Riv. Mat. Univ. Parma, Vol. **1** (2010), 1-75

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Some results on the Navier-Stokes equations with Navier boundary conditions

Abstract. I make an overview of some results concerning the Stokes and Navier-Stokes equations, supplemented with the Navier's type slip boundary conditions. I try to explain the interest for this problem, the main analytical results, and also the differences between the flat case and more general cases. Some recent results concerning the vanishing viscosity limits are also announced.

Keywords. Stokes equations, Navier-Stokes equations, existence, regularity, singular limits.

Mathematics Subject Classification (2000): 35Q30, 76F65, 76D03, 76D09.

Contents

Received: February 26, 2010; accepted: April 20, 2010.
This research was partially supported by M.I.U.R. within the project "Fluid dynamics and conservation laws."

1 - Introduction

In this paper I am collecting and enlarging the notes of a short course for PhD students and young researchers, given in September 2009 at the International School for Advanced Studies (SISSA/ISAS) of Trieste. The lectures have been given within the seventh meeting in hyperbolic conservation laws and fluid dynamics. In these lectures I took chance of presenting certain rather classical results, but also more recent and even current advances concerning a subject which is related with fluid dynamics, but also with the basic theory of elliptic systems of partial differential equations and with the behavior of solutions when singular perturbations are present. In particular -motivated by physical, numerical, and analytical insight- I am presenting some results and open problems concerning the *incompressible* Navier-Stokes system supplemented by certain slip boundary conditions. This paper cannot be considered as an exhaustive treatment of the subject, but just a limited collection of results along a research path I think most interesting, especially for young scientists, potentially oriented in doing a research in related fields. In the setting of the problem I am trying to emphasize the connections between modeling, numerical aspects, computational tools, and the mathematical analysis. Most of the results are strictly linked, but I am trying to look at them from different points of view, since taking a (limited) detour in related fields can give new insight, new inspiration, and also open new research avenues.

I am deliberately skipping many details in the various proofs, since I am trying to shed light on the ideas, with the hope of interesting the reader to this subject. In fact, in order that the reader can focus directly on the (hopefully) most relevant points, I am avoiding the most technical parts, but an extended and rather detailed bibliography is also added, where one can precisely find the missing details. In the references one can also find recent results and attempts to understand more about fluids and the fascinating research involving their mathematical analysis, modeling of turbulence, and also the numerical resolution of real-life flows.

The presentation is intended for a reader with at least some background of Sobolev spaces and of the basic variational results for elliptic and parabolic equations. The knowledge of the (nowadays) "elementary" results in mathematical fluid mechanics (existence of weak and strong solutions for the Navier-Stoke equations) is not necessary, even if for a better understanding at least a qualitative idea of the basic results is warmly welcome.

The paper is organized as follows: In Section 2 I introduce the problem I will consider and I give the main motivations for the study of viscous fluids with slip boundary conditions. In Section 3 I give some of the motivations that I believe relevant to study this problem. In Section 4 the main properties regarding the variational formulation of the linear stationary problem are recalled. In particular, the approach based on the introduction of the artificial compressibility is explained. In Section 5 I recall the formulation for the time-evolution nonlinear problem and some existence results. In Section 6 the connection with modeling of boundary conditions in LES is explained, together with some rigorous results. In Section 7 some results concerning the vanishing viscosity are recalled and some new results are announced.

2 - Setting of the problem

The incompressible (with constant density) Navier-Stokes equations read as

$$\begin{aligned} u_t - \nu \Delta u + (u \cdot \nabla)\, u + \nabla p &= f \quad \text{in } \Omega \times]0, T], \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times]0, T], \end{aligned} \tag{1}$$

where the open set $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is the physical domain. The unknown u is the velocity vector field with n components (and to avoid a too heavy notation I do not use boldface symbols for vector fields). The scalar p is the pressure, while the positive number ν denotes the kinematic viscosity.

The differential operators ∇ and Δ are the standard ones, while we observe that the convection term is properly defined in terms of coordinates as follows

$$\big[(u \cdot \nabla)\, u\big]_i := u_k \partial_k u_i, \quad \text{for} \quad i = 1, \dots, n,$$

and in the paper I use (when needed) the Einstein convention of summation over repeated indices, while ∂_k denotes partial differentiation with respect to x_k. It is also worth noting that, due to the incompressibility, we can also write the convection term as

$$(u \cdot \nabla)\, u = \nabla \cdot (u \otimes u),$$

while in some cases the notation

$$(u \cdot \nabla)\, u = \big[\nabla u\big] u$$

is used, thinking of ∇u as a linear operator acting on the vector u.

For an axiomatic derivation of the equations and for the precise assumptions underlying the process, see Serrin [**125**]. Further details and other presentations can be found in Chorin and Marsden [**48**], Batchelor [**9**], Lamb [**93**], and Landau and Lifshitz [**94**].

In almost all the paper the space dimension will be $n = 3$, corresponding to the case with more interesting physical meaning. In the 2D case the theory is much more complete (regardless of the boundary conditions). In some cases I will restrict to the 2D case in order to point out some of the simplifications occurring in two dimensions. The initial-value-problem must be supplemented with a divergence-free initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega.$$

In presence of a domain Ω with boundary $\Gamma = \partial\Omega$, one has to add suitable boundary conditions. In literature, most of the results in domain with boundaries[1] are obtained by supplementing the system with Dirichlet (*no slip*) boundary conditions

$$u(x, t) = 0, \qquad \text{on } \Gamma \times]0, T]. \tag{2}$$

The Dirichlet boundary conditions have been proposed by Stokes [**131**] since the contrary assumption

> "*...implies an infinitely greater resistance to the sliding of one portion of fluid past another than to the sliding of fluid over a solid.*"

Condition (2) corresponds to consider that fluid particles adhere to the boundary, hence they have the same velocity (generally vanishing) of the solid boundary.

The basic results on existence of weak solutions, local existence of strong solutions, partial regularity and so on (in the periodic setting and with Dirichlet boundary

[1] In the whole space case one has just to assume suitable decay at infinity and of particular interest is also the problem in the space periodic setting.

conditions) can be found in many references, see for instance the books by Constantin and Foiaş [**54**], Doering and Gibbon [**57**], Galdi [**65**, **66**, **67**], Ladyžhenskaya [**92**], Sohr [**128**], and Temam [**137, 138**]. Moreover the reader is warmly encouraged to read the original sources, especially the masterpieces written by Leray [**97**] and Hopf [**75**].

Under the Dirichlet boundary conditions, one can (formally) use the velocity itself as test function, obtaining -with suitable integration by parts- the energy balance

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx+\nu\int_{\Omega}|\nabla u|^2\,dx=\int_{\Omega}f\cdot u\,dx.$$

This gives a control of the kinetic energy, which is critical to prove existence of weak solutions. Moreover, it is well-known that there are long-standing open questions related the 3D Navier-Stokes equations: Essentially we know global (in time) existence of solutions in a class in which we are not able to prove uniqueness, and uniqueness in a class in which we are able to prove just existence for small times. The "*Million dollar problem*" concerning the Navier-Stokes equations can be found in the web site http://www.claymath.org/millennium of the Clay Institute.

I do not want to focus on a so hard and seemingly elusive-to-any-attempt open problem, but I would like to take a slightly different path, a little bit more oriented towards applications, and where there are realistic chances to obtain some new (and non trivial) results.

2.1 - *The Navier boundary conditions*

It is well known that there are situations in which the boundary condition (2) may not be valid. From the historical point of view, the slip (with friction) boundary conditions proposed by Navier [**111**] (twenty years before the work of Stokes) were

$$(3)\qquad \begin{aligned} &u\cdot\underline{\mathrm{n}}=0 && \text{on } \Gamma\times\,]0,T],\\ &\beta\,u_\tau+\underline{\mathcal{T}}(u,p)=0,\quad \beta\geq 0, && \text{on } \Gamma\times\,]0,T], \end{aligned}$$

where $\underline{\mathrm{n}}$ denotes the exterior unit normal vector to Γ, while

$$u_\tau:=u-(u\cdot\underline{\mathrm{n}})\,\underline{\mathrm{n}}\,,$$

denotes the tangential part[2] of the velocity. In addition

$$\underline{\mathcal{T}}(u,p):=\underline{t}(u,p)-(\underline{t}(u,p)\cdot\underline{\mathrm{n}})\,\underline{\mathrm{n}}$$

[2] I use this notation which is historical, but one can also write the same in terms of exterior product with the unit vector $\underline{\mathrm{n}}$, see also (46).

denotes the tangential part of the *Cauchy stress vector* $\underline{t}$ defined by

$$\underline{t}(u,p) := \underline{n} \cdot \mathbb{T}(u,p) = \sum_{k=1}^{n} \mathbb{T}_{ik}(u,p)\,\underline{n}_k,$$

and, if δ_{ij} denotes the Kronecker symbol,

$$\mathbb{T}_{ik}(u,p) := -\delta_{ik}\,p + \nu(\partial_k u_i + \partial_i u_k).$$

Since the term $\underline{\mathcal{T}}(u,p)$ in fact does not depend explicitly on the pressure, we use also the notation

$$\beta\, u_\tau + \underline{\mathcal{T}}(u) = 0, \qquad \beta \geq 0.$$

Probably Maxwell [**107**] first analyzed the two types of boundary conditions (conditions (3) proposed by Navier and condition (2) proposed by Stokes), observing that the same conditions may be derived also within the kinetic theory of gases. In fact, the Navier-Stokes equations can be obtained by taking suitable limits from the kinetic theory of gases and one obtains the slip conditions with

$$\beta \sim \frac{\text{mean free passes of molecules}}{\text{macroscopic length}}.$$

Thus, for certain range of the parameters, the no-slip condition

$$u_\tau = 0 \qquad \text{on } \Gamma \times]0,T]$$

can be recovered. In particular, the parameter β should depend on the viscosity ν and on the mean free-path λ, satisfying the pair of consistency conditions:

$$\begin{aligned} &\beta \to \infty \qquad &&\text{as } \lambda \to 0 \text{ for } \nu \text{ fixed},\\ &\beta \to 0 \qquad &&\text{as } \nu \to 0 \text{ for } \lambda \text{ fixed}. \end{aligned}$$

With the above asymptotics it is possible to recover in both cases the correct no-slip boundary conditions for viscous fluids and the no-penetration conditions for ideal fluids. Observe in fact that $u \cdot \underline{n} = 0$ is the condition supplementing the Euler system, i.e., the Navier-Stokes equations with $\nu = 0$, see Section 7.

Remark 2.1. *In contrast to Stokes* [**131**] (*in* 1845) *who employed continuum mechanics, Navier* [**111**] (*in* 1823) *derived the equations by using some formal* (*and possibly out the range of applicability*) *analogy with the elasticity theory and the assumption that molecules are animated by attractive and repulsive forces. Overview on the historical connections can be found in Cannone and Friedlander* [**46**].

Under the boundary conditions (3) one can perform integrations by parts similar to those valid with Dirichlet boundary conditions, obtaining the energy balance

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u|^2\,dx+\beta\int_{\Gamma}|u_\tau|^2\,dS+\nu\int_{\Omega}|\nabla u|^2\,dx=\int_{\Omega}f\cdot u\,dx,$$

but the most interesting features of the slip boundary conditions can be better explained when considering the vorticity field $\omega := \operatorname{curl} u = \nabla \times u$.

2.2 - *A couple of vector identities*

It is worth noting that, on flat portions of the boundary and if $\beta = 0$, the boundary conditions (3) and

$$\begin{aligned} u\cdot \underline{\mathrm{n}} &= 0 \qquad \text{on } \Gamma\times\,]0,T],\\ \omega\times \underline{\mathrm{n}} &= 0 \qquad \text{on } \Gamma\times\,]0,T], \end{aligned} \tag{4}$$

coincide. Observe that -in a certain sense- under the Dirichlet boundary conditions (see [**125**]) it holds that $\omega\cdot\underline{\mathrm{n}} = 0$, hence the slip conditions are really different from the no-slip ones in the light of behavior of the vorticity field.

This leads us to consider (4), even when the boundary is not flat. Note that the boundary conditions (4) are strongly related to the slip boundary conditions (3). In fact,

$$\underline{t}\cdot\underline{\tau}=\frac{\nu}{2}(\omega\times\underline{\mathrm{n}})\cdot\underline{\tau}-\nu u\cdot\frac{\partial\underline{\mathrm{n}}}{\partial\underline{\tau}}\quad\text{on }\Gamma,$$

for each vector $\underline{\tau}$ tangential to the boundary. Note that the last term from the right-hand side is a lower order term, and that $\omega\times\underline{\mathrm{n}}$ and $\partial\underline{\mathrm{n}}/\partial\underline{\tau}$ are tangential to Γ, while $|\partial\underline{\mathrm{n}}/\partial\underline{\tau}|$ is the normal curvature in the $\underline{\tau}$ direction. The relevance of using boundary conditions involving the vorticity is that one can try to use the vorticity equation in order to obtain estimates on the vorticity field and consequently on the gradient of the velocity. Recall in fact that for divergence free vector fields it holds

$$-\Delta u=\operatorname{curl}\omega,$$

hence, one can invert the Laplace operator showing that (at least in terms of L^q estimates) ω and ∇u are equivalent. In the case of Dirichlet boundary conditions the partial differential equations for the vorticity

$$\omega_t-\nu\Delta\omega+(u\cdot\nabla)\,\omega-(\omega\cdot\nabla)\,u=\operatorname{curl} f,$$

seem difficult to be used since in the integration by parts[3] arise some boundary integrals which we are not able to control. In particular, most of the problems are created by the Laplacian of the vorticity (this explains why for the Euler equations this problem does not appear). Consequently new problems derive from the linear part of the equations, while the well-known limitations due to the nonlinear terms remains essentially the same. We set the problem in the flat case and for instance consider $\Omega = \mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : \ x_3 > 0\}$, with boundary $\Gamma := \{x \in \mathbb{R}^3 : \ x_3 = 0\}$. On the boundary we consider, for $\beta = 0$, the conditions (3) (now equivalent to (4)) and we obtain with direct computation

$$\begin{cases} \omega_1 = \partial_2 u_3 - \partial_3 u_2 = 0 + 0, \\ \omega_2 = \partial_3 u_1 - \partial_1 u_3 = 0 + 0, \\ \partial_3 \omega_3 = -\partial_1 \omega_1 - \partial_2 \omega_2 = 0 + 0, \end{cases}$$

where the last line follows since $\nabla \cdot \omega = 0$. With this at disposal we can show that

$$-\int_\Omega \Delta\omega \cdot \omega \, dx = \int_\Omega |\nabla \omega|^2 \, dx,$$

because the boundary term $\int_\Gamma \omega \cdot \partial\omega / \partial \underline{n} \, dS$ vanishes identically, and one can also compare this Gauss-Green formula with Lemma 5.2 in the general case.

The mathematics of the Navier-Stokes under these boundary conditions presents new problems: Many results are not straightforward and need some adaption, see also Málek and Rajagopal [**105**]. As a further example, the finite element numerical analysis requires some work, due to a) the choice of the basis functions and b) the storage of the information is not the same as for the Dirichlet conditions. A study of the numerical problems related to the implementation of (3) or (4) can be found in Girault [**71**] (in this paper they are called non-standard), John [**79, 78**], Liakos [**100**], and Verfürth [**142, 143**].

Remark 2.2. *In the case $n = 2$ the situation is considerably simpler, as we will see in the next sections. In fact, for $n = 2$, the second boundary condition in equation* (4) *is simply replaced by $\omega = 0$. Furthermore,*

$$\underline{t} \cdot \underline{\tau} = \frac{\nu}{2}\omega - \nu u \cdot (k\, \underline{\tau}), \tag{5}$$

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[3] Clearly in the space-periodic case and for the Cauchy problem, the use of the vorticity represents a formidable tool. Problems in order to use the vorticity equation in the Dirichlet case turns out since we do not know the boundary values of ω, see also the results in Rautmann [**119**].

where k is the curvature of Γ. This vector identity is well-known and it can be used to construct weak solutions to the Euler equations in the 2D case by approximating the Euler equations with the Navier-Stokes equations supplemented by $u \cdot \underline{n} = \omega = 0$ on Γ, see for instance J.L. Lions [**101**], *Bardos* [8], *Clopeau, Mikelić, and Robert* [**49**], *and in the stochastic context Bessaih and Flandoli* [**37**, **38**]. *See also Section* 7.1.

3 - Some problems in which the Navier conditions naturally arise

In this section I explain some of the motivations that make the Navier-slip boundary conditions interesting from different points of view and not only a mathematical game.

3.1 - *Certain physical situations linked with slip on the boundary*

In Serrin [**125**, § 64] and Truesdell [**140**] it is pointed out that when moderate pressure and low surface stresses are involved, the adherence condition is no longer true. In this respect several authors proposed various *slip* (generally nonlinear) conditions, modeling precise physical situations. Having in mind problems with high altitude aerodynamics and the interface of porous media, Serrin [**125**], Beavers and Joseph [**10**], and Kreĭn and Laptev [**90**] proposed various slip conditions. Recently, Fujita [**64**, **123**] performed the analysis with the "slip or leak with friction" boundary conditions. These conditions are of particular interest in the study of polymers, blood flow, and flow through filters. The boundary conditions studied in [**64**], are

$$\begin{aligned}
&u \cdot \underline{n} = 0 \qquad \text{on } \Gamma \times]0,T],\\
&\text{if } |\underline{t}| < k|\underline{n} \cdot \mathbb{T} \cdot \underline{n}|, \quad \text{then } u_\tau = 0 \qquad \text{on } \Gamma \times]0,T],\\
&\text{if } |\underline{t}| = k|\underline{n} \cdot \mathbb{T} \cdot \underline{n}|, \quad \text{then } \exists\, \lambda \geq 0 : \ u_\tau = -\lambda\, \underline{t} \qquad \text{on } \Gamma \times]0,T],
\end{aligned}$$

where $k > 0$ is a coefficient of friction. This problem is treated with the techniques of variational inequalities, and turns out to be a particular case of the nonlinear boundary conditions proposed in [**125**, p. 240]. These nonlinear (unilateral) conditions are very-strictly connected to both the Navier and the no-slip boundary conditions. See also Consiglieri [**50**] for related problems.

For laminar flows the Navier boundary conditions (3) also appears in the presence of rough boundaries, see Jäger and Mikelić [**76**, **77**] and Achdou, Pironneau, and Valentin [**1**]. Among other nonstandard boundary conditions I recall those studied by Begue et al. [**11**]

$$\begin{aligned} \underline{\mathrm{n}} \times u = 0 \qquad & \text{on } \Gamma \times]0, T], \\ p = 0 \qquad & \text{on } \Gamma \times]0, T], \end{aligned}$$

and the “do-nothing” Neumann conditions, appealing for numerical studies in pipes, implemented in Heywood, Rannacher, and Turek [**74**]:

$$\frac{\partial u}{\partial \underline{\mathrm{n}}} - p\,\underline{\mathrm{n}} = K\,\underline{\mathrm{n}} \qquad \text{on } \Gamma \times]0, T].$$

Some interest for slip boundary conditions has recently appeared also in problems of shape optimization (see Bucur *et al.* [**44**, **45**]), especially in presence of rough boundaries.

My main interest about the Navier-type conditions comes from another theme and more precisely the modeling of turbulent flows. In the next section I will explain some of the results and connections with large scale approximation of fluids with very small viscosities.

3.2 - *Near wall models and turbulent flows*

My interest in non-standard boundary conditions started from the study of the numerical methods in turbulence. In order to briefly introduce the problem, I am summarizing the main points. It is not possible to compress any reasonable understanding of turbulent flows in a few pages, but I am trying to give at least motivations for the results that will follow. I also hope to interest the reader for a research field, which is still lacking of the needed mathematical rigor. For turbulent flows new insight and advances could come from the joint efforts of engineers, mathematician, and physicists. The aim is to try to attack what has been defined by Landau [**94**]

> ... *one of the great unsolved problems of classical physics.*

The reader interested in a better and deeper understanding of the field can see the books by Frisch [**63**], Pope [**117**], and Tennekes and Lumley [**139**]. As one can notice I will not define what a turbulent flow is, but one can stem on the fact that as the viscosity decreases, the flow becomes less and less stable and the motion becomes “*chaotic.*”[4] From the mathematical point of view one can say that uniqueness or stability are known only for large values of the viscosity (with respect to the velocity of the flow), while the most interesting effects appear when $\nu \sim 0$.

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[4] This word is used here in a purely qualitative way.

It is well-known that Kolmogorov's [**89**] theory predicts that simulating turbulent flows by using the Navier-Stokes Equations requires $\mathcal{N} = O(Re^{9/4})$ degrees of freedom, where $Re = UL\,\nu^{-1}$ denotes the (non dimensional) Reynolds number and U and L are a typical velocity and length, respectively. This number $\mathcal{N}$ is too large, in comparison with memory and computational capacities of actual computers, to perform a Direct Numerical Simulation (DNS). Indeed, for realistic flows -such as for instance geophysical flows- the Reynolds number is order at least of 10^8, yielding $\mathcal{N}$ of order 10^{18}....

With some technical assumptions and certain physical guessing, Kolmogorov has been able to show that (at least for some classes of velocities considered as random variables with suitable properties) the coherent structures of the flow (nowadays called *eddies*) evolve into smaller and smaller ones, leaving the total energy unaltered, till the point where they are so small to be destroyed by the viscous mechanism[5] and this happens in a proper statistical sense. The Kolmogorov length η_K at which this occurs represents the smallest scale present in the flow and it is of the order of $Re^{-3/4}$. For scales below this length, the behavior is more or less the same of solutions of the (dissipative) heat equation. Even if this is not rigorous (because to obtain this behavior one has to postulate a mathematical knowledge of solutions that we do not have) it is one of the reasons why one aims at computing at least the "mean or large scales values" of the unknowns (u, p), those not involving scales smaller than η_K. This is not enough, since numerical simulations cannot reach a so small scale, but motivated also from the fact that some gross characteristics of the flow behave in a more orderly manner, Large Eddy Simulation (LES) is about approximating (spatial) averages of turbulent flows, see Foiaş *et al.* [**61**]. Thus, LES seeks to predict the dynamics (the motion) of the organized structures in the flow (the eddies) which are larger than some user-chosen length-scale α. The length $\alpha > 0$ is related to mesh-size of the grid used in the numerical simulation. It is clear that LES is a *computational tool*, for which one tries to give a sound mathematical justification. One of the great challenges of simulating turbulence is that equations describing averages of flow quantities cannot be obtained directly from the physics of fluids. On the other hand, the equations for the point-wise flow quantities are well-known, but intractable to direct solution and sensitive to small perturbations and uncertainties in problem data.

In the spirit of the ideas speculated probably the first time by Leonardo da Vinci [**56**], the LES approach corresponds in finding a suitable computational de-

[5] This mechanism has been postulated in the early thirties by the meteorologist Richardson [**120**].

composition

$$u = \overline{u} + u' \qquad \text{and} \qquad p = \overline{p} + p',$$

where the primed variables are turbulent fluctuations around the over-lined mean fields. In many cases fluctuations can be disregarded and this can be justified because in applications knowledge of the mean flow is enough to extract relevant information on the fluid motion. The "mean values" can be defined in several ways (time or space average, statistical averages, solution of elliptic problems ...). If one denotes the means fields by $(\overline{u}, \overline{p})$, and by assuming that the averaging operation (whatever it is) commutes with differential operators, one gets the *filtered Navier-Stokes equations*

$$\begin{aligned} \partial_t \overline{u} + \nabla \cdot (\overline{u \otimes u}) - \nu \Delta \overline{u} + \nabla \overline{p} &= \overline{f}, \\ \nabla \cdot \overline{u} &= 0. \end{aligned}$$

This immediately raises the question of the *interior closure problem*, that is the modeling of the second order tensor $R(u) = \overline{u \otimes u}$ in terms of the filtered variables $(\overline{u}, \overline{p})$. Classical Large Eddy Simulations (LES) models approximate $R(u)$ by $w \otimes w - \nu_T(k/k_c)\nabla^s w$ where $w \approx \overline{u}$, and $\nabla^s w := (\nabla w + \nabla w^T)$. Here $\nu_T \geq 0$ is an *eddy viscosity* based on a "cut-off frequency" k_c (for a general discussion see [**121**]).

Remark 3.1. *I am introducing the new variable w since when using any approximation for $R(u)$, one is not writing the differential equations satisfied by $\overline{u}$, but that satisfied by another field w, which is hopefully close enough to $\overline{u}$.*

In recent years the role of Large Eddy Simulation increased and attracted the attention of mathematicians. One can find extensive overview in the nowadays classic book by Sagaut [**121**]. See also Geurts [**70**], John [**79**], Lesieur, Métais, and Comte [**98**] and -for a more theoretical approach- the monograph I wrote with my co-workers [**32**].

Here, I do not want to discuss the interior closure modeling, or other specific issues of LES. I want to focus on the problem that, even if one has a disposal a set of partial differential equations describing in some sense the mean values of the flow, there is the need to describe the boundary conditions. The classical [**89**] theory of turbulence starts with the homogeneous and isotropic case and the flow is in a periodic box. In real life boundaries do exist and they are one of the biggest source of problems. As a folklore one can recall the sentence of Heisenberg

... the boundary is the invention of the devil ...

Concerning this issue, it is also interesting to read the reprint of an old paper by von

Neumann [**145**] in one of the latest volume of the Bulletin of the American Mathematical Society.

In most cases in LES, the filtered velocity $\overline{u}$ is defined through a space-convolution

$$\overline{u}(x,t) = g_\alpha(x) * u(x,t) \tag{6}$$

with a rapidly decreasing smoothing kernel $g_\alpha(x)$ of width α. In several cases of practical interest g_α is a Gaussian, i.e.,

$$g_\alpha(x) := \left(\frac{6}{\pi}\right)^{3/2} \frac{1}{\alpha^3}\, \mathrm{e}^{-\frac{6|x|^2}{\alpha^2}}.$$

By definition, the value of $\overline{u}$ at a point x_0 on the boundary Γ will mainly depend on the behavior of u in a neighborhood of width α near that point: Even if u is extended to zero for each $x \notin \overline{\Omega}$, it is clear that in general $\overline{u}(x_0) \neq 0$.

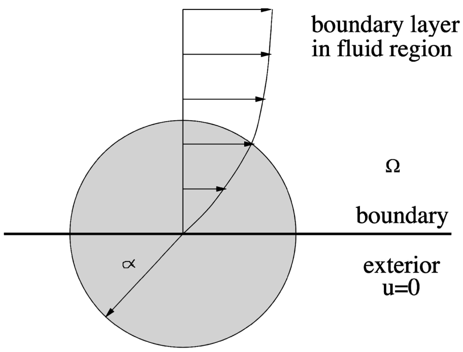

Fig. 1. Filtering the velocity does not yield homogeneous Dirichlet conditions at the boundary.

Following the approach of LES (again trying to understand the possible connections with the boundary conditions) I recall that another way of approximating the equations -avoiding eddy viscosities- consists in approaching $R(u)$ by a suitable quadratic term. It is curios that the first LES model has been introduced by Leray [**97**] with a different goal. In fact to construct weak solutions of the Navier-Stokes equations in the celebrated 1934 paper he solved (in $\mathbb{R}^3$) the differential problem

$$\begin{aligned} w_t - \nu\Delta w + \nabla\cdot(\overline{w}\otimes w) + \nabla q &= f && \text{in } \mathbb{R}^3\times\,]0,T],\\ \nabla\cdot w &= 0 && \text{in } \mathbb{R}^3\times\,]0,T]. \end{aligned} \tag{7}$$

In this model the transport is realized by $\overline{w}$, which is a field smoother than w itself. In particular in Leray's work $\overline{w}$ is defined by means of a smoothing by mollifiers, as in (6). This approximation has the same properties of the recent Leray-alpha LES model, where $\overline{w}$ satisfies[6] the elliptic equations

$$-\alpha^2 \Delta \overline{w} + \overline{w} = w.$$

One has to specify the boundary conditions for the above differential equation and this is again absolutely non-trivial. It is also relevant to observe that (in absence of boundaries) the solution w to (7) satisfies

$$w \to u, \qquad \text{as} \qquad \alpha \to 0,$$

where u is the velocity of the Navier-Stokes system. In this limit process the good properties of w (which is smooth and unique) are lost and this is Leray's proof for existence of possibly non unique weak-solutions.

Remark 3.2. *As one can understand this is not the real goal of LES, since one would like to approximate $\overline{u}$ and not a single trajectory u. A first rigorous result in this direction have been recently proved in a joint work with R. Lewandowski* [**33**] *for the so-called Stolz and Adams* [**132**, **2**] *Approximate Deconvolution Model* (*ADM*)*. Even if we are forced to consider the periodic setting, to our knowledge such a "well posedness", i.e., proving that w converges to $\overline{u}$* (*which is the average of a weak solution*)*, was not previously known for any LES model.*

In order to describe the problems arising when studying the boundary conditions for a LES model, I first recall that in the boundary-layer theory several log-law and power-law asymptotics near the boundary are introduced, together with the fictitious boundaries, in order to model turbulent flows within a small region near to Γ. Roughly speaking, appropriate nonlinear boundary conditions are imposed on an artificial boundary that lies inside the computational domain. The boundary conditions may simulate (at least in a computational approach) the behavior of the *boundary-layer*, and they are modeled to take into (partial) account of the peculiar behavior of a fluid near the boundaries. In this respect we recall that Maxwell [**107**] observed

> *... it is almost certain that the stratum of gas next to a solid body is in a very different state from the rest of the gas.*

As we pointed out before, one basic problem in LES is turbulence driven by interaction of the flow with a solid wall. Mathematically, this is the problem of *specifying*

[6] This is why it is also called a differential filter and it is generally used in the periodic setting.

boundary conditions for flow averages. Flow averages are *inherently non-local*: they depend on the behavior of the unknown turbulent flow near the boundary. On the other hand, to be guided by the mathematical theory of the equations of fluid motion and seek boundary conditions that have hope of leading to a simple enough and well-posed problem, those boundary conditions should be *local*. From the mathematical point of view more complex conditions can be analyzed, but one has to keep in mind that LES is a computational tool, hence the equations need to be implemented in a efficient way. Introducing something too complex or numerically intractable will have the only effect of moving problems from one point to another one.

In LES the question of finding boundary conditions when using a constant averaging radius α is known as *Near Wall Modeling*. This is related to the extensive literature in Conventional Turbulence Modeling (CTM) on "wall-laws." CTM seeks to approximate long-time averages of flow-quantities and, conveniently for CTM, there is a lot of experimental and asymptotic information available about time-averaged turbulent boundary-layers. One common approach in CTM is to place an artificial boundary *inside* the flow domain and *outside* the boundary-layer, together with a Dirichlet condition for the stresses. The main difference between CTM and LES is that LES would like to describe inherently dynamic phenomena, so imposing a condition that $\overline{u}$ should match some equilibrium profile is probably not correct. One challenge in LES is how to use the extensive information on *time averaged turbulent* boundary-layers to generate NWM's that allow *time fluctuating* solution's behavior near the wall.

The classical approach (first introduced for the $k - \varepsilon$ model) consists in eliminating part of the boundary-layer, see Launder and Spalding [**95**]. The boundary that is considered is not the real boundary Γ, but it is an artificial one Γ_1, lying inside the volume of the flow, where one can impose

$$\begin{aligned} &\overline{u} \cdot \underline{\mathrm{n}} = 0 && \text{on } \Gamma_1 \times [0, T], \\ &\frac{u_{ws}^2}{|\overline{u}|}\overline{u}_\tau + \underline{\mathcal{T}}(\overline{u}, \overline{p}) = 0 && \text{on } \Gamma_1 \times [0, T]. \end{aligned} \tag{8}$$

In particular, when considering the Smagorinsky model[7] (studied with the above artificial boundary conditions by Parés [**115**]) the turbulent stress-tensor in (8) is

[7] This is the oldest LES model introduced by Smagorinsky [**127**] with the intent of studying geophysical flows. The mathematical properties have been studied starting with Ladyženskaya [**91**] since they fit with the theory of monotone operators. We do not treat here these equations, but we recall that they are still an intense research field. For the treatment of the Smagorinsky model with Navier boundary conditions, especially in the context of regularity, see Beirão da Veiga [**15**].

given by

$$\mathbb{T}_{ik}(\overline{u}, \overline{p}) = -\delta_{ik}p + (\nu + \nu_T)(\partial_k \overline{u}_i + \partial_i \overline{u}_k)$$

where ν is the usual kinematic viscosity, while $\nu_T = \nu_T(\alpha, \nabla^s \overline{u})$ is the turbulent viscosity. The quantity u_{ws} appearing in Eq. (8) is the so-called *wall shear velocity* (or skin friction velocity). It has the dimension of a length divided by a time and acts as a characteristic velocity for the turbulent flow; for more details, see Landau and Lifshitz [**94**, § 42-44] and Pope [**117**, § 7.1.3].

One recurring theme in these attempts is the use of *non-local* boundary conditions to incorporate solution's behavior in a strip near Γ, *via* an extra forcing function in the strip along the boundary. The problem remains however, difficult because the behavior of $\overline{u}$ on Γ depends on the behavior of u in a α-neighborhood of Γ.

As pointed out in Galdi and Layton [**68**] the physical intuition may suggest that

> ... *large coherent structures touching a wall do not penetrate, but instead slide along the wall and lose their energy.*

Consequently the boundary conditions of Navier may be revisited by linking the micro-scale λ of the kinetic theory of gases with the radius α of the averaging filter. Many NWM have been tested in the computational approach (Sagaut [**121**] and Piomelli and Balaras [**116**]), the results are not uniformly successful, and a positive outcome is very often based on a fine tuning of parameters. This is why new models require at least a positive background from the physical hypotheses and a coherent mathematical analysis. In particular, a direct application of the Navier slip-with-friction boundary conditions (3) is prevented by

1) The presence of recirculation regions;
2) The presence of fast time-fluctuating quantities.

The first problem is motivated by the fact that in recirculation regions the local Reynolds number is very different from the main stream, and it is natural to expect that β should depend (possibly in a nonlinear way) on a local Reynolds number related to the local slip speed, i.e., if u_τ is the local tangential velocity

$$\beta = \beta(\alpha, |u_\tau|).$$

Preliminary analysis has been performed by John, Layton, and Sahin [**80**] and Dunca et al. [**102**], and an appropriate power-law choice of β seems promising to improve the estimation of reattachment points. To emphasize the role of recirculation in real life flows I want to stress that in the simulation of the blood flow within the carotid-bifurcation recirculation has a prominent effect, see Quarteroni [**118**]. The limitation of the Navier law (3) in a boundary-layer theory is that it can well-describe time-averaged flow profiles, but also the information coming from fluctuating quantities in the wall-normal direction can play an important role in triggering separation and detachment. To try to overcome these limitations, Layton [**96**] recognized a particular class of boundary conditions, leading to conditions similar "in spirit" to the so-called *vorticity seeding methods*. We will see more details about this in Section 6. Observe in fact, that in the 2D case identity (5) implies the generation of vorticity at the boundary, proportional to the tangential velocity. In particular, in [**96**] the following boundary conditions are proposed to simulate the boundary effects

$$(9)\qquad \begin{aligned} &u\cdot \underline{\mathrm{n}} = \alpha^2 \mathcal{G}(x,t) && \text{on } \Gamma\times]0,T],\\ &\frac{\nu}{\alpha U} u_\tau + \underline{\mathcal{T}}(u) = 0 && \text{on } \Gamma\times]0,T], \end{aligned}$$

where $\mathcal{G}$ is a *highly oscillating* function in the time variable (hopefully a random variable in numerical tests), while it may be very smooth in the space variables and should satisfy the natural compatibility condition

$$(10)\qquad \int_\Gamma \mathcal{G}(x,t)\, dS = 0 \qquad \forall\, t\in (0,T),$$

which is required by the normal trace of a divergence-free vector field.

This way of reasoning is also similar to the introduction of stochastic fluctuations to simulate the micro-scale effects. A comprehensive introduction to stochastic partial differential equations in fluid mechanics can be found in Monin and Yaglom [**109**], Bensoussan and Temam [**27**], Višik and Fursikov [**144**], and Flandoli [**60**].

3.3 - *Vanishing viscosity limits*

To conclude the introduction I observe that another motivation for the study of slip boundary conditions are the recent advances on the vanishing viscosity limits obtained by Xiao and Xin [**148**] and by Beirão da Veiga and Crispo [**25**, **26**]. The main idea is that under Dirichlet boundary conditions one cannot expect to have convergence (in strong norms) of the solutions to the Navier-Stokes equations, towards those of the Euler equations with the same data, as $\nu \to 0$: There is the *boundary-layer*, characterized by large gradients and this prevents from proving results of convergence, see Constantin [**52**]. It is also well-known that in presence of boundaries one cannot expect convergence (or one can expect convergence only in special settings, see the review in Mazzucato [**108**]) if the boundary-layer has some effect, see also Asano [**6**] and Sammartino and Caflisch [**122**]. In 3D the situation is complicated, also for the fact that we do not know the existence of reasonably weak solutions to the Euler equations. Anyway, in the case of weak solutions u^ν (corresponding to the positive viscosity ν) to the Navier-Stokes equations Kato [**83**] proved that

$$u^\nu \to u^E \quad \text{in} \quad L^2(\Omega), \text{ uniformly in } t \in [0,T], \quad \text{as} \quad \nu \to 0,$$

$$\text{if and only if}$$

$$\nu \int_0^T \|\nabla u^\nu(\tau)\|_{L^2(\Omega^\nu)}\, d\tau \to 0, \quad \text{as} \quad \nu \to 0, \tag{11}$$

where u^E is the solution to the Euler equation and Ω^ν is a boundary strip of width $1/\nu$. Recent results have been also proved by Kelliher [**86, 87**] and Wang, Wang, and Xin [**147**].

On the other hand in the whole space or in the periodic case it is well-known that convergence takes place, see Section 7. Probably the most difficult part is showing strong convergence with respect to the norm of the initial datum, part of the so-called *sharp* convergence result.

The Navier conditions represent an intermediate path between having no boundaries and having a solid boundary (in fact they have been used also for the free-boundary problem). To understand why the Navier conditions allow to simplify the problem, one can observe that if $\beta = 0$ and in the half-space case, one can extend the solution (u, p) to the whole space by extending in a even (with respect to $\{x_3 = 0\}$) way u_1, u_2, and p, while extending in a odd way u_3. With this procedure one obtains a new couple $(\widetilde{u}, \widetilde{p})$ which is a solution the Navier-Stokes equations in $\mathbb{R}^3$. With this trick the problem is reduced to the Cauchy one and this is tractable with the standard tools, see for instance [**81**]. On the other hand, the generic non-flat case is much more difficult, because the reflection technique does not seem to work and one has to introduce new tools also to study the existence of strong solutions to the Navier-Stokes problem. Moreover, the 2D theory is much more complete also in the non-flat case, but some sharp results were still lacking, see Section 7, where recent developments are explained. On the other hand, the approach to the 3D case is particularly new. There have been important advances in the last three years [**148, 25, 26**] and I review some of the new results. In addition, I announce some recent results concerning the well-posedness of the boundary value problem in the non-flat case and I recall that in the general 3D case some questions remain still open, and others are subject of the current research.

4 - The linear stationary problem

In this section I present the basic results concerning the linear and stationary Stokes problem

$$
\begin{aligned}
-\nu\Delta u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{aligned}
\tag{12}
$$

with the slip boundary conditions (3). I am starting with this simplified problem since its understanding represents one of the main building blocks to deal with the Navier-Stokes equations under the slip boundary conditions.

In particular, we will see that existence of weak solutions derives from a fairly standard application of the Lax-Milgram lemma, once the precise functional setting has been introduced. We observe that the property of uniqueness is not straightforward: It depends on the geometry of the domain and on the values of the parameters entering in the boundary conditions. Moreover, the regularity of weak solutions (if the data of the problem are smooth enough) represents a non-trivial result. The $H^2(\Omega)$ regularity of solutions, which represents the counterpart of the Cattabriga [**47**] results for the Dirichlet problem, has been first proved by Solonnikov and Ščadilov [**130**]. Here, I will present a summary of the self-contained proof provided by Beirão da Veiga [**14, 16**], which is particularly important for the estimates on the pressure (I wish also to mention the results in [**13**] for the Dirichlet problem). Based on a subtle treatment of the pressure, the regularity is obtained by means of the usual Nirenberg translations method [**114**]. I also observe that the proof I am presenting does not make any use of the vorticity equation, which is one of the main tools at disposal when dealing with problems without boundary or with the Navier boundary conditions. Next, I will briefly present also some simplifications and extension obtained in the Master Thesis of Borselli [**39**] and by myself [**30**] in the context of very-weak solutions, and which make substantial use of the vorticity equation.

4.1 - *Notation*

Here and in the sequel Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{1,1}$ (Lipschitz-continuous first derivatives). We use the classical Sobolev spaces $W^{k,q}(\Omega)$ with norm $\|\,.\,\|_{k,q}$ and we also write $H^k(\Omega) = W^{k,2}(\Omega)$ (we use standard notation and symbols, see also Adams [**3**] and Brezis [**42**]). If $k = 0$, we write simply $\|\,.\,\|_q := \|\,.\,\|_{0,q}$ and, since the Hilbert case $q = 2$ represent the cornerstone, to simplify the notation we set

$$\|\,.\,\| := \|\,.\,\| = \|\,.\,\|_{L^2}.$$

The number k can also be a real one and we will not distinguish between scalar, vector, or tensor valued function spaces (the reader can extrapolate from the context the correct meaning of the symbols). We also use fractional trace spaces on Γ and we denote by $\|\,.\,\|_{k,p,\Gamma}$ its norm (with $\|\,.\,\|_{s,\Gamma} := \|\,.\,\|_{s,2,\Gamma}$). As usual $W_0^{k,p}(\Omega)$ denotes the closure, with respect to the norm of $W^{k,p}(\Omega)$, of smooth and with compact support functions. Since in the sequel we will deal many times with tangential vector fields, we use the following symbol

$$H^1_\tau := \big\{v \in H^1(\Omega) : (v \cdot \underline{\mathrm{n}})_{|\Gamma} = 0\big\},$$

and we also remark that, as for the standard Poincaré inequality, $\|\nabla v\|$ is a norm in H^1_τ equivalent to the canonical norm $\|v\|_{1,2}$, see e.g. Galdi [**65**] (the same holds also in the non Hilbertian case).

We use also the symbol $(X)'$ to denote the topological dual of the linear space X and $\langle\,.\,,\,.\,\rangle$ denotes the duality paring. Generally we will denote the duality pairing of spaces of functions defined on the boundary Γ by $\langle\,.\,,\,.\,\rangle_\Gamma$.

The pressure in the Stokes system enters only with first derivatives and hence it is determined up to an additive constant. To uniquely determine the pressure generally one imposes a vanishing mean value. In the sequel we denote by $X_\#$ the subspace of functions of X with vanishing mean value.

4.2 - *On a generalized Stokes system*

Contrary to the Dirichlet case, the geometry of the domain is very important when studying boundary values problems with Navier slip boundary conditions. In the next sections we will also understand some of the substantial differences between the flat and non-flat case. Here, we recall some of the restrictions which are needed in order to study the linear case. In particular, the presence of symmetries can give rise problems of non uniqueness. These issues are treated in great detail in [**14**] and we report here the main conclusions. We say that the domain Ω is *axially symmetric* if it can be generated by a revolution around a given axis l_1 (or even around by two orthogonal axes l_1 and l_2). We also assume that the origin of the coordinates belongs to both axes and denote by $\underline{l}_i$ the unit vector with the same direction of l_i. Define the linear space (zero, one, or two dimensional)

$$Z := \{z : \; z = k_i\, \underline{l}_i \times x, \quad k_i \in \mathbb{R}\},$$

with summation over repeated indices. The summation is taken over zero, one or two indices, depending on the symmetries of the domain (obviously if there are no symmetries $Z = \{0\}$).

We say that we are in the special case if Ω is symmetric and $\beta = 0$ and otherwise in the generic case. We also use the following notation

$$H^1_z := \{v \in H^1 : \langle v, z\rangle = 0, \quad \forall\, z \in Z\}$$

and

$$H^1_{z,\tau} := H^1_z \cap H^1_\tau.$$

Clearly it follows that $H^1_\tau = H^1_{z,\tau} \oplus Z$, while in the generic case $H^1(\Omega) = H^1_z$.

I start considering the following "generalized Stokes" system

$$\begin{aligned} -\nu\Delta u - \mu\,\nabla(\nabla\cdot u) + \nabla p &= f(x) \quad \text{in } \Omega,\\ \lambda\, p + \nabla\cdot u &= g(x) \quad \text{in } \Omega, \end{aligned} \tag{13}$$

under the general non homogeneous boundary conditions

$$\begin{aligned} &u\cdot \underline{\mathrm{n}} = a(x) && \text{on } \Gamma,\\ &\beta\, u_\tau + \underline{T}(u) = b(x) && \text{on } \Gamma, \end{aligned} \tag{14}$$

where $a(x)$ and $b(x)$ are a given scalar field and a given tangential vector field on Γ and the constants μ, ν, and λ satisfy the assumptions

$$\nu > 0, \qquad \mu + \nu > 0 \qquad \text{and} \qquad \lambda \geq 0.$$

The (only apparent) complication coming from the study of a more general system is motivated by the fact that it allows to give a better understanding of the role of the pressure and of boundary data. Moreover, it can be used also to study problems involving compressible fluids and clearly when $\mu = \lambda = g(x) = 0$ one re-obtains the classical Stokes system.

If $\lambda > 0$ then in the "generic case" there is a unique solution to (13)-(14). On the other hand, if $\lambda = 0$ the necessary and sufficient condition for existence is the compatibility condition

$$\int_\Omega g\, dx = \int_\Gamma a\, dS, \tag{15}$$

which derives from the Gauss-Green formula applied to $(13)_2$ with $(14)_1$. The velocity u is uniquely determined, while p is determined up to an additive constant, which is generally set in such a way[8] that

$$\overline{p} := p - \frac{1}{|\Omega|}\int_\Omega p\, dx = 0.$$

[8] In this section the over-lined variable has nothing to do with the LES variables of the previous sections.

On the contrary, in the "special case" there are non-zero solutions to the homogeneous problem, that are rigid motions. In fact, the kernel of the linear problem coincides with the linear space Z. If we take the couple $(z, 0)$ with $z \in Z$, this is a solution of the problem (13)-(14) with $a = f = 0$ and $b = g = 0$, since $\Delta z = 0$, $\nabla \cdot z = 0$, $(z \cdot \underline{\mathrm{n}})_{|\Gamma} = 0$ and $\underline{\mathcal{T}}(z)_{|\Gamma} = 0$. The converse is also true but the proof (see [**14**, App. 1]) requires some care: If (u, p) is a weak solution of the homogeneous problem, then necessarily $u \in Z$ and $p = 0$ (or a constant if $\lambda = 0$). In the special case the solution can be decomposed as $u_0 = u + z$, where u is the particular solution to the non-homogeneous problem such that $\int_\Omega u\, z\, dx = 0$ for all $z \in Z$. Moreover, the function u (which is unique) exists if and only if the compatibility condition

$$\int_\Omega f \cdot (\underline{l}_i \times x)\, dx = -\nu \int_\Gamma b \cdot (\underline{l}_i \times x)\, dS \tag{16}$$

is satisfied.

Remark 4.1. *The proof of the existence of weak solutions I am presenting is based on some kind of* <u>*artificial compressibility*</u> *method. In fact, the term λp has been added to the divergence equation. This tool will greatly simplify the problem for two reasons:*

1. *The resulting variational problem is coercive over the set of all tangential vector fields, and not only on the subspace of divergence-free functions;*
2. *The use of functions without prescribed divergence allows us to use the same as test functions. This will be of particular interest for proving higher regularity, since the classical Nirenberg translation method* [**114**] *can be used directly.*

Obviously the price to be paid is that the estimates (*cf.* (18)) *we obtain directly are not independent of λ. One has to use in a clever way some well-known inequalities concerning a function and its derivatives* (*as those proved by Duvaut and Lions* [**58**]) *in order to recover results independent of $\lambda > 0$.*

Remark 4.2. *The use of function spaces without a constraint on the divergence is particularly important also in view of the numerical analysis and implementation of numerical schemes for the Stokes problem. The reader can find details on the numerical implementation and additional* (*with respect to the Laplacian, which seems apparently similar*) *problems involving the Stokes system, e.g., in Girault and Raviart* [**72**] *and Brezzi and Fortin* [**43**].