Riv. Mat. Univ. Parma (8) **2** (2009), 81-90

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Very ampleness of balanced line bundles on stable curves

Abstract. Here we study the spannedness and the very ampleness of balanced line bundles (in the sense of L. Caporaso) on stable and quasi-stable curves.

Keywords. Stable curve, very ample line bundle, balanced line bundle, torsion free sheaf.

Mathematics Subject Classification (2000): 14H51, 14H10, 14H20.

1 - Introduction

L. Caporaso defined, constructed and studied a universal compactified Picard variety over $\overline{\mathcal{M}}_g$ ([**1**]). In her construction she considered the following notion, first introduced by D. Gieseker ([**6**]). Let X be a genus g semistable curve. Let L be a degree d line bundle on X and C a proper subcurve of X. The pair (L, C) satisfies the Basic Inequality if and only if

$$(1) \qquad |(\deg(L|C) - d(p_a(C) - 1 + \sharp(C \cap \overline{X \backslash C})/2)/(g-1)| \le \sharp(C \cap \overline{X \backslash C})/2$$

L is called semibalanced ([**2**], Definition 4.6, [**7**], Definition 1.1) or satisfying the Basic Inequality ([**1**], p. 611) if (1) is satisfied for all proper subcurves C of X. Now assume that X is quasi-stable. The line bundle L is called *balanced* if it is semibalanced and $\deg(L|E) = 1$ for every exceptional component E of X, i.e. for every irreducible component E of X such that $E \cong \mathbb{P}^1$ and $\sharp(E \cap \overline{X \backslash E}) = 2$. A point P of a nodal and connected projective curve X is called a *separating point* of X if $X \backslash \{P\}$ is not connected. To state our results we introduce the following short-hands. For any sub-

Received: February 20, 2009; accepted in revised form: May 13, 2009.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

curve $Z \subseteq X$ set $w_Z := \deg(\omega_X|Z)$ and $g_Z := 1 - \chi(\mathcal{O}_Z)$. Set $w := w_X = 2g - 2$. For any proper subcurve $Z \subsetneq X$ set $\delta_Z := \sharp(Z \cap \overline{X \backslash Z})$. Hence $w_Z = 2g_Z - 2 + \delta_Z$. For any quasi-stable curve X such that its stable reduction is reducible let $\widetilde{w}$ denote the minimal integer w_Z among all proper connected subcurves Z of X which are not exceptional. Let X be a reducible quasi-stable curve with a disconnecting node, i.e. with at least one proper connected subcurve Z such that $\delta_Z = 1$; let $\widetilde{w}_1$ denote the minimal w_Z among all such Z. Notice that $\widetilde{w}_1$ is always an odd positive integer.

The following result is related to [4], Theorem 2.2.1. Even its proof is only a small modification of the proof given in [4].

Theorem 1. *Let X be a quasi-stable curve of genus $g \geq 3$ with a separating point. and L a balanced line bundle of degree $d \geq 2g$. Assume*

$$(d - 2g + 2)\widetilde{w}_1 \geq g. \tag{2}$$

Then L is spanned.

Remark 1. The inequality (2) is satisfied if either $d \geq 3g - 2$ or $d \geq g/3 + 2g - 2$ and $\widetilde{w}_1 \neq 1$, i.e. if $d \geq g/3 + 2g - 2$ and X has no connected proper subcurve Z such that $\delta_Z = 1$ and $p_a(Z) = 1$.

Theorem 2. *Let X be a quasi-stable curve of genus $g \geq 3$ and L a balanced line bundle of degree d. Set $\beta := d - 2g + 2$. Assume $\beta \geq 3$ and $\beta w_B + \delta_B(g - 1) \geq 4g - 3$ for every connected proper subcurve B of X, which is not an exceptional component of X. Then L is very ample.*

Assume that X has at least one exceptional component, T. We have $w_T = 0$ and $\delta_T = 2$. Hence $\beta w_T + \delta_T(g - 1) < 4g - 3$ even if β is arbitrarily large.

The different cases considered in the proof of Theorem 2 give the following observation.

Remark 2. Let X be a quasi-stable curve of genus $g \geq 3$ and L a balanced line bundle of degree d. Set $\beta := d - 2g + 2$ and assume $\beta \geq 3$. If either $\delta_Z \geq 4$ or $\delta_Z = 3$ and $\beta w_Z \geq g$ or $\delta_Z = 2$ and $\beta w_Z \geq 2g - 1$ or $\delta_Z = 1$ and $\beta w_Z \geq 3g - 2$ for all proper connected subcurves Z of X which are not exceptional components, then the restriction to X_{reg} of the morphism induced by the base point free linear system $|L|$ is injective and unramified.

R. Pandharipande proved that Caporaso's compactification has another interpretation ([8], theorem 10.3.1). Fix $X \in \overline{\mathcal{M}}_g$ and $d \in \mathbb{Z}$. In [1] L. Caporaso used as the fiber over X of her moduli scheme the set of all degree d balanced line bundles on all quasi-stable curves Y with X as stable model. Since X is stable, ω_X is ample. R. Pandharipande took as the fiber over X of his moduli scheme all equivalence classes of ω_X-semistable depth 1 sheaves on X with pure rank 1 and degree d ([**9**], chapters VII and VIII). Fix any such sheaf F. It is obviously interesting to know if F is spanned. In the non-locally free case there are several different notions related to "very ampleness". We explore this topic in Section 3.

We thank the referee for several essential corrections, remarks and improvements.

2 - Proofs of Theorems 1 and 2

Remark 3. Let U be a reduced projective curve. Fix an ordinary node P of U and let $\alpha : E \to U$ be the partial normalization of U in which we normalize only the point P. Set $\{Q_1, Q_2\} := \alpha^{-1}(P)$. Let $W \subset U$ be a zero-dimensional subscheme of U such that $W_{red} = \{P\}$ and lenght(W) = 2. Let $m_{U,P}$ be the maximal ideal of the local ring $\mathcal{O}_{U,P}$ and $J = \mathcal{I}_{W,P}$ the ideal of $\mathcal{O}_{U,P}$ associated to W. Thus $\mathcal{O}_{U,P}/J \cong \mathcal{O}_W$. Since length($W$) = 2, $m^2_{U,P} \subseteq J$. Hence the pair $(\mathcal{O}_{U,P}, J)$ depends only from the formal completion of U at P. Identify $\hat{\mathcal{O}}_{U,P}$ with $\mathbb{K}[[x,y]]/(xy)$. Either W is contained in a formal branch of U at P or not. In the former case J contains either x or y. The latter case occurs if and only if W is a Cartier divisor of U (use the intersection multiplicity at a point P of two plane curves, one of them singular at P ([**5**], property (5) at p. 75)). In both cases there is an integer $j \in \{1, 2\}$ such that $\mathcal{I}_W$ contains the image of the natural map $\alpha_*(\mathcal{I}_{Q_1+Q_2+Q_j,E}) \to \mathcal{O}_U$, where we see $Q_1 + Q_2 + Q_j$ as an effective Cartier divisor of E.

We fix a balanced $L \in \mathrm{Pic}(X)$ and write $d_Z := \deg(L|Z)$ for any subcurve Z of X, $d := d_X$ and $\beta := d - 2g + 2$. Since L is balanced,

$$d_Z \geq 2g_Z - 2 + \beta w_Z/w + \delta_Z/2 \tag{3}$$

for every proper subcurve Z of X.

Proof of Theorem 1. Since $d \geq 2g$, the proof of [4], Theorem 2.2.1, shows that it is sufficient to prove $d_Z \geq 2g_Z$ for all proper connected subcurves Z of X such that $\delta_Z = 1$. Assume that Z is a proper connected subcurve with $\delta_Z = 1$. Thus $w_Z \geq \widetilde{w}_1$. Hence (2) gives $(d - 2g + 2)w_Z/w > 1/2$. Since $\delta_Z = 1$ and $d_Z - 2g_Z$ is an integer, (3) gives $d_Z \geq 2g_Z$. □

Proof of Theorem 2. The line bundle L is very ample if and only if $h^0(X, \mathcal{I}_W \otimes L) = h^0(X, L) - 2$ for all zero-dimensional schemes $W \subset X$ such that length$(W) = 2$. We distinguish the following four cases (a), (b), (c) and (d). In step (f) we will handle the exceptional components (if any) for steps (b), (c) and (d).

(a) Here we assume $W \subset X_{reg}$. As in the proof of [4], Theorem 2.2.1, $h^0(X, \mathcal{I}_W \otimes L) = h^0(X, L) - 2$ if $d_Z \geq 2g_Z + 1$ for every subcurve Z of X. It is sufficient to consider the connected subcurves. The case $Z = X$ is true, because we assumed $\beta \geq 3$. Now assume $Z \neq X$. If Z is an exceptional component, then $d_Z = 1$ and $g_Z = 0$. Hence we may assume that Z is not an exceptional component, i.e. we may assume $w_Z \geq 1$. Since d_Z is an integer, (3) shows that it is sufficient to prove the inequality $\beta w_Z/w + \delta_Z/2 > 2$, which is true if either $\delta_Z \geq 4$ and $\beta > 0$ (but we even assumed $\beta \geq 3$) or $\delta_Z \geq 3$, $w_Z \geq 2$ and $\beta \geq g$ or $\delta_Z \geq 2$ and $\beta w_Z \geq 2g - 1$ or $\delta_Z \geq 1$ and $\beta w_Z \geq 3g - 2$.

(b) Here we assume that W_{red} is a point $P \in \mathrm{Sing}(X)$. We will see that in this case it is sufficient to assume

$$(d - 2g + 2)w_B + \delta_B(g - 1) > 2g - 2 \tag{4}$$

for every proper connected subcurve B of X which is not an exceptional component. Let $v : C \to X$ be the partial normalization of X in which we only normalize the point P. Set $\{P_1, P_2\} := v^{-1}(P)$ and $M := v^*(L)$. It is sufficient to prove $h^1(C, M(-P_1 - P_2 - P_j)) = 0$ for all $j \in \{1, 2\}$ (see Remark 3)). It is sufficient to prove $\deg(M(-P_1 - P_2 - P_j)) \geq 2p_a(A) - 1$ for all $j \in \{1, 2\}$ and every connected subcurve A of C ([4], Lemma 2.2.2). Since $\deg(M(-P_1 - P_2 - P_j)) > -2\chi(\mathcal{O}_C)$, it is sufficient to consider the proper subcurves of C. Let A be a proper connected subcurve of C. Set $B := v(A)$. If $P \notin B$, then $\{P_1, P_2\} \cap A = \emptyset$. Hence $M(-P_1 - P_2 - P_j)|A \cong M|A$, and $v|A : A \to B$ induces an isomorphism such that $((v|A)^{-1})^*(L|B) \cong M|A$. Hence in this case we have $\deg(M(-P_1 - P_2 - P_j)|A) = d_B \geq 2g_B - 1 = 2g_A - 1$ (use the inequality $\beta w_B/w + \delta_B/2 > 0$). Hence we are done in this case. Now assume $P \in B, P_1 \in A, P_2 \notin A$ and $j = 1$. Hence A is smooth at P_1, $v|A : A \to B$ is an isomorphism and $\deg(M(-2P_1 - P_2)|A) = d_B - 2 \geq 2g_B - 1$ (use (4)). In the same way we handle the case $P_1 \in A$, $P_2 \notin A$ and $j = 2$ and the two subcases with $P_2 \in A$ and $P_1 \notin A$. Now assume $\{P_1, P_2\} \subset A$, i.e. $P \in B$ and $g_B = g_A + 1$. Since $\deg(M(-P_1 - P_2 - P_j)|A) = d_B - 3$, it is sufficient to check the inequality $d_B \geq 2g_B$, which is true (since d_B and $2g_B$ are integers) if $\beta w_B/w + \delta_B/2 > 1$, i.e. if (4) is satisfied.

(c) Here we assume W reduced, say $W = \{P, Q\}$, and $\sharp(W \cap \mathrm{Sing}(X)) = 1$, say $P \in \mathrm{Sing}(X)$. Let $v : C \to X$ be the partial normalization of X in which we only normalize the point P. Set $\{P_1, P_2\} := v^{-1}(P)$, $Q' := v^{-1}(Q)$ and $M := v^*(L)$. It is sufficient to prove $h^1(C, M(-P_1 - P_2 - Q')) = 0$. Since $\deg(M(-P_1 - P_2 - Q')) > -2\chi(\mathcal{O}_C)$, it is sufficient to prove $\deg(M(-P_1 - P_2 - Q')) \geq 2p_a(A) - 1$ for every proper connected subcurve A of C ([4], Lemma 2.2.2). Let A be any proper connected subcurve of C. Set $B := v(A)$. First assume $\{P_1, P_2, Q'\} \subset A$. In this case $\deg(M|A) = d_B - 3$ and $g_B = g_A - 1$. In this case we conclude if $d_B \geq 2g_B$, i.e. if $d_B > 2g_B - 1$. Hence in this case (3) shows that we only need to require $\beta w_B + \delta_B(g-1) \geq 2g - 1$. Now assume $W \subset B$ and $\sharp(\{P_1, P_2\} \cap A) = 1$. In this case $\deg(M|A) = d_B - 2$ and $g_A = g_B$. Hence in this case it is sufficient to have $d_B > 2g_B$. Hence by (3) it is sufficient to have $\beta w_B + \delta_B(g-1) \geq 4g - 3$. The other cases, i.e. $\sharp(W \cap B) \leq 1$, require weaker assumptions (see step (b)).

(d) Here we assume $W = \{P, Q\}$ with $P \neq Q$ and $W \subseteq \mathrm{Sing}(X)$. Let $v : C \to X$ be the partial normalization of X in which we normalize only the points P and Q. Set $\{P_1, P_2\} := v^{-1}(P)$, $\{Q_1, Q_2\} = v^{-1}(Q)$ and $M := v^*(L)$. It is sufficient to prove $h^1(C, M(-P_1 - P_2 - Q_1 - Q_2)) = 0$. Since $\deg(M(-P_1 - P_2 - Q_1 - Q_2)) > -2\chi(\mathcal{O}_C)$, it is sufficient to prove $\deg(M(-P_1 - P_2 - Q_1 - Q_2)) \geq 2p_a(A) - 1$ for every proper connected subcurve A of C ([4], Lemma 2.2.1). Let A be any proper connected subcurve of C. Set $B := v(A)$. As in steps (b) and (c) the worst bound arises if $W \subset B$ and $\sharp(A \cap v^{-1}(W)) = 2$. In this case $g_A = g_B$ and $\deg(M(-P_1 - P_2 - Q_1 - Q_2)) = d_B - 2$. Hence in this case to have $\deg(M(-P_1 - P_2 - Q_1 - Q_2)) \geq 2p_a(A) - 1$ we need to check $d_B > 2g_B$. Hence by (3) it is sufficient to have $\beta w_B + \delta_B(g-1) \geq 4g - 3$.

(e) In the statement of Theorem 2 we assumed no inequality for the exceptional components (if any). Hence in the previous steps (b), (c) and (d) we cannot conclude if the connected subcurve B is an exceptional component. We could do all steps with as W a point and get at least that L is spanned. Assume the existence of at least one exceptional component and fix an exceptional component T of X. Since L is spanned, $T \cong \mathbb{P}^1$ and $\deg(L|T) = 1$, the morphism $h_L|T$ induces an embedding of T onto a line of $\mathbb{P}^{d-g}$. Thus the restriction map $H^0(X, L) \to H^0(T, L|T)$ is surjective. First assume $W \subset T$. Since $h_L|T$ is an embedding, the zero-dimensional scheme $h_L(W)$ is not a point. Hence $h^0(\mathbb{P}^{d-g}, \mathcal{I}_{h_L(W)}(1)) \leq d - g - 1$, i.e. $h^0(Y, \mathcal{I}_W \otimes L) \leq h^0(Y, L) - 2$. Since $h^1(Y, L) = 0$ and $\mathrm{length}(W) = 2$, we get $h^1(Y, \mathcal{I}_W \otimes L) = 0$. Thus from now on we assume that W is not contained in T.

First assume that $\{P\} := W_{red} \in \mathrm{Sing}(X)$ (as in step (b)) with $P \in T \cap \mathrm{Sing}(X)$ and take $B := T$. Hence A is the unique component of C such that $v(A) = T$. Thus $A \cong \mathbb{P}^1$, i.e. $g_A = 0$. Since T does not contain W, we have $\sharp(A \cap \{P_1, P_2\}) = 1$. Hence $\deg(M(-P_1 - P_2)|A) \geq 2g_A$, concluding this case.

Now assume that we are in the set-up of (c) with $B = T$. If $W \subset T$, then we get $h^1(X, \mathcal{I}_W \otimes L) = 0$ as above, because $h_L|T$ is an embedding and $h^1(X, L) = 0$. Hence $\sharp(B \cap \{P, Q\}) \leq 1$. Hence $\deg(M(-P_1 - P_2 - Q_1 - Q_2)|A) \geq \deg(M|A) - 2 = -1$. Hence $\deg(M(-P_1 - P_2 - Q_1 - Q_2)|A) \geq 2g_A - 1$.

Now assume that we are in the set-up of (d). If $\sharp(W \cap T) \le 1$, then we get $\deg(M(-P_1 - P_2 - Q_1 - Q_2)|A) \ge 2g_A - 1$. If $W \subset T$, then we get $h^1(X, \mathcal{I}_W \otimes L) = 0$ as above. □

3 - The non-locally free case

Remark 4. Let X be a stable curve of genus g and F a depth 1 sheaf on X with pure rank 1. Set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. Notice that $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. Let $v : C \to X$ be the partial normalization of X in which we only normalize the points of $S := \mathrm{Sing}(F)$. Hence $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \sharp(S)$. Set $M := v^*(F)/\mathrm{Tors}(v^*(F))$. The coherent sheaf M is a line bundle on C such that $\deg(M) = \deg(F) - \sharp(S)$ and $v_*(M) \cong F$. Hence $h^i(C, M) = h^i(X, F)$, $i = 0, 1$. For any zero-dimensional scheme $Z \subset X \backslash S$ we have $v^{-1}(Z) \cong Z$ and $h^i(C, M(-v^{-1}(Z)) = h^i(X, \mathcal{I}_Z \otimes F)$, $i = 0, 1$. Let Y be the unique quasi-stable curve with X as its stable reduction and with S as image of the exceptional components. Let $u : Y \to X$ denote the stable reduction of Y. For any $P \in S$ let $E_P := u^{-1}(P) \subset Y$ denote the associated exceptional component. Thus the curves $\{E_P\}_{P \in S}$ are the exceptional components of Y. There is an inclusion $j : C \to Y$ such that $\cup_{P \in S} E_P = \overline{Y \backslash j(C)}$ and $v = u \circ j$. For any $P \in S$ we have $E_P \cap j(C) = j(v^{-1}(P))$. A standard property of the semistable reduction says that there is a unique $L \in \mathrm{Pic}(Y)$ such that $j^*(L|j(C)) \cong M$ and $\deg(L|E_P) = 1$. The proof of [8], Theorem 10.3.1, says that L is balanced if and only if F is ω_X-semistable.

Remark 5. Let U be a reduced projective curve, F a depth 1 sheaf on U with pure rank 1 and $W \subset U$ a zero-dimensional subscheme. By tensoring with F the exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{O}_U \to \mathcal{O}_W \to 0$$

we get a map $\psi : \mathcal{I}_W \otimes F \to F$ whose cokernel is supported by the finite set W_{red}. Hence the subsheaf $\mathrm{Im}(\psi)$ of F has pure rank 1. Set $\mathcal{I}_W F := \mathrm{Im}(\psi)$. The sheaf $\mathcal{I}_W F$ is the kernel of the restriction map $F \to F|W$. The coherent sheaf $\mathrm{Ker}(\psi)$ is supported by the finite set $\mathrm{Sing}(F) \cap W_{red}$. Hence $h^1(U, \mathrm{Ker}(\psi)) = 0$. Thus the exact sequence

$$0 \to \mathrm{Ker}(\psi) \to \mathcal{I}_W \otimes F \to \mathcal{I}_W F \to 0$$

gives $h^1(U, \mathcal{I}_W \otimes F) = h^1(U, \mathcal{I}_W F)$.

Lemma 1. *Take the set-up of Remark 4. The sheaf F is spanned if and only if the line bundle M is spanned and $h^0(C, \mathcal{I}_{v^{-1}(P)} \otimes M) = h^0(C, M) - 2$ for every $P \in S$.*

Proof. First assume that F is spanned. Since the tensor product is a right exact functor, $v^*(F)$ is spanned. Hence M is spanned. Fix $P \in S$. Since the fiber $F|\{P\}$ of F at P is a 2-dimensional vector space and F is spanned at P, $h^0(X, \mathcal{I}_P F) = h^0(X, F) - 2$. Since the natural map $H^0(X, F) \to H^0(C, M)$ is an isomorphism (Remark 4), we get $h^0(C, \mathcal{I}_{v^{-1}(P)} \otimes M) = h^0(C, M) - 2$.

Now assume that M is spanned. Let G be the subsheaf of F spanned by $H^0(X, F)$. Since we saw that the functor v^* induces an isomorphism $H^0(X, F) \to H^0(C, M)$ whose inverse is induced by v_* (Remark 4) and M is spanned, F is spanned outside S. Hence the support of F/G is contained in S. Fix $P \in S$ and assume $h^0(C, \mathcal{I}_{v^{-1}(P)} \otimes M) = h^0(C, M) - 2$. Since the map $H^0(X, F) \to H^0(C, M)$ is an isomorphism, we get $h^0(X, \mathcal{I}_P F) = h^0(X, F) - 2$. Since the fiber $F|\{P\}$ of F at P is a 2-dimensional vector space, the last equality implies that F is spanned at P. □

Remark 6. Take the set-up of Remark 4. Assume M spanned and $h^1(C, M) = 0$, i.e. $h^1(X, F) = 0$ (Remark 4). Fix $P \in \mathrm{Sing}(X)$. Set $\{P_1, P_2\} := v^{-1}(P)$. Let $h_M : C \to \mathbb{P}^r$, $r := h^0(X, F) - 1$, be the morphism induced by the complete linear system $|M|$. We have $h_M(P_1) \neq h_M(P_2)$, h_M is unramified at P_1 and at P_2 and the tangent lines to the curve $h_M(C)$ at $h_M(P_1)$ and at $h_M(P_2)$ span a 3-dimensional linear subspace if and only if $h^1(C, M(-2P_1 - 2P_2)) = 0$, i.e. if and only if $h^0(C, M(-2P_1 - 2P_2)) = h^0(C, M) - 4$.

Lemma 2. *Let B be a reduced projective curve and E an irreducible component of B. Set $T := \overline{B \backslash E}$. Assume $E \cong \mathbb{P}^1$, $\sharp(T \cap E) = 2$ and that each point of $E \cap T$ is an ordinary node of B. Fix $R \in Pic(B)$ such that $R|E$ has degree 1, $R|T$ is spanned and $h^0(T, (R|T)(-T \cap E)) = h^0(T, R|T) - 2$. Then R is spanned and the restriction map $\rho : H^0(B, R) \to H^0(T, R|T)$ is bijective.*

Proof. Since $\sharp(T \cap E) = 2$ and each point of $T \cap E$ is an ordinary node of B, $\mathrm{length}(T \cap E) = 2$. Look at the following Mayer-Vietoris exact sequence on B:

$$0 \to R \to R|T \oplus R|E \to R|T \cap E \to 0. \tag{5}$$

Since $E \cong \mathbb{P}^1$, $\mathrm{length}(T \cap E) = 2$ and $\deg(E|T) = 1$, the restriction map $\rho' : H^0(E, R|E) \to H^0(E \cap T, R|T \cap E)$ is an isomorphism. Hence (5) gives the bijectivity of ρ. Thus R is spanned at each point of T. Since $\mathrm{length}(T \cap E) = 2$, the condition " $h^0(T, (R|T)(-T \cap E)) = h^0(T, R|T) - 2$" is equivalent to the surjectivity of the restriction map $\rho'' : H^0(T, R|T) \to H^0(T \cap E, R|T \cap E)$. Since ρ' and ρ are bijective, the surjectivity of ρ'' gives the surjectivity of the restriction map $\rho_1 : H^0(B, R) \to H^0(E, R|E)$. Since $R|E$ is spanned, the surjectivity of ρ_1 gives that R is spanned at each point of E. □

Lemma 3. *Take the set-up of Remark* 4.

(a) $h^i(Y,L) = h^i(C,M) = h^i(X,F)$, $i = 0,1$.

(b) *F is spanned if and only if L is spanned.*

Proof. Consider the Mayer-Vietoris exact sequence of coherent sheaves on Y:

$$0 \to L \to L|j(C) \oplus L|(\cup_{P\in S} E_P) \to L|j(C) \cap (\cup_{P\in S} E_P) \to 0. \tag{6}$$

Since $E_P \cong \mathbb{P}^1$ and $\deg(L|E_P) = 1$, $h^1(E_P, L|L_P) = 0$ for all $P \in S$. Notice that $E_P \cap E_Q = \emptyset$ for all $P, Q \in S$ such that $P \neq Q$. Since $E_P \cong \mathbb{P}^1$, $\deg(L|E_P) = 1$ and $\sharp(j(C) \cap E_P) = 2$ for all $P \in S$, the restriction map $H^0(\cup_{P\in S} E_P, L| \cup_{P\in S} E_P) \to H^0(j(C) \cap (\cup_{P\in S} E_P), L|j(C) \cap (\cup_{P\in S} E_P))$ is an isomorphism. Hence (6) gives that the restriction map $\rho_i : H^i(Y,L) \to H^i(j(C), L|j(C)) \cong H^i(C,M)$, $i = 0,1$, is an isomorphism, proving the first equality of part (a). The second equality of part (a) was proved in Remark 4.

Now assume that F is spanned. Lemma 1 allows us to iterate $\sharp(S)$ times Lemma 2, each time inserting a different exceptional component E_P, $P \in S$. At the end we get that L is spanned.

Now we check the "if" part of (b). Assume that L is spanned. Since $j^*(L|j(C)) \cong M$, M is spanned. Since the natural map $H^0(X,F) \to H^0(C,M)$ is an isomorphism, F is spanned at each point of $X\backslash S$. Fix $P \in S$ and set $\{P_1, P_2\} := v^{-1}(P)$. Assume that F is not spanned at P. Lemma 1 gives that any section of M vanishing at P_1 vanishes at P_2. Hence every section of L vanishing at $j(P_1)$ vanishes at $j(P_2)$. Since L is spanned and $\deg(L|E_P) = 1$, the restriction map $H^0(Y,L) \to H^0(E_P, L|E_P)$ is surjective. Hence the very ampleness of $L|E_P$ gives the existence of a section of L vanishing at $j(P_1)$, but not at $j(P_2)$, contradiction. □

Remark 7. Take the set-up of Remark 4. Fix $P \in \mathrm{Sing}(F)$ and set $\{P_1, P_2\} := v^{-1}(P)$. Let $W \subset X$ be a zero-dimensional scheme such that $W_{red} = \{P\}$ and $\mathrm{length}(W) = 2$. If W is a Cartier divisor of X, then the natural map $\mathcal{I}_W \otimes F \to \mathcal{I}_W F$ introduced in Remark 5 is an isomorphism. Hence $\mathcal{I}_W F$ is not locally free at P. Thus $\mathcal{I}_W F \cong v_*(M(-P_1 - P_2))$ (apply Remark 4 to the sheaf $\mathcal{I}_W F$). Hence $h^i(X, \mathcal{I}_W F) = h^i(C, M(-P_1 - P_2))$, $i = 0,1$. Now assume that W is not a Cartier divisor, i.e. assume that, identifying $\hat{O}_{X,P}$ with $\mathbb{K}[[x,y]]/(x,y)$ as in Remark 4, the ideal J of W contains either x or y. Thus there is $j \in \{1,2\}$ such that $v_*(M(-P_1 - P_2 - P_j)) \subseteq \mathcal{I}_W F$ (Remark 3). Thus $h^1(X, \mathcal{I}_W F) \le h^1(C, M(-P_1 - P_2 - P_j))$.

Theorem 3. *Take the set-up of Remark* 4 *with $S := Sing(F)$. Assume that F is ω_X-semistable, spanned and $h^1(X,F) = 0$ or, equivalently, assume L balanced,* *spanned, and* $h^1(Y,L)=0$. *Let* $h_L : Y \to \mathbb{P}^{d-g}$, $d = \deg(L)$, *be the morphism induced by* $|L|$.

(a) $h^i(X,\mathcal{I}_W \otimes F) = h^i(X,\mathcal{I}_W F) = h^i(Y,\mathcal{I}_{u^{-1}(W)} \otimes L)$, $i=0,1$, *for every zero-dimensional scheme* $W \subset X \backslash S$.

(b) *Fix* $P,Q \in X$ *such that* $P \neq Q$. *We have* $h^1(X,\mathcal{I}_{\{P,Q\}}F)=0$ (*or, equivalently, the restriction map* $\rho_{\{P,Q\}} : H^0(X,F) \to H^0(\{P,Q\},F|\{P,Q\})$ *is surjective*) *if and only if the restriction of* h_L *to* $u^{-1}(\{P,Q\})$ *is injective.*

(c) *Fix* $P \in Sing(F)$ *and a zero-dimensional scheme* $W \subset X$ *such that* $W_{red}=\{P\}$, *length*$(W)=2$ *and* W *is a Cartier divisor of* X. *Set* $\{P_1,P_2\} := v^{-1}(P)$. *Then* $h^i(X,\mathcal{I}_W \otimes F) = h^i(X,\mathcal{I}_W F) = h^i(C,M(-P_1-P_2))$, $i=0,1$. *If* L *is very ample, then* $h^1(X,\mathcal{I}_W F)=0$ *and the restriction map* $H^0(X,F) \to H^0(W,F|W)$ *is surjective.*

(d) *Fix* $P \in Sing(F)$ *and a zero-dimensional scheme* $W \subset X$ *such that* $W_{red}=\{P\}$ *and length*$(W)=2$. *Set* $\{P_1,P_2\} := v^{-1}(P)$. *Then there is* $j \in \{1,2\}$ *such that* $h^1(X,\mathcal{I}_W \otimes F) = h^1(X,\mathcal{I}_W F) \le h^1(C,M(-P_1-P_2-P_j))$.

(e) *Fix* $P \in Sing(F)$ *and a zero-dimensional scheme* $W \subset X$ *such that* $W_{red}=\{P\}$ *and length*$(W)=2$. *Set* $\{P_1,P_2\} := v^{-1}(P)$. *If* $h^1(C,M(-2P_1-P_2)) = h^1(C,M(-P_1-2P_2))=0$, *then* $h^1(X,\mathcal{I}_W \otimes F) = h^1(\mathcal{I}_W F)=0$. *If*

$$(d-2g+2)\deg(\omega_Y|B) + (g-1)\cdot \sharp(B \cap \overline{Y\backslash B})/2 > 2g-2$$

for every proper connected subcurve B *of* Y *which is not an exceptional component of* Y, *then* $h^1(X,\mathcal{I}_W \otimes F) = h^1(X,\mathcal{I}_W F)=0$.

Proof. The sheaf F is ω_X-semistable if and only if L is balanced ([8], Theorem 10.3.1). Lemma 3 gives the equivalence of the other conditions listed in the second sentence of the statement of the theorem. Part (a) is true by Lemma 3. In the proof below we always identify $H^0(X,F)$ and $H^0(Y,L)$.

(i) Here we check part (b). Fix $P,Q \in X$ such that $P \neq Q$. Part (a) gives the case $\{P,Q\} \in X\backslash S$. Now assume $P \in S$ and $Q \notin S$. Hence $F|\{P,Q\}$ is a 3-dimensional vector space. Since $h_L(E_P)$ is a line, up to the identification of $H^0(X,F)$ and $H^0(Y,L)$, the surjectivity of $\rho_{\{P,Q\}}$ is equivalent to $h_L(u^{-1}(Q)) \notin h_L(E_P)$, i.e. to the injectivity of $h_L|u^{-1}(\{P,Q\})$. Now assume $P \in S$ and $Q \in S$. Hence $F|\{P,Q\}$ is a 4-dimensional vector space. Since $h_L(E_P)$ and $h_L(E_Q)$ are lines, they are disjoint (i.e. $h_L|(E_P \cup E_Q)$ is injective) if and only if they span a 3-dimensional vector space, i.e. if and only if $h^0(Y,\mathcal{I}_{E_P \cup E_Q} \otimes L) = h^0(Y,L)-4$, i.e. (up to the identification of $H^0(X,F)$ and $H^0(Y,L)$) if and only if $\mathrm{Im}(\rho_{\{P,Q\}})$ has dimension 4, i.e. if and only if $\rho_{\{P,Q\}}$ is surjective.

(ii) Parts (c) and (d) follows from Remark 7. The first assertion of part (e) follows from parts (c) and (d). The second assertion of part (e) follows from the first one and step (b) of the proof of Theorem 2. □

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