar product $\langle \cdot, \cdot \rangle_B$, basis of $\mathbb{R}^n$. By Eqs. (92),(96) ,(97), (113) we know that the elements of this basis are such that

$$\langle w_r^+, w_r^+ \rangle_B = 1 > 0, \ \langle u_i, u_i \rangle_B = \lambda_i > 0 \text{ and } \langle y_j, y_j \rangle_B = \gamma_j > 0 \tag{114}$$

for $r = 1, \dots, l$, $i = 1, \dots, m^+$, and $j = 1, \dots, k^+$, and

$$\begin{aligned} &\langle w_r^-, w_r^- \rangle_B = -1 < 0, \ \langle u_{i+m^+}, u_{i+m^+} \rangle_B = \lambda_{i+m^+} < 0 \text{ and} \\ &\langle y_{j+k^+}, y_{j+k^+} \rangle_B = \gamma_{j+k^+} < 0 \end{aligned} \tag{115}$$

for $r = 1, \dots, l$, $i = 1, \dots, m^-$, and $j = 1, \dots, k^-$, where $k^- = k - k^+$, with $q + k + 2l = n$ and $m^- = q - m^+$. Formulas (28) follows by Sylvester's Theorem and Eqs. (112), (114), (115) and so also the theorem.

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Abstract

In this paper we study typical half-space problems of rarefied gas dynamics, including the problems of Milne and Kramer, for the discrete Boltzmann equation. The discrete Boltzmann equation reduces to a system of ODEs for plane stationary problems. These systems are studied, and for general boundary conditions at the "wall" a classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete Boltzmann equation is made. Applications for axially symmetric models are studied in more detail. Exact solutions of a (simplified) linearized kinetic model of BGK type are also found as a limiting case of the corresponding discrete models.

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