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Nonlinear ergodic theory

Abstract. J.-B. Baillon is considered the father of nonlinear ergodic theory and its deep applications in nonlinear analysis. In this paper, we present a development of this relatively young and exciting theory.

Keywords. Fixed point, Cesàro means, ergodic theorem, almost-convergent, nonexpansive maps, asymptotically nonexpansive maps, asymptotically nonexpansive maps in the intermediate sense, semigroups, amenable semigroups, nonexpansive retraction.

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1 - Introduction

Although a result of M. Edelstein [**15**] may be called the first nonlinear ergodic theorem (cf. MR 81c:47056), it is J.-B. Baillon [**3, 4**] who is credited for establishing the first nonlinear ergodic theorem using Cesàro means for a nonexpansive map in Hilbert spaces.

Baillon's theorem gives a new dimension in nonlinear functional analysis and, at present, nonlinear ergodic theory is one of the frontier activities in operator theory and summability methods.

The significance of Baillon's theorem can be realized in the light of the well-known Brouwer's theorem which uses only the continuity of the map. However, if a map is nonexpansive then the sequence of iterates need not converge to a fixed point of the map. For example, take $X = [0, 1]$ with the absolute value metric and

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$T : X \to X$ such that $Tx = 1 - x$. Then, for $x_0 \neq \frac{1}{2}$, the Picard sequence of iterates $\{T^n x_0\}$ is not convergent. However, the sequence $\{T^n x_0\}$ is $(C, 1)$ summable to the fixed point 1/2 of T.

In the present paper, we present a survey and a brief historical development of nonlinear ergodic theorems. However, owing to the diversity of the field and limitations in accessibility of literature, we do not claim it to be exhaustive.

2 - Preliminaries

Definition 1. A Banach space E is called *uniformly convex* if, given any $\varepsilon > 0$ there exist a $\delta > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, we have $\frac{1}{2}\|x + y\| < 1 - \delta$.

The $L_p (1 < p < \infty)$ spaces and all Hilbert spaces are examples of uniformly convex Banach spaces (see, for instance, Baillon [**7**], Berinde [**8**]).

Definition 2. A sequence $\{x_n\}$ in a Banach space E is said to be *strongly* (*weakly*) *almost-convergent* to $x \in E$ if the strong (weak) $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} x_{i+k} = x$ uniformly in $k \geq 0$.

Example 1 ([**21**]). Let us take a space

$$l^1 = \left\{ x = (x_1, x_2, \ldots) : \sum_{i=1}^{\infty} |x_i| < \infty, \ \ x_i \in R \right\}.$$

This space with norm $\|x\| = \sum_{i=1}^{\infty} |x_i|$ is a Banach space. Let $C = \{x \in l^1 : \|x\| \leq 1\}$ and let $T : C \to C$ be the mapping $T(x_1, \ x_2, \ \ldots) = (0, \ x_1, \ x_2, \ \ldots)$. This is a nonexpansive map having only one fixed point $Q = (0, 0, 0, \ldots)$. We can choose a point x_0 in the set C such that the sequence $\{T^n x_0\}_{n=0}^{\infty}$ is not almost convergent to the point Q. Indeed, for $x_0 = e_1 = (1, 0, 0, \ldots)$, it is noticeable that $Te_1 = e_2 = (0, 1, 0, \ldots,)$, $T^2 e_1 = e_3$, etc. Let $A = [a_{nk}]_{0 \leq n,k < \infty}$ be a real strongly ergodic matrix, wherein

$$a_{nk} = \begin{cases} \dfrac{1}{n+1} & \text{for } n \geq k, \\ 0 & \text{for } n < k, \end{cases} \qquad n, \ k = 0, 1, 2, \ldots \ .$$

Then

$$S_n(e_1) = \sum_{k=0}^{\infty} a_{nk}\, T^k e_1 = \frac{1}{n+1}\sum_{k=0}^{n} T^k e_1$$

$$= \left(\frac{1}{n+1}, \ldots (n+1)-\text{components} \ldots, \frac{1}{n+1}, 0, 0, \ldots\right), \quad n = 0, 1, 2, \ldots .$$

Notice that, for each $n = 1, 2, \ldots$, $\|S_n(e_1)\| = 1$ so $S_n e_1 \to Q$ as $n \to \infty$. This means that the sequence $\{T^n e_1\}_{n=0}^{\infty}$ is not almost convergent to the fixed point Q.

Definition 3. Let E be a real Banach space and $B = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Fréchet differentiable* if for $x \in B$, $\lim\limits_{t\to 0} \dfrac{\|x+ty\| - \|x\|}{t}$ is attained uniformly for y in B.

The $L_p(1<p<\infty)$ spaces and Hilbert spaces are examples of uniformly convex Banach spaces which have Fréchet differentiable norms (see Baillon [**7**]).

Definition 4. Let $\{x_n\}$ be a sequence of real numbers and $A = (a_{n,k})$ be an infinite matrix. If $\sigma_n = \sum\limits_k a_{n,k}\, x_k$ converges for $n \geq 0$, then the sequence $\{\sigma_n\}$ is the A-transform of the sequence $\{x_n\}$. A is called *regular* if it is limit preserving over the space of convergent sequences.

Definition 5. An infinite matrix A is said to be *strongly regular* if A assigns a limit to each almost convergent sequence. Necessary and sufficient conditions for A to be strongly regular are that A be regular and satisfy $\lim\limits_n \sum\limits_{x=0}^{\infty} \left|a_{n,k+1} - a_{n,k}\right| = 0$ (see Lorentz [**39**]).

Definition 6. A matrix A is called a *weighted mean matrix* if A is a lower triangular matrix with nonzero entries $a_{nk} = p_k/P_n$, where $\{p_k\}$ is a nonnegative sequence with p_o positive and $P_n = \sum\limits_{k=0}^{n} p_k \to \infty$.

Definition 7 (cf. Zeidler [**80**, p. 54]). Let E be a Banach space. A self-map T of E is called *compact* if T is continuous and T maps bounded sets into precompact (also called relatively compact) sets.

It is well known that a compact map is always continuous. However, a continuous map need not be compact. For example, the identity map is continuous but is not compact (see also Singh [**64**]).

Definition 8. Let E be a Banach space. A selfmap T of E is said to be *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all x, $y \in X$.

Translation, identity, isometry maps and orthogonal projections are examples of nonexpansive maps (see also Agarwal *et al.* [**1**], Berinde [8], Singh [**64**] and Zeidler [**80**]).

Definition 9 ([**19**]). Let C be a nonempty subset of Banach space E. A selfmap T of C is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive real numbers with $\lim_n k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for $n \geq 1$ and $x, y \in X$.

An asymptotically nonexpansive map is nonexpansive if $k_n = 1$ for all $n \geq 1$ (see Górnicki [**20**], Tan and Xu [**77, 78**]).

Definition 10 ([**13**], see also Oka [**50**]). Let C be a closed convex subset of a Banach space E. A self- map T of C is called *asymptotically nonexpansive in the intermediate sense* if T is continuous and

$$\lim_n \sup \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0, \text{for any bounded subset } K \subset \ C.$$

Definition 11 ([**16**]). Let E be a Banach space and $\{x_n\}$ a bounded sequence in a convex subset C of E. Define $r_m(u) = \sup\{\|x_n - u\| : n \geq m\}$, and denote by c_m the unique point in C with the property that $r_m(c_m) = \inf\{r_m(u) : u \in C\}$. Then $\lim_n c_n = c$, and c is called the *asymptotic center* of $\{x_n\}$ (see also Berinde [8]).

Definition 12 ([**51**], see also Tan and Xu [**78**]). A Banach space E is said to *satisfy the Opial's condition* if for any sequence $x_n \subset X$ weakly convergent to a point $x_0 \in X$ and for all $x \neq x_0$, $\lim_n \inf \|x_n - x_0\| < \lim_n \inf \|x_n - x\|$, or equivalently

$$\lim_n \sup \|x_n - x_0\| < \lim_n \sup \|x_n - x\|.$$

The example of Banach spaces which satisfy Opial condition is every Hilbert space, and the l_p spaces $(1 < p < \infty)$. But the spaces $L_p (1 < p < \infty)$, $p \neq 2$ do not satisfy the Opial condition (see Górnicki [**21**] Tan and Xu [**77**]).

3 - Ergodic theorems for nonexpansive and asymptotically nonexpansive maps

Nonlinear ergodic theory is an interesting area of study in nonlinear analysis. It has wide applications in different fields of mathematics and mathematical sciences.

There is a close connection between fixed point iteration processes and summability methods. We begin with a brief discussion on the role of the Mann iterative procedure in summability methods.

Let C be a closed convex subset of a Banach space E and $T: C \to C$. The Mann iteration process (or MIP in brief) is defined by

$$x_0 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0,\ 1,\ 2, \ldots,$$

where the sequence $\{\alpha_n\}$ satisfies (i) $\alpha_0 = 1$, (ii) $0 \le \alpha_n < 1$ and (iii) $\sum \alpha_n = \infty$.

Mann [**40**] proved that, if a continuous self-map T of a closed interval $[a, b]$ has a unique fixed point then the MI scheme converges to a fixed point. Franks and Marzec [**18**] extended this result to continuous functions possessing more than one fixed point in the interval. Rhoades [**58**] (see also Rhoades [**60**]), in turn, extended Frank and Marzec's result to any continuous self-map of an interval $[a, b]$. Precisely, he proved the following.

Theorem 1. *Let A be a regular weighted mean method satisfying*

$$\lim_n \sum_{k=0}^{\infty} |a_{n,k} - a_{n-1,k}| = 0,$$

and T a continuous mapping from $[a, b]$ into itself. Then the MI scheme converges to a fixed point of T.

Unless otherwise stated, let H be a Hilbert space, K a bounded closed convex subset of H, K^C a closed convex subset of H, N the set of natural numbers and $F(T)$ the set of fixed points of a map T.

The following is the first nonlinear ergodic theorem, essentially due to Baillon [**3**] obtained by using the Cesàro means for nonexpansive maps in Hilbert spaces.

Theorem 2. *Let K be a subset of H and T a nonexpansive self-map of K. Then, for each $x \in K$, the Cesàro mean $S_n x = \dfrac{1}{n}\sum_{i=0}^{n-1} T^i x$ converges weakly to some fixed point $y \in F(T)$.*

Baillon [**5**] (see also Baillon [**6, 7**]) further obtained the following strong convergence theorem for an odd map.

Theorem 3. *Let K^C be a subset of H and T be a nonexpansive and odd self-map of K^C, i.e., $-K^C = K^C$ and $T(-u) = Tu$ for $x \in K^C$. Then for every $x \in K^C$, $\{T^n x\}$ is strongly almost convergent to a fixed point of T.*

We remark that it may happen that the Cesàro means, $S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converge to a point which is not the fixed point of T even if T is nonexpansive (see Wittmann [**79**]). The following example supports this observation.

Let R_+ be the set of non-negative reals.

Example 2 ([**79**]). We define

$$T(x_1, x_2) = \begin{cases} \dfrac{\max(x_1^2, x_2^2)}{x_1^2 + x_2^2}(x_2, x_1) & \text{if}(x_1, x_2) \in R_+^2 \setminus \{(0,\ 0)\}, \\ 0 & \text{if}(x_1, x_2) = (0,\ 0). \end{cases}$$

Again we have

$$\|T(x_1,\ x_2) + T(y_1,\ y_2)\| \le \|(x_1,\ x_2) + (y_1,\ y_2)\|,\ \big((x_1,\ x_2),\ (y_1,\ y_2) \in R_+^2\big),$$

and T is continuous on R_+^2 with $(0, 0)$ as the only fixed point.

On the other hand, if $x > 0$ the averages $\frac{1}{n+1}\sum_{i=0}^{n} T^i(x, 0)$ converge to (x, x) which is not a fixed point.

The power of Theorem 2, in terms of its applications, attracted several mathematicians and researchers to work along these lines. Subsequently, it was extended and generalized in various ways. Among some of the immediate generalizations of Theorem 2 are those due to Brézis and Browder [**9, 10**], Bruck [**11, 12**], Hirano and Takahashi [**26**], Massa [**41**], Pazy [**52, 53**] and Reich [**54-56**].

Brézis and Browder [**9**] replaced the Cesàro means in Theorem 2 by a more general summability method and established the following theorem for matrix transform of iterates of T.

Theorem 4. *Let T be a nonexpansive self-map of K. Suppose that A is an infinite matrix with zero column limits and satisfies $\lim_n \sum_{k=0}^{\infty}(a_{n,k+1} - a_{n,k}) = 0$. Then, for each $x \in K$, $\sum_{k=0}^{\infty} a_{n,k} T^k n$ converges weakly to a fixed point of T.*

Bruck [**11**] obtained the following interesting weak convergence theorem.

Theorem 5. *Let T be a nonexpansive self-map of C with a fixed point. Then for any $x \in K^C$ and any strongly regular matrix A, the A-transform of $\{T^n x\}$ converges weakly to a fixed point p of T, which is the asymptotic center of $\{T^n x\}$.*

In order to establish a strong convergence theorem for a nonexpansive map on a Hilbert space, Pazy [**52**] required map T to be compact and obtained the following result.

Theorem 6. *Let T be a nonexpansive selfmap of K^C. If T is compact then $F(T) \neq \emptyset$ and, for every $x \in K^C$, $S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converges strongly to a fixed point p of T.*

Nonlinear ergodic theorems, for nonexpansive maps, were extended to the Banach space setting, among others, by Baillon [**6, 7**], Bruck [**12**], Górnicki [**20**], Hirano [**22, 23**], Kobayasi and Miyadera [**30**], Reich [**56, 57**] and Samanta [**62**]. We may divide these theorems into two categories. The first category covers those theorems for which a uniformly convex Banach space has a Fréchet differentiable norm (see, for instance, Baillon [**6, 7**], Bruck [**12**], Hirano [**22**], Kobayasi and Miyadera [**30**], Reich [**56, 57**] and references therein). The results in the second category are those theorems for which the uniformly convex Banach space satisfies Opial's condition (see, for instance Górnicki [**20**], Hirano [**23**], Samanta [**62**] and references thereof).

Throughout this section, let E be a uniformly convex Banach space, C a bounded closed convex subset of E and C^C a closed convex subset of E.

The following result, due to Bruck [**12**], is obtained for nonexpansive maps in a uniformly convex Banach space with Fréchet differentiable norm.

Theorem 7. *Let T: $C \to C$ be a nonexpansive self-map of C and E has a Fréchet differentiable norm. Then $\{T^n x\}$ is weakly almost-convergent to a fixed point of T.*

Reich [**56**] (see also Hirano [**22**]) obtained the following analogous result.

Theorem 8. *Let the norm of E be Fréchet differentiable and T: $C \to C$ a nonexpansive self-map of C with a fixed point. Then $\{T^n x\}$ is weakly almost convergent to a fixed point of T.*

In the case of uniformly Banach spaces which satisfy the Opial condition, Hirano [**23**] established the following.

Theorem 9. *Let E satisfy Opial's condition and T a nonexpansive self-map of C^C with a fixed point $x \in C^C$. Then the sequence $\{T^n x\}$ is weakly almost convergent to a fixed point of T.*

The concept of nonexpansive maps was generalized by Goebel and Kirk [**19**]. They introduced a new class of maps, namely asymptotically nonexpansive maps. Hirano and Takhashi [**26**] were the first to extend Theorem 2 for this class of maps in the framework of Hilbert spaces. Subsequently, several papers proposing generalizations and extensions of their theorem and Theorem 2 have appeared (see, for instance, Górnicki [**20**], Krüppel and Górnicki [**31**], Miyadera [**42-44**], Oka [**47, 48**], Rhoades [**59**] and Tan and Xu [**77-78**]).

The following result due to Rhoades [**59**] extends Hirano and Takahashi's result [**26**] to a strongly regular matrix in Hilbert spaces.

Theorem 10. *Let T be an asymptotically nonexpansive self-map of K^C such that $\{T^n x\}$ is bounded for each $x \in K^C$. Let A be a strongly regular matrix. Define $A_n x = \sum_{k=0}^{\infty} a_{n,k} T^k x$. Then, for each $x \in K^C$, $\{A_n x\}$ converges weakly to a fixed point p, which is the asymptotic center of $\{T^n x\}$.*

Let $\omega_w(\{x_n\})$ be the set of the weak subsequential limits of $\{x_n\}$ and $\operatorname{clco} M$ be the closed convex hull of M.

A sequence $\{x_n\}$ in C is said to be an almost orbit of T if

$$\lim_n \left[\sup_m \|x_{n+m} - T^n x_n\| \right] = 0.$$

Giving an affirmative answer to the question that whether Baillon's result [**3**] is valid for asymptotically nonexpansive maps in a Banach space, Oka [**47**] established the following theorem for an asymptotically nonexpansive map in uniformly convex Banach spaces with a Fréchet differentiable norm.

Theorem 11. *Let T be an asymptotically nonexpansive self map of C and $\{x_n\}$ be an almost-orbit of T. If the norm of E is Fréchet differentiable then, $\{x_n\}$ is weakly almost convergent to the unique point of $F(T) \cap \operatorname{clco} \omega_w(\{x_n\})$.*

Now we cite the following result obtained by Górnicki [**20**] (see also Tan and Xu [**78**]) for an asymptotically nonexpansive map in uniformly convex Banch spaces satisfying Opial's condition.

Theorem 12. *If E is a uniformly convex Banach space satisfying Opial's condition and C a nonempty bounded closed convex subset of E, then, for each asymptotically nonexpansive map T: $C \to C$, the sequence of iterates $\{T^n x\}$ is weakly almost-convergent to a fixed point of T.*

Tan and Xu [**77**] obtained the following weak convergence theorem for a uniformly convex Banach space with a Fréchet differential norm.

Let co(A) denote the closure of the convex hull of A.

Theorem 13. *Let T be an asymptotically nonexpansive self map of C and the norm of E be Fréchet differentiable norm. Then for each $x \in C$, $\{T^n x\}$ is weakly almost convergent to the unique point of the set* $\bigcap_{n=0}^{\infty} \overline{\text{co}}\{T^i x : i \geq n\} \cap F(T)$.

Miyadera [**43**] (see also Miyadera [**42**]) established the following strong convergence theorem using a condition weaker than asymptotic nonexpansivity of map T (see condition (i) below). Further, he relaxed the closedness and convexity of the subset K of Hilbert space H.

Let K be a nonempty subset of Hilbert space H.

Theorem 14. *Suppose for every $u, v \in K$ and integer $k \geq 0$, there exists a $\delta_x(u, v) \geq 0$ with $\lim_k \delta_k(u, v) = 0$ such that*

$$\text{(i)} \qquad \begin{aligned} &\|T^k u - T^k v\|^p \\ &\leq a_k\|u - v\|^p + c\Big[a_k\|u\|^p - \|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p\Big] + \delta_k(u, v), \end{aligned}$$

where a_k, c and p are nonnegative constants independent of u and v such that $\lim_k a_k = 1$ and $p \geq 1$. If either $F(T) \neq \emptyset$ or $c > 0$ in (i), *and if $x \in K$ satisfies*

$$\text{(ii)} \qquad \overline{\lim_m}\ \overline{\lim_n}\ \sup_{i\geq 0}\Big[\big\|T_x^{m+i} - T^n x\big\|^2 - \big\|T^{n+i} - T_x^n\big\|^2\Big] \leq 0.$$

Then $\{T^n x\}$ is strongly almost convergent to its asymptotic center.

If T: $K \to K$ is asymptotically nonexpansive and odd, then it satisfies the following condition:

$$\text{(iii)} \qquad \|T^n x + T^n y\| \leq k_n\|x + y\|, \quad \textit{for } x, y \in C \textit{ and } k \geq 0,$$

where $\{k_n\}$ is a sequence of positive real numbers with $\lim_n k_n = 1$.

Wittmann [**79**] obtained the following interesting result for an asymptotically nonexpansive odd map in a Hilbert space.

Theorem 15. *Let K be a subset of H and $T : K \to K$ satisfy the condition* (iii). *Then for any $x \in K$,* $\left(\frac{1}{n}\right)\sum_{i=0}^{n-1} T^i x$ *is norm convergent.*

This theorem generalizes Theorem 3. Further, notice that "closedness" and "convexity" of K are not required in the above theorem.

Replacing condition (iii) of Theorem 15 by the weaker condition (iv) (see Theorem 16 below), Miyadera [**42**] obtained the following theorem for an odd map.

Theorem 16. *Suppose that for every bounded set $B \subset K$ and integer $k \geq 0$, there exists $\delta_k(B) \geq 0$ with $\lim_k \delta_k(B) = 0$ such that*

$$\text{(iv)} \qquad \begin{aligned} &\left\|T^k u + T^k v\right\|^p \\ &\leq a_k\|u+v\|^p + c\left[a_k\|u\|^p - \left\|T^k u\right\|^p + a_k\|v\|^p - \left\|T^k v\right\|^p\right] + \delta_k(B) \end{aligned}$$

for $u, v \in B$, where a_k, c and p are nonnegative constants independent of B such that $\lim_k a_k = 1$ and $p \geq 1$. Then, for $x \in K$, $\{T^n x\}$ is strongly almost convergent to its asymptotic center y, i.e., $\lim_n \left(\frac{1}{n}\right) \sum_{i=0}^{n-1} T^{i+k}x = y$ uniformly in $k \geq 0$.

In the above theorem, condition (iv) with $a_k = 1$, $p = 2$ and $\delta_k(B) = 0$ reduces to that of Brézis and Browder [**9**], Theorem 2. Further, condition (iii) of Theorem 15 is condition (iv) with $c = 0$, $\delta_k(B) = 0$ and $p = 1$.

In his subsequent paper, Miyadera [**43**] further improved Theorem 16 by using the following condition in place of (iv).

For every bounded subset $B \subset C$, $v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B, v) \geq 0$ with $\lim_k \delta_k(B, v) = 0$ such that

$$\begin{aligned} &\left\|T^k u + T^k v\right\| \\ &\leq a_k\|u+v\|^p + c\left[a_k\|u\|^p - \left\|T^k u\right\|^p + a_k\|v\|^p - \left\|T^k v\right\|^p\right] + \delta_k(B, v) \end{aligned}$$

for $u \in B$, where a_k, c and p are nonnegative constants independent of B and v such that $\lim_k a_k = 1$ and $p \geq 1$.

In due course of time, the asymptotic behavior of asymptotically nonexpansive maps has been studied by many researchers (see, for instance, Bruck, Kuczumow and Reich [**13**], Miyadera [**42-44**], Oka [**50**], and Tan and Xu [**77, 78**]). Bruck, Kuczumow and Reich [**13**] defined asymptotically nonexpansive maps in the intermediate sense. If $T : C \to C$ is asymptotically nonexpansive then it is asymptotically nonexpansive in the intermediate sense. However, the converse is not true (see Miyadera [**44**]).

We cite here the following result of Oka [**50**] obtained for an asymptotically nonexpansive map in the intermediate sense.

Theorem 17. *Let C^C be a subset of E. Suppose that $T : C^C \to C$ is asymptotically nonexpansive in the intermediate sense with $F(T) \neq \emptyset$, and that $\{x_n\}$ is* *an almost-orbit of T. If the norm of E is Fréchet differentiable, then* $\{x_n\}$ *is weakly almost convergent to the unique point of* $F(T) \cap \operatorname{clco} \omega_w(\{x_n\})$.

The following result is due to Miyadera [**44**].

Theorem 18. *Let C be a subset of E,* $T : C \to C$ *be an asymptotically nonexpansive in the intermediate sense and let* $x \in C$. *If* $\lim_n \|T^{n+1}x - T^n y\|$ *exists uniformly in* $i \geq 1$, *then* $\{T^n x\}$ *is strongly almost convergent to a fixed point of T.*

4 - Nonlinear ergodic theorems for semigroups of maps

In their recent work, Lau and Takahashi [**35**] have presented a nice discussion of nonlinear ergodic theorems for semigroups of nonexpansive maps. The intent of this section is to present some fundamental work concerning ergodic theorems for nonlinear maps. Our survey includes some very recent results as well.

In due course of time, analogous results for nonlinear ergodic theorems on semigroups were obtained by several authors (see, for instance, [**4, 10, 17, 24, 27-29, 32-38, 49, 61, 65, 66, 70-73** and **76**]).

Let C be a nonempty closed convex subset of a real Banach space E and $S = \{S(t) : t \geq 0\}$ be a family of nonexpansive mappings of C into itself such that $S(0) = \mathrm{I}$, $S(t+s) = S(t)S(s)$ for all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. Then S is said to be a nonexpansive semigroup on C (see [**27**] and [**46**]). Let E^* be the dual of real Banach space E, S be an abstract semigroup and $\mathrm{m}(S)$ the Banach space of all bounded real valued functions on S with the supremun norm. For each $s \in S$ and $f \in \mathrm{m}(S)$, we define two operators $\ell(s)$ and $r(s)$ on $\mathrm{m}(S)$ by $(\ell(s)f)(t) = f(st)$ and $(r(s)f)(t) = f(ts)$ for each $t \in S$. Let X be a subspace of $\mathrm{m}(S)$ containing 1. An element μ of the topological dual X^* of X is said to be a mean of X if $||\mu|| = \mu(1) = 1$. A mean μ is called left (resp. right) invariant if $\mu(\ell(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. An invariant mean is a left and right invariant mean (cf. [**27, 35** and **46**]).

A semigroup which has an invariant mean is called amenable (see [**35** and **46**]). It is known that if S is commutative, then X is amenable (see Day [**14**]). Let $\{\mu_\alpha\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is said to be asymptotically invariant if for each $s \in S$, both $\ell(s)^*\mu_\alpha - \mu_\alpha$ and $r(s)^*\mu_\alpha - \mu_\alpha$ converge to 0 in the weak star topology, where $\ell(s)^*$ and $r(s)^*$ are the adjoint operators of $\ell(s)$ and $r(s)$ respectively.

The semigroup S is called semitopological if S is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \to a.s$ and s $\to s.a$ from S to S are continuous. S is called right reversible if any two closed left ideals of S have nonempty intersection. The left reversibility is defined analogously. Further, S is called reversible if it is both left and right reversible (see [**33, 35** and **80**]).

As an immediate generalization of Baillon's result in the domain of commutative semigroups, Brézis and Browder [**10**], Hirano and Takahashi [**26**] and others obtained nonlinear ergodic theorems in the framework of a Hilbert space.

With reference to Theorem 2, putting $y = Px$ for each $x \in K$, P is a nonexpansive retraction from K onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in K$. Such a retraction is called an ergodic retraction (see Miyake and Takahashi [**46**] and Lau and Takahashi [**35**]). The existence of such a retract for an amenable semigroup of nonexpansive maps on a Hilbert space was obtained by Takahashi [**70**]. Precisely, he obtained the following theorem.

Theorem 19. *Let K^C be a closed convex subset of a Hilbert space H and S an amenable semigroup of nonexpansive maps t of K^C into itself. Suppose*

$$F(S) = \cap\{F(t) : T \in S\} \neq \emptyset.$$

Then there exists a nonexpansive retraction P of K^C onto $F(S)$ such that $Pt = tP = P$ for all $t \in S$ and $Px \in \overline{\mathrm{co}} Sx$ for all $x \in K^C$, where $Sx = \{tx : t \in S\}$ and $\overline{\mathrm{co}} A$ is the closure of the convex hull of A.

Rodé [**61**], studied the ergodic behavior of a noncommuting semigroup of nonexpansive maps in Hilbert spaces. Finding a sequence of means on the semigroups, he generalized the Cesàro means and obtained the following result (see also [**35**]).

Theorem 20. *Let S be an amenable semigroup, K^C a subset of a Hilbert space H, T a nonexpansive map from K^C into itself, $\mathbf{S} = \{T_t : t \in S\}$ a nonexpansive semigroup of K^C and $\{\mu_\alpha\}$ an asymptotically invariant net of means. Then for each $x \in K^C$, $\{T\mu(\alpha)x\}$ converges weakly to an element of $F(S)$. Further, for each $x \in K^C$, the limit point of $\{T\mu(\alpha)x\}$ is the same for all asymptotically invariant nets $\{\mu_\alpha\}$ of means.*

Takahashi's and Rodé's results were extended to the Banach space setting, among others, by Eshita and Takahashi [**17**], Hirano, Kido and Takahashi [**24, 25**], Hirano and Takahashi [**27**], Lau, Shioji and Takahashi [**32**], Lau and Takahashi [**34**], Li [**36**], Li and Ma [**38**], Shioji and Takahashi [**63**], Suzuki [**66-68**] and Suzuki and Takahashi [**69**]. The following result was obtained by Hirano and Takahashi [**27**].

Theorem 21. *Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let S be an amenable semigroup of non-*

expansive mappings t of C into itself. Suppose $F(S) = \cap\{F(t) : t \in S\} \neq \emptyset$. *Then the following conditions are equivalent:*

(1) *for each* $x \in C$, $\cap_s \overline{\mathrm{co}}\,\{tx : t \geq s\} \cap F(S) \neq \emptyset$;
(2) *there exists a nonexpansive retraction P of C onto* $F(S)$ *such that* $Pt = tP = P$ *for all* $t \in S$ *and* $Px \in \cap_s \overline{\mathrm{co}}\,\{tx : t \geq s\}$ *for each* $x \in C$.

Let S be a semitopological semigroup, $C(S)$ the Banach algebra of all continuous bounded real-valued functions on S with the supremum norm. Let $\mathrm{RUC}(S)$ be the space of bounded right uniformly continuous functions on S and $\pounds = \{T_a : a \in S\}$ a continuous representation of S as nonexpansive maps from a nonempty closed convex subset C of a Banach space E into C.

Lau and Takahashi [**33**] obtained the following theorem for semitopological semigroups in a Banach space with a Fréchet differentiable norm.

Theorem 22. *Let C be a closed convex subset of a uniformly convex Banach space with a uniformly Fréchet differentiable norm and S a reversible semitopological semigroup. If* $\mathrm{RUC}(S)$ *has a right invariant mean, then the following are equivalent:*

(1) $\cap_s \overline{\mathrm{co}}\,\{tx : t \geq s\} \cap F(\pounds) \neq \emptyset$ *for each* $x \in C$;
(2) $F(\pounds)$ *is nonempty and there is a nonexpansive retraction P of C onto* $F(\pounds)$ *such that* $PT_t = T_tP = P$ *for every* $t \in S$ *and* $Px \in \overline{\mathrm{co}}\,\{T_tx : t \in S\}$ *for each* $x \in C$.

Let C be a subset of a Banach space E, T be a nonexpansive mapping on C and τ a Hausdorff topology on E. A family of mappings $\{T(t) : t \geq 0\}$ is called a one-parameter strongly continuous semigroup of nonexpansive maps on C if the following conditions are satisfied; (i) for each $t \geq 0$; $T(t)$ is a nonexpansive map on C; (ii) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$; and (iii) for each $x \in C$, the mapping $t \to T(t)x$ from $[0, \infty)$ into C is strongly continuous (see Suzuki [**66, 68**]).

Suzuki [**68**] obtained the following theorem for one-parameter continuous semigroup of maps.

Theorem 23. *Let C be a subset of a Banach space E,* τ *a Hausdorff topology on E, and* $\{T(t) : t \geq 0\}$ *be a one-parameter* τ*-continuous semigroup of maps on C. Let* α *and* β *be positive real numbers satisfying* $\alpha/\beta \notin \mathbb{Q}$ *(rationals). Then* $\bigcap_{t\geq 0} F(T(t)) = F(T(\alpha)) \cap F(T(\beta))$ *holds.*

Convergence theorems for one parameter nonexpansive semigroups were obtained by several authors (see, for instance, [**2, 4, 45, 65-69**]). In [**68**], Suzuki proved that many convergence theorems to a common fixed point of a one-parameter continuous semigroup of nonexpansive maps can be obtained by using Theorem 23 (see also Takahashi [**74**]). Suzuki and Takahashi [**69**] obtained the following theorem.

Theorem 24. *Let C be a compact convex subset of a Banach space E and $\{T(t) : t \geq 0\}$ a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let $x \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\lambda}{t_n} \int_0^{t_n} T(s)\, x_n \, dS + (1 - \lambda)\, x_n \ \textit{ for } n \in N,$$

where λ is a constant in $(0, 1)$ and $\{t_n\}$ is a sequence in $(0, \infty)$ satisfying $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

Miyake and Takahashi [**46**] obtained the following strong convergence theorem.

Theorem 25. *Let C be a compact convex subset of a Banach space E, T a nonexpansive mapping of C into itself and $\mathbf{Q} = \{q_{n,m}\}$ a weighted mean matrix. Let $x_0 \in C$ and let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n \sum_{m=0}^{\infty} q_{n,m}\, T^m\, x_n \ + (1 - \alpha_n)\, x_n, \ \textit{ for each } n \in N,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Zeng [**81**] obtained the following result for semitopological semigroups which was previously proved by Takahashi and Zeng [**75**] for a Hilbert space and by Li [**36**] for a uniformly convex Banach space with $T = \{T_t : t \in S\}$ as a semitopological semigroup of asymptotically nonexpansive type maps.

Let C be a bounded closed convex subset of a Banach space E then $T = \{T_t : t \in S\}$ is said to be of nearly asymptotically nonexpansive type if for each x, $y \in C$, there exist two functions $\gamma(\cdot, x) : S \to [0, \infty]$ and $\rho(\cdot, y) : S \to [0, \infty]$ such that

$$\|T_t x - T_t y\| \leq \|x - y\| + \gamma(t, x) + \rho(t, y) \quad \text{for all } t \in S,$$

where $\lim_{t\in S} \gamma(t, u(s)) = 0$, for all $s \in S$, and $\lim_{s\in S} \limsup_{t\in S} \rho(t, u(s)) = 0$ for each almost-orbit $u(\,\cdot\,) : S \to C$ of T, and $\rho(t, \cdot) : C \to [0, \infty)$ is convex for each $t \in S$.

Theorem 26. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space with Fréchet differentiable norm. Let $T = \{T_t : t \in S\}$ be a semitopological semigroup of a nearly asymptotically nonexpansive type mappings on $C, F(T) \neq \emptyset$, and let $u(\cdot)$ be an almost orbit of T. Then the set*

$$\cap_s \overline{\text{co}}\{u(t) : t \geq s\} \cap F(T)$$

consists of at most one point.

We close our exposition with the remark that we are not aware of any ergodic theorem for multivalued maps, a versatile research area in waiting.

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