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On some special type of trans-Sasakian manifolds

Abstract. We obtained necessary and sufficient condition for a 3-dimensional trans-Sasakian manifold of type (α, β) to be η-Einstein. In particular expressions for Ricci-tensor, curvature tensor and Ricci-operator obtained in 3-dimensional trans-Sasakian and η-Einstein trans-Sasakian manifolds. We also prove that in a three dimensional trans-Sasakian manifold of type (α, β), $Q\phi = \phi Q$ if $\phi(\text{grad}\,\alpha) = \text{grad}\,\beta$. It is also proved that every 3-dimensional α-Sasakian, β-Kenmotsu and (α, β) trans-Sasakian manifolds, where α, β are constants, are always η-Einstein manifolds. Under certain conditions eigenvalues and eigenvectors are also discussed. We also obtained condition for projectively flat trans-Sasakian manifold to be Einstein.

Keywords. Trans-Sasakian manifold, conharmonically flat, conformally flat, projectively flat, $\eta-$Einstein manifold, curvature tensor, Ricci-curvature, Ricci-operator.

Mathematics Subject Classification (2000): 53B05, 53B15.

1 - Introduction

It is known that a trans-Sasakian structure of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are cosymplectic [**1**], α-Sasakian [**2**], [**11**] and β-Kenmotsu [**2**], [**7**] respectively. Sasakian, α-Sasakian, Kenmotsu, β-Kenmotsu are particular cases of trans-Sasakian manifold of type (α, β). Oubiña [**10**] studied a new class of almost contact Riemannian manifold known as trans-Sasakian manifold which generalizes both α-Sasakian and β-

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Kenmotsu structure. Szabo [**12**], [**13**] has obtained some curvature identities and Nomizu [**9**] has studied some curvature properties. Concept of nearly trans-Sasakian was introduced by C. Gherghe [**4**]. It is also known that a locally trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic or α-Sasakian or β-Kenmotsu manifold [**8**]. On other hand, three-dimensional proper trans-Sasakian manifold are constructed by Marrero [**8**]. Jeong-Sik Kim et al [**6**] has studied generalized Ricci-recurrent trans-Sasakian manifold.

However, the curvature tensor, Ricci-tensor and Ricci-operator for 3-dimensional trans-Sasakian and η-Einstein trans-Sasakian manifolds are almost not discussed so far. It is also not discussed that under what condition a three-dimensional trans-Sasakian manifold becomes η-Einstein. So it is worthwhile to study three-dimensional trans-Sasakian and η-Einstein trans-Sasakian manifolds. The present work is organized as under. Section-1 is introductory. Section-2 contains necessary details about trans-Sasakian manifolds. Some basic results are also given in Section-2. The relation between α and β of trans-Sasakian manifold of type (α, β) is discussed in Section-2 as well. In Section-3 the necessary and sufficient condition for three-dimensional trans-Sasakian manifold becomes η-Einstein is given. In Section-4 the Ricci-tensor, curvature tensor and Ricci-operator for η-Einstein trans-Sasakian manifold are found. In Section-5 Sasakian and trans-Sasakian manifold of 3-dimension, which are conharmonically flat are discussed. In Section-6 eigenvalues and eigenvectors are discussed. Lastly in Section-7 projectively flat trans-Sasakian manifolds are discussed.

2 - Trans-Sasakian Manifold

Let M be a $(2n+1)$-dimensional almost contact metric manifold equipped with almost contact metric structure (ϕ, ξ, η, g), where ϕ is $(1,1)$ tensor field, ξ is a vector field, η is 1- form and g is compatible Riemann metric such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X).\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$. A manifold M is called trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\} \tag{2.4}$$

where ∇ is Levi-Civita connection of Riemannian metric g and α and β are smooth functions on M. From equation (2.4) and equations (2.1), (2.2) and (2.3)

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

Some important results:

Lemma 2.1. *In a trans-Sasakian Manifold* [**3**], [**6**]

$$\begin{aligned} R(X,Y)\xi = &(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \beta d\eta(X,Y)\xi \\ &+ 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X \\ &- (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi,Y)X = &(\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(X)Y] \\ &+ 2\alpha\beta[g(\phi X,Y)\xi - \eta(X)\phi Y] \\ &+ (X\alpha)\phi Y + g(\phi X,Y)(\text{grad}\,\alpha) \\ &+ X\beta[Y - \eta(Y)\xi] - g(\phi X,\phi Y)(\text{grad}\,\beta), \end{aligned} \tag{2.8}$$

$$R(\xi,X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X] \tag{2.9}$$

and

$$2\alpha\beta + \xi\alpha = 0, \tag{2.10}$$

where R is the curvature tensor.

Equation (2.10) implies that in trans-Sasakian manifold of type (α,β), α and β are not arbitrary functions but related to each other by structural vector field ξ. Equation (2.10) also implies that α and β are not non-zero constants simultaneously. If $\xi\alpha = 0$ and $\alpha \neq 0$, we have $\beta = 0$ and we can state the following.

Corollary 2.1. *A trans-Sasakian manifold of type (α,β) is α-Sasakian if α is constant only on integral curves of ξ.*

Now we shall give two proper examples of trans-Sasakian manifold of type (α,β) which are neither α-Sasakian nor β-Kenmotsu and both the examples satisfy equation (2.10).

Example 2.1. *Let* (x, y, z) *be Cartesian coordinate in* R^3, *then* (ϕ, ξ, η, g) *given by* $\xi = \dfrac{\partial}{\partial z}$, $\eta = dz - y dx$, $\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}$, $g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$ *is a trans-Sasakian structure of type* $\left(-\dfrac{1}{2\,e^z}, \dfrac{1}{2}\right)$ *in* R^3 [**6**].

Example 2.2. *In fact in three-dimensional K-contact manifold with structure tensors* (ϕ, ξ, η, g), *for a non-constant function f, defined* $g' = fg + (1 - f)\eta \otimes \eta$ *then* (ϕ, ξ, η, g') *is a trans-Sasakian structure of type* $\left(\dfrac{1}{f}, \dfrac{1}{2}\xi(\ln f)\right)$ [**6**].

It is easy to verify that Examples 2.1 and 2.2 satisfy the condition

$$2\alpha\beta + \xi\alpha = 0.$$

Lemma 2.2. *In a* $(2n+1)$*-dimensional trans-Sasakian manifold* (α, β), *we have from* [**6**]

$$S(X, \xi) = \big(2n(\alpha^2 - \beta^2) - \xi\beta\big)\eta(X) - (2n-1)X\beta - (\phi X)\alpha, \tag{2.11}$$

$$Q\xi = \big(2n(\alpha^2 - \beta^2) - \xi\beta\big)\xi - (2n-1)\,\mathrm{grad}\,\beta + \phi(\mathrm{grad}\,\alpha), \tag{2.12}$$

where S is the Ricci-curvature and Q is the Ricci-operator of trans-Sasakian manifold of type (α, β). *S and Q are related to each other by*

$$S(X, Y) = g(QX, Y). \tag{2.13}$$

Corollary 2.2. *In a trans-Sasakian manifold of type* (α, β) *of dimension-*$(2n+1)$ *if* $\phi\,(\mathrm{grad}\,\alpha) = (2n-1)(\mathrm{grad}\,\beta)$, *then*

$$\xi\beta = g(\xi, \mathrm{grad}\,\beta) = \frac{1}{2n-1}\,g(\xi, \phi(\mathrm{grad}\,\alpha)) = \frac{1}{2n-1}\,\eta(\phi(\mathrm{grad}\,\alpha)) = 0$$

and then, we also have

$$R(\xi, X)\xi = \big(\alpha^2 - \beta^2\big)(\eta(X)\xi - X),$$
$$S(X, \xi) = 2n\big(\alpha^2 - \beta^2\big)\eta(X).$$

3 - Three-dimensional trans-Sasakian manifolds

In a three-dimensional trans-Sasakian manifold, we have from Lemma 2.2

$$S(X, \xi) = \big(2(\alpha^2 - \beta^2) - \xi\beta\big)\eta(X) - (X\beta) - (\phi X)\alpha, \tag{3.1}$$

$$Q\xi = \big(2(\alpha^2 - \beta^2) - \xi\beta\big)\xi - \mathrm{grad}\,\beta + \phi(\mathrm{grad}\,\alpha). \tag{3.2}$$

Definition 3.1. *The Weyl conformal curvature tensor C of type $(1,3)$ of an $(2n+1)$-dimensional manifold (M,g) is defined by*

$$
\begin{aligned}
C(X,Y)Z = & R(X,Y)Z \\
& - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\
& + \frac{r}{(2n)(2n-1)}[g(Y,Z)X - g(X,Z)Y],
\end{aligned} \tag{3.3}
$$

where R, S, Q, r denotes respectively the Riemannian curvature tensor, Ricci-tensor of type $(0,2)$, the Ricci-operator and the scalar curvature of the manifold.

Lemma 3.1. *In a three-dimensional trans-Sasakian manifold, the Ricci-operator is given by*

$$
\begin{aligned}
QX = & \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)X - \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\xi \\
& + \eta(X)(\phi(\operatorname{grad}\alpha) - (\operatorname{grad}\beta)) - (X\beta + (\phi X)\alpha)\xi.
\end{aligned} \tag{3.4}
$$

Proof. We know that the Weyl conformal curvature tensor vanishes in three-dimensional Riemannian manifold, therefore from equation (3.3)

$$
\begin{aligned}
R(X,Y)Z = & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\
& - S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y).
\end{aligned} \tag{3.5}
$$

For any three-dimensional trans-Sasakian manifold, from equations (3.5) and (3.1), we have

$$
\begin{aligned}
R(X,Y)\xi = & \eta(Y)QX - \eta(X)QY \\
& - \Big(\frac{r}{2} + \xi\beta - 2\Big(\alpha^2 - \beta^2\Big)\Big)(\eta(Y)X - \eta(X)Y) \\
& - (Y\beta + (\phi Y)\alpha)X + (X\beta + (\phi X)\alpha)Y.
\end{aligned} \tag{3.6}
$$

From equations (2.7) and (3.6), we have

$$
\begin{aligned}
& \eta(Y)\Big[QX - \Big(\frac{r}{2} + \xi\beta - \Big(\alpha^2 - \beta^2\Big)\Big)X - 2\alpha\beta\phi X + (X\beta)\xi\Big] \\
& + \beta d\eta(X,Y)\xi - (Y\alpha)\phi X - ((\phi Y)\alpha)X \\
= & \eta(X)\Big[QY - \Big(\frac{r}{2} + \xi\beta - \Big(\alpha^2 - \beta^2\Big)\Big)Y - 2\alpha\beta\phi X + (Y\beta)\xi\Big] \\
& - (X\alpha)\phi Y - ((\phi X)\alpha)Y.
\end{aligned}
$$

Putting $Y = \xi$, we get equation (3.4). □

Proposition 3.1. *Let M be a three-dimensional trans-Sasakian manifold of type (α,β). If* $\operatorname{grad}\beta = \phi(\operatorname{grad}\alpha)$ *then the Ricci operator and the structure tensor commute, i.e.* $Q\phi = \phi Q$.

Proof. Replace X by ϕX in equation (3.4), we get

$$(Q\phi)X = \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\phi X - ((\phi X)\beta - X\alpha + \eta(X)(\xi\alpha))\xi, \tag{3.7}$$

again from equation (3.4)

$$\phi(QX) = \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\phi X + \eta(X)\Big(\phi^2(\operatorname{grad}\alpha) - \phi(\operatorname{grad}\beta)\Big),$$

using equation (2.1), we have

$$\begin{aligned}(\phi Q)X =&\Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\phi X \\ &+ \eta(X)(-\operatorname{grad}\alpha + \eta(\operatorname{grad}\alpha)\xi - \phi(\operatorname{grad}\beta)).\end{aligned} \tag{3.8}$$

From equations (3.7) and (3.8), $Q\phi \neq \phi Q$ in general but if $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$, then $Q\phi = \phi Q$. □

Lemma 3.2. *In a three-dimensional trans-Sasakian manifold, Ricci-tensor is given by*

$$\begin{aligned}S(X,Y) = &\Big(\frac{r}{2} + \xi\beta - \Big(\alpha^2 - \beta^2\Big)\Big)g(X,Y) \\ &- \Big(\frac{r}{2} + \xi\beta - 3\Big(\alpha^2 - \beta^2\Big)\Big)\eta(X)\eta(Y) \\ &- (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y).\end{aligned} \tag{3.9}$$

Proof. Using equations (2.13) and (3.4), we get the equation (3.9). □

Definition 3.2. *An almost contact metric manifold M is said to be η-Einstein if its Ricci-tensor S is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.10}$$

equivalently an almost contact metric manifold M is said to be η-Einstein if its Ricci-operator is of the form

$$Q(X) = aX + b\eta(X)\xi, \tag{3.11}$$

where a and b are smooth functions on M. It is also known that for any η-Einstein K-contact manifold of dimension ≥ 5, a and b are constant.

Lemma 3.3. *If in a three-dimensional trans-Sasakian manifold of type* (α, β), $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$, *then*

$$X\beta + (\phi X)\alpha = 0, \tag{3.12}$$

$X \in TM$ *and*

$$\xi\beta = 0. \tag{3.13}$$

Proof. We know that

$$X\beta = g(X, \operatorname{grad}\beta) = g(X, \phi(\operatorname{grad}\alpha)) = -\,g(\phi X, \operatorname{grad}\alpha) = -\,(\phi X)\alpha,$$

which implies equation (3.12) and on putting $X = \xi$ in equation (3.12), we get the equation (3.13)

Theorem 3.1. *A sufficient condition for any three-dimensional trans-Sasakian manifold of type* (α, β) *to be an* η*-Einstein manifold is* $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$.

Proof. If $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$, then by using Lemma 3.3 and Definition 3.2, we get the result. □

Theorem 3.2. *A three-dimensional trans-Sasakian manifold of type* (α, β) *to be an* η*-Einstein manifold if and only if*

$$X\beta + (\phi X)\alpha = (\xi\beta)\eta(X), \tag{3.14}$$

$\forall X \in TM$.

Proof. Let us suppose that a three-dimensional trans-Sasakian manifold is η-Einstein. So from equation (3.9) and Definition 3.2, we must have an equation of the form

$$(X\beta + (\phi X)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(X) = cg(X, Y) + d\eta(X)\eta(Y), \tag{3.15}$$

$\forall X, Y \in TM$, where c and d are smooth functions which are to be determined. Replace X by ϕX and Y by ϕY in equation (3.15), we get

$$cg(\phi X, \phi Y) = 0 \Rightarrow c = 0.$$

Again putting $X = \xi$ and $Y = \xi$ in equation (3.15), we get

$$2\xi\beta = d.$$

Hence we have

$$(X\beta + (\phi X)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(X) = 2(\xi\beta)\eta(X)\eta(Y).$$

On taking $X = Y$ we get the equation (3.14).

Conversely, let the equation (3.14) is satisfied. With the help of equation (3.9), we get

$$S(X,Y)=\Big(\frac{r}{2}+\xi\beta-\alpha^2+\beta^2\Big)g(X,Y)$$
$$-3\Big(\frac{r}{6}+\xi\beta-\alpha^2+\beta^2\Big)\eta(X)\eta(Y), \tag{3.16}$$

which is the condition for η-Einstein manifold, where the values of a and b of equation (3.10) are $\Big(\frac{r}{2}+\xi\beta-\alpha^2+\beta^2\Big)$ and $-3\Big(\frac{r}{6}+\xi\beta-\alpha^2+\beta^2\Big)$ respectively. □

Corollary 3.1. *Every three-dimensional α-Sasakian manifold is η-Einstein iff α is a constant.*

Proof. Proof of this Corollary follows from Theorem 3.2. If we take $\beta=0$ in equation (3.14), we get $\phi(\operatorname{grad}\alpha)=0$. From $\phi(\operatorname{grad}\alpha)=0$ and $\xi\alpha=0\Rightarrow\alpha=$ constant. And if α is constant then proof is obvious. □

Corollary 3.2. *Every three-dimensional Sasakian manifold or Kenmotsu manifold or (α,β) trans-Sasakian manifold where α and β are some constants are η-Einstein.*

Proof. For Sasakian manifold, Kenmotsu manifold and (α,β) trans-Sasakian manifold, where α and β are constants, the condition

$$X\beta+(\phi X)\alpha=(\xi\beta)\eta(X),\qquad \forall X\in TM,$$

is satisfied. Hence the corollary is proved. □

Lemma 3.4. *Let M be a trans-Sasakian manifold of type (α,β) and of dimension three. Then the following conditions are equivalent*

(1) *M is η-Einstein.*
(2) $\phi(\operatorname{grad}\alpha)-\operatorname{grad}\beta=-\xi(\beta)\xi$.
(3) *The Ricci-operator of M is given by*

$$QX=\Big(\frac{r}{2}+\xi\beta-\big(\alpha^2-\beta^2\big)\Big)X-\Big(\frac{r}{2}+3\xi\beta-3\big(\alpha^2-\beta^2\big)\Big)\eta(X)\xi.$$

Proof. A manifold M is η-Einstein if Ricci-operator is of the form

$$QX=aX+b\eta(X)\xi, \tag{3.17}$$

for all $X \in TM$, where a and b are smooth functions. It is clear from equation (3.4) that a three-dimensional trans-Sasakian manifold of type (α, β) become η-Einstein if and only if

$$X\beta + (\phi X)\alpha = (\xi\beta)\eta(X),$$

$\forall X \in TM$. Let

$$\phi(\operatorname{grad}\alpha) - \operatorname{grad}\beta = a_1\xi,$$

where a_1 is to be determined by operating η both sides, we get

$$-\xi\beta = a_1.$$

Converse part can be proved easily. Conditions (2) and (3) are equivalent because

$$\begin{aligned} X\beta + (\phi X)\alpha &= (\xi\beta)\eta(X), \\ g(X, \operatorname{grad}\beta) + g(\phi X, \operatorname{grad}\alpha) &= (\xi\beta)\eta(X) \qquad \forall X \in TM, \\ g(X, \operatorname{grad}\beta - \phi(\operatorname{grad}\alpha)) &= g(X, (\xi\beta)\xi) \quad \forall X \in TM, \\ \Longrightarrow \phi(\operatorname{grad}\alpha) - \operatorname{grad}\beta &= -(\xi\beta)\xi. \end{aligned}$$

□

4 - η-Einstein trans-Sasakian manifold of dimension three

Let M be a $(2n+1)$-dimensional η-Einstein trans-Sasakian manifold with almost contact metric structure (ϕ, ξ, η, g). Let $\{e_1, e_2, ...e_{2n}, e_{2n+1} = \xi\}$ be orthonormal basis, then scalar curvature

$$r = \sum_{i=1}^{2n+1} S(e_i, e_i).$$

From equation (3.10), we have

$$r = (2n+1)a + b.$$

For a three-dimensional η-Einstein trans-Sasakian manifold

$$r = 3a + b. \tag{4.1}$$

Putting $Y = \xi$ and $X = \xi$ in equation (3.10), we get

$$S(\xi, \xi) = a + b.$$

From equation (3.9), we have

$$S(\xi, \xi) = 2\big(\alpha^2 - \beta^2 - \xi\beta\big)$$

so

$$2\left(\alpha^2-\beta^2-\xi\beta\right)=a+b. \tag{4.2}$$

From equations (4.1) and (4.2)

$$a=\frac{r}{2}-\left(\alpha^2-\beta^2-\xi\beta\right),\quad b=-\frac{r}{2}+3\left(\alpha^2-\beta^2-\xi\beta\right).$$

Hence we have the following lemma.

Lemma 4.1. *In three-dimensional η-Einstein trans-Sasakian manifold the Ricci-tensor S is given by*

$$\begin{aligned} S(X,Y)=&\left(\frac{r}{2}-\alpha^2+\beta^2+\xi\beta\right)g(X,Y)\\ &-3\left(\frac{r}{6}-\alpha^2+\beta^2+\xi\beta\right)\eta(X)\eta(Y), \end{aligned} \tag{4.3}$$

$\forall\, X,Y\in TM.$

Lemma 4.2. *In a three-dimensional trans-Sasakian manifold M of type* (α,β)*, curvature tensor is given by*

$$\begin{aligned} &R(X,Y)Z\\ &=\left(\frac{r}{2}+2\xi\beta-2\left(\alpha^2-\beta^2\right)\right)(g(Y,Z)X-g(X,Z)Y)\\ &-g(Y,Z)\Big\{\left(\frac{r}{2}+\xi\beta-3\left(\alpha^2-\beta^2\right)\right)\eta(X)\xi\\ &-\eta(X)(\phi(\mathrm{grad}\,\alpha)-\mathrm{grad}\,\beta)+(X\beta+(\phi X)\alpha)\xi\Big\}\\ &+g(X,Z)\Big\{\left(\frac{r}{2}+\xi\beta-3\left(\alpha^2-\beta^2\right)\right)\eta(Y)\xi\\ &-\eta(Y)(\phi(\mathrm{grad}\,\alpha)-\mathrm{grad}\,\beta)+(Y\beta+(\phi Y)\alpha)\xi\Big\}\\ &-\Big\{(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z)\\ &+\left(\frac{r}{2}+\xi\beta-3\left(\alpha^2-\beta^2\right)\right)\eta(Y)\eta(Z)\Big\}X\\ &+\Big\{(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z)\\ &+\left(\frac{r}{2}+\xi\beta-3\left(\alpha^2-\beta^2\right)\right)\eta(X)\eta(Z)\Big\}Y, \end{aligned} \tag{4.4}$$

$\forall X,Y,Z\in TM.$

Proof. For three-dimensional manifold

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y).$$

Using equations (3.4) and (3.9) in the above equation, we get the equation (4.4). □

Lemma 4.3. *In a 3-dimensional η-Einstein trans-Sasakian manifold the expression for curvature tensor R is*

$$\begin{aligned} R(X,Y)Z = &\Big(\frac{r}{2} + 2\xi\beta - 2\big(\alpha^2 - \beta^2\big)\Big)(g(Y,Z)X - g(X,Z)Y) \\ &- \Big(\frac{r}{2} + 3\xi\beta - 3\big(\alpha^2 - \beta^2\big)\Big)\{g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{4.5}$$

Proof. For η-Einstein trans-Sasakian manifold

$$X\beta + (\phi X)\alpha = (\xi\beta)\eta(X), \qquad \forall X \in TM$$

or

$$\phi(\operatorname{grad}\alpha) - \operatorname{grad}\beta = -(\xi\beta)\xi.$$

Using equation (4.4), we get the result. □

Lemma 4.4. *In a three-dimensional trans-Sasakian manifold M of type* (α,β) *if* $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$, *then curvature tensor R is given by*

$$\begin{aligned} R(X,Y)Z = &\Big(\frac{r}{2} - 2\big(\alpha^2 - \beta^2\big)\Big)(g(Y,Z)X - g(X,Z)Y) \\ &- \Big(\frac{r}{2} - 3\big(\alpha^2 - \beta^2\big)\Big)\Big[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &- \{\eta(Y)X - \eta(X)Y\}\eta(Z)\Big], \end{aligned}$$

$\forall X, Y, Z \in TM$.

Proof. From equation (4.4) and $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$, we get the result. □

Example 2.1 is a non-trivial example of trans-Sasakian manifold which satisfy the condition $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$.

5 - Three-dimensional conharmonically flat trans-Sasakian manifold

It is well known that a conformally flat manifold is conharmonically flat if and only if the scalar curvature vanishes i.e. if a manifold is conformally flat then it is conformally flat with zero scalar curvature. Conharmonically flat Kenmotsu manifold and non locally Euclidian Sasakian manifold do not exist [5].

The conharmonic curvature tensor $\overline{C}$ of type $(1,3)$ on a Riemannian manifold (M,g) of dimension-$(2n+1)$ is defined by

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{5.1}$$

$\forall\, X,Y,Z \in TM$, where Q and S are the Ricci-operator and Ricci-tensor respectively and related to each other as $S(X,Y) = g(QX,Y)$. If $\overline{C}$ vanishes identically then it is said that the manifold is conharmonically flat.

First we consider the manifold M which is conharmonically flat. From equation (5.1), it follows that

$$R(X,Y)Z = \frac{1}{(2n-1)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}. \tag{5.2}$$

Taking $Z = \xi$ in the above equation, we get

$$\begin{aligned} R(X,Y)\xi = \frac{1}{(2n-1)}\Big[&\Big(2n\Big(\alpha^2-\beta^2\Big) - \xi\beta\Big)(\eta(Y)X - \eta(X)Y) \\ &+ (2n-1)((X\beta)Y - (Y\beta)X) + ((\phi X)\alpha)Y - ((\phi Y)\alpha)X \\ &+ \eta(Y)QX - \eta(X)QY\Big]. \end{aligned}$$

For three-dimensional trans-Sasakian manifold taking $n = 1$ in above equation, we get

$$\begin{aligned} R(X,Y)\xi = &\Big(2\Big(\alpha^2-\beta^2\Big) - \xi\beta\Big)(\eta(Y)X - \eta(X)Y) - ((X\beta)Y - (Y\beta)X) \\ &+ ((\phi X)\alpha)Y - ((\phi Y)\alpha)X + \eta(Y)QX - \eta(X)QY. \end{aligned} \tag{5.3}$$

Taking $X = \xi$ and using equations (2.9) and (3.2), we have

$$\begin{aligned} QY = &\Big(3\Big(\alpha^2-\beta^2\Big) - \xi\beta\Big)\eta(Y)\xi - \Big(\Big(\alpha^2-\beta^2\Big) - \xi\beta\Big)Y \\ &- \eta(Y)(\operatorname{grad}\beta - \phi(\operatorname{grad}\alpha)) - ((Y\beta) + (\phi Y)\alpha)\xi. \end{aligned} \tag{5.4}$$

It is known

$$
\begin{aligned}
(5.5)\qquad S(X,Y) &= g(QX,Y)\\
&= \Big(\xi\beta - \Big(\alpha^2-\beta^2\Big)\Big)g(X,Y) - \Big(\xi\beta - 3\Big(\alpha^2-\beta^2\Big)\Big)\eta(X)\eta(Y)\\
&\quad - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y).
\end{aligned}
$$

Using equation (5.2), we get

$$
\begin{aligned}
(5.6)\qquad & R(X,Y)Z\\
&= \Big(2\xi\beta - 2\Big(\alpha^2-\beta^2\Big)\Big)(g(Y,Z)X - g(X,Z)Y)\\
&\quad - g(Y,Z)\Big\{\Big(\xi\beta - 3\Big(\alpha^2-\beta^2\Big)\Big)\eta(X)\xi\\
&\quad - \eta(X)(\phi(\mathrm{grad}\,\alpha) - \mathrm{grad}\,\beta) + (X\beta + (\phi X)\alpha)\xi\Big\}\\
&\quad + g(X,Z)\Big\{\Big(\xi\beta - 3\Big(\alpha^2-\beta^2\Big)\Big)\eta(Y)\xi\\
&\quad - \eta(Y)(\phi(\mathrm{grad}\,\alpha) - \mathrm{grad}\,\beta) + (Y\beta + (\phi Y)\alpha)\xi\Big\}\\
&\quad - \Big\{(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z)\\
&\quad + \Big(\xi\beta - 3\Big(\alpha^2-\beta^2\Big)\Big)\eta(Y)\eta(Z)\Big\}X\\
&\quad + \Big\{(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z)\\
&\quad + \Big(\xi\beta - 3\Big(\alpha^2-\beta^2\Big)\Big)\eta(X)\eta(Z)\Big\}Y.
\end{aligned}
$$

Theorem 5.1. *In a three-dimensional conharmonically flat trans-Sasakian manifold M of type (α,β) Ricci-operator, Ricci-tensor and curvature tensor are given by equations* (5.4), (5.5) *and* (5.6) *respectively.*

Lemma 5.1. *The three-dimensional conharmonically flat trans-Sasakian manifold M has zero scalar curvature.*

Proof. It is well known that every three-dimensional Riemannian manifold is conformally flat and it is given that M is conharmonically flat, so scalar curvature $r = 0$. □

6 - Eigenvalues and eigenvectors

Let $\{e_1, e_2, \xi\}$ be an orthonormal basis of a three-dimensional trans-Sasakian manifold. Taking $X = \xi$ in equation (3.4), we have

$$Q\xi = 2(\alpha^2 - \beta^2)\xi + (\phi(\text{grad}\,\alpha) - \text{grad}\,\beta) - (\xi\beta)\xi. \tag{6.1}$$

Now for η-Einstein trans-Sasakian manifold, using Lemma 3.4

$$Q\xi = 2(\alpha^2 - \beta^2 - \xi\beta)\xi,$$

which implies $2(\alpha^2 - \beta^2 - \xi\beta)$ is eigenvalue corresponding to eigenspace $\{\xi\}$. In the same fashion if $\phi(\text{grad}\,\alpha) = \text{grad}\,\beta$, then $2(\alpha^2 - \beta^2)$ is eigenvalue corresponding to eigenspace $\{\xi\}$. Now taking $X = e_i,\ i = 1, 2$ in equation (3.4), we have

$$Qe_i = \Big(\frac{r}{2} - \Big(\alpha^2 - \beta^2\Big)\Big)e_i; \qquad i = 1, 2,$$

which implies $\Big(\frac{r}{2} - \Big(\alpha^2 - \beta^2\Big)\Big)$ is eigenvalue corresponding to eigenspace $\{e_1, e_2\}$ i.e. $\Big(\frac{r}{2} - \Big(\alpha^2 - \beta^2\Big)\Big)$ is eigenvalue corresponding to contact distribution D.

Lemma 6.1. *In any three-dimensional Kenmotsu manifold with scalar curvature $r = -6$, the Ricci-operator Q has only one real eigenvalue -2.*

Proof. The Ricci-operator Q in any three-dimensional Kenmotsu manifold is obtained from equation (3.4) as

$$QX = \Big(\frac{r}{2} + 1\Big)X - \Big(\frac{r}{2} + 3\Big)\eta(X)\xi.$$

Taking $r = -6$, we get

$$QX = -2X.$$

Hence the result. □

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of vector field in M. If $r = -6$, then $Qe_i = -2e_i,\ i = 1, 2$ and $Q\xi = -2\xi$. So multiplicity of eigenvalue -2 is 3. If $r = -2$, then $Qe_i = 0,\ i = 1, 2$ and $Q\xi = -2\xi$. So multiplicity of eigenvalues -2 and 0 are 1 and 2 respectively.

7 - Projectively flat trans-Sasakian manifolds

In this section a necessary condition for projectively flat trans-Sasakian manifold to be Einstein is obtained.

The Weyl projective curvature tensor P of type $(1, 3)$ on a Riemannian manifold

(M,g) of dimension-$(2n+1)$ is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{7.1}$$

for any $X,Y,Z \in TM$. The manifold (M,g) is said to be projectively flat if P vanishes identically on M.

For projectively flat trans-Sasakian manifold $P=0$, from equation (7.1), we get

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{7.2}$$

From equation (7.2), we have

$$\begin{aligned} \overline{K}(X,Y,Z,W) &= g(R(X,Y)Z,W) \\ &= \frac{1}{2n}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]. \end{aligned} \tag{7.3}$$

Taking $W=\xi$ in equation (7.3), we get

$$\eta(R(X,Y)Z) = \frac{1}{2n}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)],$$

again taking $X=\xi$ and using equations (2.8) and (2.11), we get

$$\begin{aligned} S(Y,Z) =& 2n\Big(\alpha^2 - \beta^2 - \xi\beta\Big)g(Y,Z) - (2n-1)(\xi\beta)\eta(Y)\eta(Z) \\ &+ ((2n-1)(Z\beta) + (\phi Z)\alpha)\eta(Y), \end{aligned} \tag{7.4}$$

if

$$(2n-1)(d\beta - \xi(\beta)\eta) + d\alpha \circ \phi = 0, \tag{7.5}$$

then

$$S(Y,Z) = 2n\Big(\alpha^2 - \beta^2 - \xi\beta\Big)g(Y,Z). \tag{7.6}$$

Hence we have following theorem.

Theorem 7.1. *A Weyl projectively flat trans-Sasakian manifold is an Einstein manifold iff equation* (7.5) *is satisfied.*

Proposition 7.1. *A Weyl projectively flat trans-Sasakian manifold of type* (α,β) *is an Einstein manifold If* $\phi(\operatorname{grad}\alpha) = (2n-1)\operatorname{grad}\beta$.

Proof. Consider

$$\xi\beta = g(\xi,\operatorname{grad}\beta) = \frac{1}{2n-1}g(\xi,\phi(\operatorname{grad}\alpha)) = -\frac{1}{2n-1}g(\phi\,\xi,\operatorname{grad}\alpha) = 0. \tag{7.7}$$

Also consider

$$(2n-1)Z\beta = (2n-1)g(Z, \operatorname{grad}\beta)$$

$$= \frac{(2n-1)}{(2n-1)} g(Z, \phi(\operatorname{grad}\alpha)) = -g(\phi Z, \operatorname{grad}\alpha) = -(\phi Z)\alpha,$$

i.e.

$$(2n-1)Z\beta + (\phi Z)\alpha = 0. \tag{7.8}$$

Using equations (7.7) and (7.8) in equation (7.4), we get

$$S(Y, Z) = 2n\left(\alpha^2 - \beta^2\right)g(Y, Z).$$

Hence the Proposition. □

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