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L^∞ solutions for 2×2 systems of conservation laws

Abstract. We consider a 2×2 hyperbolic system of conservation laws with genuinely nonlinear characteristic fields and L^∞ data. We extend the existence result of Glimm–Lax removing their assumption on the geometry of the Lax curves.

Keywords. Hyperbolic conservation laws with L^∞ data.

Mathematics Subject Classification (2000): 35L65.

1 - Introduction

In this paper we briefly present the main result of [**4**]. We consider a non-linear 2×2 system of conservation laws, i.e.:

$$\partial_t u + \partial_x [f(u)] = 0 \tag{1}$$

and the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x [f(u)] = 0 \\ u(0,x) = \bar{u}(x) . \end{cases} \tag{2}$$

Our aim is to extend the result [**10**, Theorem 5.1] relaxing the assumptions taken therein on the geometry of the shock–rarefaction curves. Nevertheless, the result of Theorem 1.1 is the same of that in [**10**, Theorem 5.1], namely the existence of a weak entropy solution to (2) for all initial data with sufficiently small L^∞ norm.

On the flow f in (1) we assume the following Glimm-Lax condition, analogously to [**10**, formula (1.4)]:

(GL) $f : B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and with both characteristic fields genuinely non linear

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where $B(0, r)$ is the ball of $\mathbb{R}^2$ with center 0 and radius r. The main result of this paper is the following:

Theorem 1.1. *Under the assumption* **(GL)**, *there exists a sufficiently small* $\eta > 0$ *such that for every initial condition* $\bar{v} \in L^1_{loc}(\mathbb{R}; \mathbb{R}^2)$ *with:*

$$\|\bar{v}\|_\infty \leq \eta \tag{3}$$

the Cauchy problem (2) *admits a weak entropy solution for all* $t \geq 0$.

The solution is constructed as limit of the ε–approximations v^ε constructed through the front tracking algorithm used in [**6**], suitably adapted to the present situation. First, as in [**10**], careful decay estimates on a trapezoid (see Figure 2) allow to bound the positive variation and the L^∞ norm of v^ε on the upper side of the trapezoid. Under the further assumption that a suitable L^∞ estimate on v^ε holds, see condition **(A)** in Paragraph 4.3, a technique based on the hyperbolic rescaling allows to extend the previous bound to any positive time. The approximate solutions can hence be defined globally in time.

A key point is now to provide estimates that allow to abandon condition **(A)**. This is achieved through L^∞ estimates essentially based on the conservation form of (1) and on the previous results on the trapezoids. It is here that the integral estimates in Section 6 allow us to extend the result in [**10**].

As a byproduct, we also obtain Theorem 3.3, under the standard Lax condition

(L) $f: B(0, r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and each characteristic field is either genuinely non linear or linearly degenerate.

Indeed, Theorem 3.3 is an existence result valid for all initial data having small L^∞ norm and bounded, not necessarily small, total variation.

In this connection, we recall that in the case of systems with coinciding shock and rarefaction waves, the well posedness of (2) in L^∞ was proved in [**3**] under condition **(L)**. A continuous semigroup of L^∞ solutions for Temple systems under condition **(GL)** was constructed in [8].

2 - Notations

Denote by $A(u)$ the 2×2 hyperbolic matrix $Df(u)$, by λ_1, λ_2 its eigenvalues and by l_1, l_2 (resp. r_1, r_2) its left (resp. right) eigenvectors, normalized so that

$$\|r_i(u)\| = 1, \qquad \langle l_j(u), r_i(u) \rangle = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \qquad i, j = 1, 2\,.$$

If the i–th characteristic field is genuinely nonlinear, we choose r_i oriented so that

$$D\lambda_i(u)\, r_i(u) \geq c > 0 \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad u \in B(0, r) \tag{4}$$

for a suitable c. In the linearly degenerate case, we do not need to specify this orientation. By **(L)**, $\sup_{B(0,r)} \lambda_1 < \inf_{B(0,r)} \lambda_2$.

By a linear change of coordinates, we can assume that $f(0) = 0$, $A(0) = \mathrm{diag}(\lambda_1(0), \lambda_2(0))$ and that $\lambda_1(0) = -1$, $\lambda_2(0) = 1$. We are thus led to assume that f can be written as follows:

$$\begin{aligned} f_1(u) &= -u_1 + \frac{1}{2} a_{11}\, u_1^2 + a_{12}\, u_1\, u_2 + \frac{1}{2} a_{22}\, u_2^2 + \mathcal{O}(1)\, \|u\|^3 \\ f_2(u) &= u_2 + \frac{1}{2} \beta_{11}\, u_1^2 + \beta_{12}\, u_1\, u_2 + \frac{1}{2} \beta_{22}\, u_2^2 + \mathcal{O}(1)\, \|u\|^3 \end{aligned} \tag{5}$$

with $a_{ij} := \dfrac{\partial^2 f_1}{\partial u_i\, \partial u_j}(0)$ and $\beta_{ij} := \dfrac{\partial^2 f_2}{\partial u_i\, \partial u_j}(0)$.

Following [**5**, formula (5.38)], introduce the Lax curves as the gluing of the shock and rarefaction curves:

$$L_i(u, \sigma) := \begin{cases} S_i(u, \sigma) & \sigma < 0\,, \\ R_i(u, \sigma) & \sigma \geq 0\,. \end{cases} \tag{6}$$

As in [**5** formula (7.36)], calling $E = E(u^-, u^+)$ the map giving the sizes of the waves in the solution to the Riemann problem for (1) with data u^- and u^+, it holds:

$$(\sigma_1, \sigma_2) = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = L_2(L_1(u^-, \sigma_1), \sigma_2)\,.$$

For a $u \in L^1_{loc}(\mathbb{R}; \mathbb{R}^2)$, define its total variation by:

$$\mathrm{TV}(u) := \sup \left\{ \sum_{i=1}^{2} \sum_{l=1}^{N} |u_i(x_l) - u_i(x_{l-1})| \colon \begin{array}{l} x_1, \ldots, x_N \in \mathbb{R} \text{ with} \\ x_1 < \cdots < x_N \end{array} \right\}. \tag{7}$$

In the following, for $L > 0$, it will be useful also the notation:

$$\mathrm{TV}(u; L) := \sup_{x \in \mathbb{R}} \mathrm{TV}\left(u_{|[x, x+L]}\right)$$

where $u_{|[x,x+L]}$ is the restriction of u to the interval $[x, x + L]$.

For a function $u \colon \mathbb{R} \to B(0, r)$, we use below the L^∞ norm

$$\|u\|_\infty := \operatorname*{supess}_{x \in \mathbb{R}} |u_1(x)| + \operatorname*{supess}_{x \in \mathbb{R}} |u_2(x)|\,.$$

Below, $\hat{\lambda}$ denotes an upper bound for the moduli of the characteristic speeds in $B(0, r)$, i.e.

$$\hat{\lambda} > \sup_{i=1,2;\, \|u\| \leq r} |\lambda_i(u)|\,. \tag{8}$$

3 - Construction of solutions with bounded total variation and small L^∞ norm

In this section, we modify the wave front tracking algorithm in [**6**, Section 2] to construct a solution to (2) under the assumption that the initial datum has bounded total variation and small L^∞ norm. More precisely, let $\bar{u}$ belong to

$$\mathcal{D}(\eta, \bar{K}) := \left\{ u \in L^1_{loc}(\mathbb{R}; B(0,\eta)) \colon \mathrm{TV}(u) \leq \bar{K} \right\}, \tag{9}$$

where $\bar{K}, \eta$ are positive constants.

Moreover, in the first two paragraphs below, it is not necessary to assume that both characteristic fields be genuinely nonlinear. The standard Lax [**11**, Section 9] condition **(L)** is sufficient.

3.1 - *The algorithm*

Fix $\varepsilon > 0$. Denote by v the Riemann coordinates of (1), see [**9**, Definition 7.3.2], and call $\mathcal{L}_i$, $\mathcal{R}_i$ and $\mathcal{S}_i$ the Lax, the rarefaction and the shock curves in the Riemann coordinates. In these variables, as in [**6**], we parametrize the rarefaction and the shock curves as follows:

$$\begin{aligned} \mathcal{R}_1(v,\sigma) &= (v_1+\sigma, v_2), \quad \mathcal{S}_1(v,\sigma) = \left(v_1+\sigma, v_2+\hat{\psi}_2(v,\sigma)\,\sigma^3\right) \\ \mathcal{R}_2(v,\sigma) &= (v_1, v_2+\sigma), \quad \mathcal{S}_2(v,\sigma) = \left(v_1+\hat{\psi}_1(v,\sigma)\,\sigma^3, v_2+\sigma\right) \end{aligned} \tag{10}$$

where $\hat{\psi}_1$ and $\hat{\psi}_2$ are suitable smooth functions of their arguments. First, the initial datum $\bar{v}$ is substituted by a piecewise constant $\bar{v}^\varepsilon$ such that:

$$\lim_{\varepsilon\to 0+} \|\bar{v}^\varepsilon - \bar{v}\|_{L^1} = 0\,, \quad \mathrm{TV}(\bar{v}^\varepsilon) \leq \mathrm{TV}(\bar{v}) \leq \bar{K}\,, \quad \|\bar{v}^\varepsilon\|_\infty \leq \eta\,.$$

At each point of jump in $\bar{v}^\varepsilon$, the resulting Riemann problem is solved as in [**6**, Section 2]. Let $\varphi \in C^\infty(\mathbb{R};\mathbb{R})$ be such that

$$\begin{array}{rclcrcl} \varphi(\sigma) &=& 1 & \text{for} & \sigma &\leq& -2 \\ \varphi(\sigma) &=& 0 & \text{for} & \sigma &\geq& -1 \\ \varphi'(\sigma) &\in& [-2,0] & \text{for} & \sigma &\in& [-2,-1] \end{array}$$

and introduce the ε-approximate Lax curves

$$\mathcal{L}_i^\varepsilon(v,\sigma) = \varphi(\sigma/\sqrt{\varepsilon})\,\mathcal{S}_i(v,\sigma) + \left(1-\varphi(\sigma/\sqrt{\varepsilon})\right)\mathcal{R}_i(v,\sigma) \quad \text{for } i = 1,2\,.$$

An ε–solution to the Riemann problem for (1) with data v^-, v^+ is obtained gluing ε–rarefactions and ε–shocks. ε–rarefactions of the first, respectively second, family are substituted by rarefaction fans attaining values in $\varepsilon\mathbb{Z} \times \mathbb{R}$, respectively $\mathbb{R} \times \varepsilon\mathbb{Z}$, traveling with the characteristic speed of the state on the right of each wave, see [**6**, formulæ (2.13)–(2.16)]. A 1–shock with left state v^- and size σ_1, such that $\sigma_1 < -2\sqrt{\varepsilon}$, travels with the exact Rankine–Hugoniot speed $\lambda_1^s(v^-, \sigma_1)$. When $\sigma_1 > -2\sqrt{\varepsilon}$, we assign to this jump an interpolated speed λ_1^φ defined as an average between the exact Rankine-Hugoniot speed $\lambda_1^\varphi(v, \sigma)$ and an approximate characteristic speed, see [**6**, formulæ (2.17), (2.18) and (2.19)]. For every $\sigma_i < 0$, it holds

$$\lambda_i(\mathcal{L}_i(v^-, \sigma_i)) < \lambda_i^\varphi(v^-, \sigma_i) < \lambda_i(v^-) . \tag{11}$$

2-shocks are treated similarly, we refer to [**6**, Section 2] for further details.

If the i–th characteristic family is linearly degenerate, the shock, the rarefaction and the ε–approximate Lax curves coincide. Moreover, the characteristic speed is constant along these curves, so that the interpolation related to the shocks speed is trivial. Gluing the solutions to the Riemann problems at the points of jump in $\bar{v}^\varepsilon$ we obtain an ε–solution defined on a non trivial time interval $[0, t_1]$, t_1 being the first time at which two or more waves interact. Any interaction yields a new Riemann problem, so that a piecewise constant ε–solution of the form

$$v^\varepsilon = \sum_a v^a \chi_{]x_a, x_{a+1}]} \qquad \text{with} \qquad v^{a+1} = \mathcal{L}_2^\varepsilon\big(\mathcal{L}_1^\varepsilon(v^a, \sigma_{1,a}), \sigma_{2,a}\big) \tag{12}$$

is recursively extended in time. Hence, we obtain a sequence of ε–approximate solutions. Here, the meaning of ε–approximate solutions is slightly different from that in [**6**, Definition 1]. The wave speed of the approximate solution is here simplified, possibly loosing the Lipschitz continuous dependence that has a key role in [**6**], but allowing for simpler proofs in the present situation. We refer to [**4**, § 3.1] for further details.

3.2 - *Existence and properties of the approximate solutions*

In this paragraph we show that the ε–approximate solutions constructed by the previous algorithm are well defined, see Theorem 3.1.

Throughout, by C we denote a positive constant dependent only on f and r as in **(L)**.

The following Lemma provides the standard interaction estimates.

Lemma 3.1. *There exists a positive C such that for any interaction resulting in the waves σ_1^+ and σ_2^+, the following estimates hold.*

1. *If the interacting waves are σ_1^- of the first family and σ_2^- of the second family,*
$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| = C\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right) .$$
2. *If the interacting waves σ' and σ'' both belong to the first family, we have*
$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| = C|\sigma'\sigma''|(|\sigma'| + |\sigma''|) .$$

3. *If the interacting waves σ' and σ'' both belong to the second family, we have*

$$\left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| = C|\sigma'\sigma''|(|\sigma'| + |\sigma''|)\,.$$

The proof is in [**6**, Lemma 2 and Lemma 3].

Assume now that the ε–approximate solution v^ε is defined up to time $T > 0$. For $i = 1, 2$, $t \in [0, T]$ and $x \in \mathbb{R}$, introduce the quantities

$$\check{\lambda}_i(t,x) := \min\{\lambda_i(v^\varepsilon(t, x-)), \lambda_i(v^\varepsilon(t, x+))\}$$
$$\hat{\lambda}_i(t,x) := \max\{\lambda_i(v^\varepsilon(t, x-)), \lambda_i(v^\varepsilon(t, x+))\}\,.$$

For any $X \in \mathbb{R}$, the generalized i-th characteristic through (T, X) is an absolutely continuous solution $x(t)$ to the differential inclusion

$$\begin{cases} \dot{x} \in \left[\check{\lambda}_i(t,x)\,,\ \hat{\lambda}_i(t,x)\right] \\ x(T) = X\,. \end{cases}$$

The *minimal* backward i-th characteristic through (T, X) is the generalized i-th characteristic such that, for $t \in [0, T]$,

$$y_i(t) := \min\{x(t) \colon x \text{ is a generalized } i\text{-th characteristic through } (T, X)\}\,,$$

where we omit the dependence of $y_i(t)$ from (T, X). It is clear that $y_i(t)$ is well defined, for v^ε piecewise constant, see [**1**, Theorem 2, Chapter 2, § 1].

As a reference about minimal backward characteristics on exact solutions, see [**9**, Paragraph 10.3]. Backward characteristics on wave front tracking solutions were used, for instance, in [**7**, Section 4].

To estimate the norm $\|v^\varepsilon(T)\|_\infty$, for $T > 0$, we follow backward the i–coordinate v_i^ε along the minimal characteristic $y_i(t)$ through (T, X), for all $X \in \mathbb{R}$. Using the Lax inequality (11) and the choice adopted for the speed of rarefaction waves, we can conclude that y_i does not interact with any i–shock with size $\sigma < -\sqrt{\varepsilon}$, it can coincide on a non-trivial time interval with an i–wave with size $\sigma \geq -\sqrt{\varepsilon}$, it can cross a wave of the other family or pass through an interaction point where a rarefaction of its family arises, see Figure 1. The total size of the j-waves, with $j \neq i$, which may potentially interact with $y_i(t)$ after time t is given by the functionals

$$\tilde{Q}_1(t) := \sum_{a : x_a < y_1(t)} \left|\sigma_{2,a}\right| \quad \text{and} \quad \tilde{Q}_2(t) := \sum_{a : x_a > y_2(t)} \left|\sigma_{1,a}\right| \tag{13}$$

where we referred to the form (12) of v^ε. Now we also define, as usual, the *total strength of waves* and the *interaction potential*:

$$V(v^\varepsilon) := \sum_{i,a} \left|\sigma_{i,a}\right|\,, \qquad Q(v^\varepsilon) := \sum_{(\sigma_{i,a}, \sigma_{j,\beta}) \in \mathcal{A}} \left|\sigma_{i,a}\sigma_{j,\beta}\right|\,, \tag{14}$$

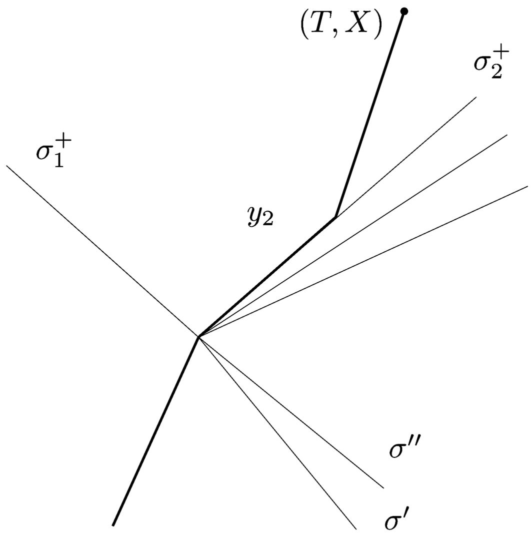

Fig. 1. Two 1-shock σ' and σ'' interact resulting in a 1-shock σ_1^+ and a 2-rarefaction σ_2^+. A 2-characteristic y_2 (thick line) is superimposed to the 2-rarefaction and passes through the interaction point.

where $\mathcal{A}$ is the set of all couples of approaching wave-fronts, see [**5**]. Using the interaction estimates of the Lemma 3.1, we prove the following result.

Proposition 3.1. *Fix a positive M'. Let the ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ be defined up to time $t > 0$. At time t an interaction between two waves σ' and σ'' takes place. If* $\mathrm{TV}(v^\varepsilon(t^-)) < M'$ *and* $\|v^\varepsilon(t^-)\|_\infty$ *is sufficiently small, then v^ε can be defined beyond time t and*

$$\Delta Q(v^\varepsilon(t)) \leq -\frac{|\sigma'\sigma''|}{2} \, .$$

We introduce now the following two functionals:

$$\Upsilon^\varepsilon(t) := V(v^\varepsilon(t)) + K\, Q(v^\varepsilon(t)) \tag{15}$$

$$\Theta_i^\varepsilon(t) := \left(\left|v_i^\varepsilon(t, y_i(t))\right| + \|\bar{v}^\varepsilon\|_\infty\right) e^{\tilde{H}\,\tilde{Q}_i(t) + H\,Q(v^\varepsilon(t))} \tag{16}$$

where $i = 1, 2$, Θ_i^ε depends on y and $\tilde{H}, H$ and K are positive constants precisely defined in the next proposition.

Proposition 3.2. *Fix positive M, M'. Choose an initial datum $\bar{v}^\varepsilon$ such that $\|\bar{v}^\varepsilon\|_\infty < \eta$. Assume that the ε-approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ is defined up to time $t > 0$. If η is sufficiently small,* $\mathrm{TV}(v^\varepsilon(t^-)) < M'$ *and* $\|v^\varepsilon(t-)\|_\infty < M\|\bar{v}^\varepsilon\|$, *then,* *choosing* $K \geq 2C(|\sigma'| + |\sigma''|)$, $\tilde{H} = CM^2 \|\bar{v}^\varepsilon\|_\infty$ *and* $H = 2\tilde{H}(|\sigma'| + |\sigma''|)$, *we have that:*

$$\Delta\Upsilon^\varepsilon(t) \leq 0 \tag{17}$$

$$\Delta\Theta_i^\varepsilon(t) \leq 0 \text{ for } i = 1{,}2. \tag{18}$$

A detailed proof can be found in [4].

Proposition 3.3. *There exist positive* M *and* C_2 *such that, for all* η, ε *sufficiently small, if the* ε*-approximate solution* $v^\varepsilon = v^\varepsilon(t,x)$ *corresponding to the initial datum* $\bar{v}^\varepsilon \in \mathcal{D}(\eta, \bar{K})$ *is defined up to time* T*, then, for all* $t \in [0, T]$,

$$\mathrm{TV}(v^\varepsilon(t)) \leq C_2\bar{K} \qquad \text{and} \qquad \|v^\varepsilon(t)\|_\infty \leq M\|\bar{v}^\varepsilon\|_\infty .$$

Proof. Let $t \in [0, T]$. To bound the L^∞ norm, for any $x \in \mathbb{R}$, first choose $\tilde{H} = CM^2\eta$ and $H = 4CM^3\eta^2$, as in Proposition 3.2. Then, recursively,

$$\begin{aligned}
\left\|v_i^\varepsilon(t)\right\| &\leq \Theta_i^\varepsilon(t) && \text{by (16)}\\
&\leq \Theta_i^\varepsilon(0) && \text{by Proposition 3.2}\\
&\leq 2\eta e^{CM^2\eta\left(\tilde{Q}(0)+4M\eta Q(0)\right)} && \text{by (16).}\\
&\leq 2\eta e^{CM^2\eta(1+M\eta)} && \text{for a suitably large } C\\
&\leq M\eta && \text{for } M = 2e^2 \text{ and } \eta < 1/(CM^2)
\end{aligned}$$

for $i = 1, 2$. Taking the supremum with respect to x, we obtain the desired bound.

Similarly, to bound the total variation, apply recursively the previous results:

$$\begin{aligned}
\mathrm{TV}(v^\varepsilon(t)) &\leq C_1\,\Upsilon^\varepsilon(t) && \text{by (15)}\\
&\leq C_1\Upsilon^\varepsilon(0) && \text{by Proposition 3.2}\\
&\leq C_2\,\mathrm{TV}(\bar{v}^\varepsilon) && \text{by (15)}\\
&\leq C_2\bar{K} && \text{by (9)}
\end{aligned}$$

completing the proof. □

Hence, by the Proposition 3.3, if $\bar{v}^\varepsilon \in \mathcal{D}(\eta, \bar{K})$ and if the approximate solution v^ε can be constructed on some initial interval $[0, T]$, then $v^\varepsilon(t, \cdot) \in \mathcal{D}(M\eta, C_2\bar{K})$ for all $t \in [0, T]$. In order to prove that v^ε can actually be defined for all $t > 0$, it remains to show that the total number of wave fronts and of points of interaction remains finite. The proof is deferred to [**6**].

Hence, we have the following theorem.

Theorem 3.1. *Let* **(L)** *hold. Fix a positive* $\bar K$. *Then, there exist positive* η *and* M *such that for every initial condition* $\bar v \in \mathcal{D}(\eta, \bar K)$ *and for every sufficiently small* $\varepsilon > 0$, *the Cauchy problem* (2) *admits an* ε*–approximate solution* $v^\varepsilon = v^\varepsilon(t,x)$ *such that*

$$\|v^\varepsilon(t)\|_\infty \le M \|\bar v\|_\infty . \tag{19}$$

Under condition **(GL)**, we also have the following decay estimate.

Theorem 3.2. *Let* **(GL)** *hold. Fix a positive* $\bar K$. *Then, there exist positive* η *and* $\mathcal{M}$ *such that for every initial condition* $\bar v \in \mathcal{D}(\eta, \bar K)$ *and for every sufficiently small* $\varepsilon > 0$, *the* ε*–approximate solution* $v^\varepsilon = v^\varepsilon(t,x)$ *to the Cauchy problem* (2) *constructed in Theorem* 3.1 *satisfies for all* $t > 0$, *for all* $a, b \in \mathbb{R}$ *and for* $i = 1, 2$:

$$\mathrm{TV}^+\left(v_i^\varepsilon(t);[a,b]\right) \le \frac{b-a}{c\,t} + \mathcal{M}\left(\|\bar v\|_\infty \mathrm{TV}\Big(\bar v;[a-\hat\lambda t, b+\hat\lambda t]\Big) + \varepsilon\right) \tag{20}$$

with c *as in* (4) *and* $\hat\lambda$ *as in* (8).

3.3 - *Existence of solutions*

For the sake of completeness, we pass the ε–approximate solutions to the limit $\varepsilon \to 0$. This standard application of Helly compactness Theorem yields a slight extension of the wave front tracking construction exhibited in [**6**]. Indeed, the mere existence of solutions to (2) is here obtained under the assumptions that the total variation of the initial datum be bounded.

Theorem 3.3. *Let* **(L)** *hold. Fix a positive* $\bar K$. *Then, there exist positive* η, M *such that for all* $\bar u \in \mathcal{D}(\eta, \bar K)$, *the Cauchy problem* (2) *admits a weak entropy solution, which is the limit of the wave front tracking approximate solutions constructed above and satisfying*

$$\|v(t)\|_\infty \le M \|\bar v\|_\infty .$$

Moreover, if also **(GL)** *holds, then there exists a positive* $\mathcal{M}$ *such that for all* $t > 0$, *for all* $a, b \in \mathbb{R}$ *and for* $i = 1,2$,

$$\mathrm{TV}^+(v_i(t);[a,b]) \le \frac{b-a}{c\,t} + \mathcal{M}\|\bar v\|_\infty \mathrm{TV}\Big(\bar v;[a-\hat\lambda t, b+\hat\lambda t]\Big)$$

with c *as in* (4) *and* $\hat\lambda$ *as in* (8).

Thanks to the estimates proved above, the proof is standard and, hence, omitted.

4 - Construction of a Solution with small L^∞ norm

We now prove Theorem 1.1 in the case of initial data satisfying the stronger conditions

$$\bar{v} \in C^1(\mathbb{R}; B(0,\eta)) \quad \text{with} \quad \left\|\frac{d\bar{v}}{dx}\right\|_\infty \le \mathcal{L}\,, \tag{21}$$

see [**10**, i), ii) and iii) in Section 5].

We are going to use an inductive method. Define, for $m = 0, 1, 2, \ldots$ and for every $L > 0$, the m-trapezoid by

$$\triangle_m := \left\{ (t,x) \in [0,+\infty[\, \times \mathbb{R} \colon \begin{array}{ccl} t & \in & [t_m, t_m + \varDelta t_m] \text{ and} \\ x & \in & [\,-2^m L + \hat{\lambda}(t - t_m), 2^m L - \hat{\lambda}(t-t_m)] \end{array} \right\} \tag{22}$$

see Figure 2, where:

$$t_m = (2^m - 1)L/2\hat{\lambda} \quad \text{and} \quad \varDelta t_m = 2^{m-1} L/\hat{\lambda}\,. \tag{23}$$

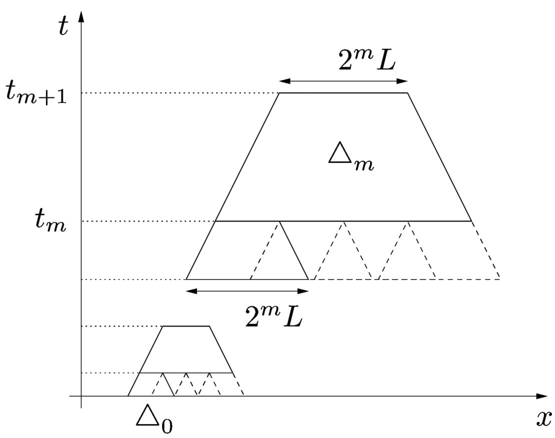

Fig. 2. Construction of the trapezoids.

The upper side of $\triangle_m$ measures $2^m L$ and the lower one $2^{m+1}L$. The upper bases of 4 trapezoids $\triangle_{m-1}$ cover the lower basis of $\triangle_m$. We denote by $\triangle_m(x)$ the translation of the m–trapezoid: $\triangle_m(x) := (0, x) + \triangle_m$. Correspondingly, we introduce the domains

$$\mathcal{D}_m\left(\delta, 20\,\frac{\hat{\lambda}}{c}\right) := \left\{ v \in L^1_{loc}(\mathbb{R}; B(0,\delta)) \colon \mathrm{TV}(v; 2^{m+1}L) \le 20\,\frac{\hat{\lambda}}{c} \right\}. \tag{24}$$

4.1 - *Construction in the* $0-$*trapezoid*

In this paragraph we show that we are able to construct a solution in $\triangle_0(x)$, for all $x \in \mathbb{R}$. In fact, since the initial datum satisfies (21), we can always choose $L > 0$ such that

$$\mathrm{TV}(\bar{v}, 2L) \le 20\hat{\lambda}/c\,. \tag{25}$$

Then, with reference to (24), we prove the following result.

Proposition 4.1. *Let* **(GL)** *and* (21) *hold. Then, there exist a sufficiently small* $\eta > 0$ *and positive* $M, \mathcal{M}$ *such that for every initial condition* $\bar{v} \in \mathcal{D}_0(\eta, 20\hat{\lambda}/c)$, *the Cauchy problem* (2) *admits a weak entropy solution* $v = v(t,x)$ *defined for all* $t \in [0, L/2\hat{\lambda}]$ *and*

$$\begin{aligned} &\|v(t)\|_\infty \le M\, \|\bar{v}\|_\infty \\ &\mathrm{TV}^+\Big(v_i(t); 2(L-\hat{\lambda}t)\Big) \le \frac{2}{c}\,\frac{L-\hat{\lambda}t}{t} + \mathcal{M}\, \|\bar{v}\|_\infty\, \mathrm{TV}(\bar{v}; 2L)\,. \end{aligned}$$

The proof follows directly from Theorem 3.3 and the finite speed of propagation.

4.2 - *Construction in the* $m-$*trapezoid*

Now we prove that, if a solution v to (2) satisfies suitable conditions at time $t = t_m$, then this solution can be extended on all the interval $[t_m, t_{m+1}]$. We also provide suitable estimates for later use.

Proposition 4.2. *Let* **(GL)** *hold. Then, there exists a sufficiently small* $\eta > 0$ *and positive* $M, \mathcal{M}$ *such that if* $v(t_m) \in \mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$, *then the problem* (1) *with datum* $v(t_m)$ *admits a weak entropy solution* $v = v(t,x)$ *defined for* $t \in [t_m, t_{m+1}]$ *satisfying*

$$\|v(t)\|_\infty \le M\, \|v(t_m)\|_\infty \tag{26}$$

$$\mathrm{TV}^+\Big(v_i(t); 2(2^m L-\hat{\lambda}t)\Big) \le \frac{2}{c}\,\frac{2^m L-\hat{\lambda}t}{t-t_m} + \mathcal{M}\, \|v(t_m)\|_\infty\, \mathrm{TV}(\bar{v}; 2^{m+1}L). \tag{27}$$

Above, $\mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$ is defined in (24). The proof is entirely similar to that of Proposition 4.1.

4.3 - *Existence of a global solution*

In this paragraph we assume the following a priory bound:

(A) Whenever it is possible to define up to time t_m a solution v to (2) with an initial datum satisfying (21), then there exists $K > 0$ such that, for all $m \in \mathbb{N}$, $\|v(t_m)\|_\infty \leq K\sqrt{\eta}$, where η is an upper bound for $\|\bar{v}\|_\infty$.

It is motivated by the recursive proof of Theorem 1.1 and by the following Proposition.

Proposition 4.3. *Suppose there exists up to time t_m a weak entropy solution $v = v(t,x)$ to* (2) *with an initial datum satisfying* (21)*. Let* **(GL)**, (25) *and* **(A)** *hold. Then, for all sufficiently small $\eta > 0$, if $\|\bar{v}\|_\infty \leq \eta$, for all $m \in \mathbb{N}$ we have the estimate*

$$\mathrm{TV}\big(v(t_m); 2^{m+1}L\big) \leq 20\frac{\hat{\lambda}}{c} .$$

Now, we are able to prove the Theorem 1.1.

Assume first that the initial data satisfies (21). By an application of Proposition 4.1, we are able to construct a solution for all $t \in [0, L/2\hat{\lambda}]$. Now, assume that a solution exists up to time t_m, with $m \geq 1$. Then, by **(A)**, we may apply Proposition 4.3 to obtain the TV bound at time t_m. Therefore, again thanks to **(A)**, we apply Proposition 4.2 to extend the solution up to time t_{m+1}. The proof is thus obtained inductively.

Consider now a general initial datum satisfying only (3). As in [**10**, Section 5], we approximate the initial datum $\bar{v}$ by a sequence of mollified data $\bar{v}_n$ such that each $\bar{v}_n$ satisfies (21). So, we are able to construct a sequence of solutions v_n to (1) related to the initial data $\bar{v}_n$. Then by [**9**, Theorem 1.7.3] we can select a subsequence that converges to a limit v, which is a weak entropy solution to (2).

5 - The L^∞ estimate

The next step consists in proving that the a priori bound **(A)** is in fact a consequence of the other assumptions in Theorem 1.1 when the initial datum satisfies (21).

Proposition 5.1. *There exists a positive K such that for all initial datum $\bar{v}$ in* (2), *satisfying* (3) *and* (21)*, the following estimate holds, for all $m \in \mathbb{N}$, on the* *solution* $v = v(t,x)$ *to* (2):

$$\|v(t_m)\|_\infty \leq K\sqrt{\eta}\,,$$

where t_m *is defined in* (23).

Proof. For $m = 0$ the thesis holds, provided $K > \sqrt{\eta}$. Now, by induction, suppose that the theorem holds true up to $m-1$.

The lower basis of $\triangle_m$ is covered exactly by the upper basis of 4 $(m-1)$–trapezoids. Denote by T_{m-1} the union of these trapezoids. Then, divide T_{m-1} by horizontal segments $b^0_{m-1}, \dots, b^N_{m-1}$ into N sub-trapezoids, say $T^1_{m-1}, \dots, T^N_{m-1}$.

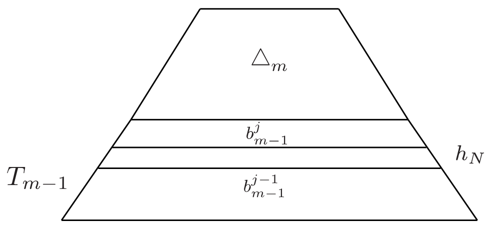

Fig. 3. The union T_{m-1} of the trapezoids $\triangle_{m-1}$.

At least one of these trapezoids, call it T^n_{m-1}, is such that

$$\begin{aligned}(28)\qquad & Q\Big(v(t_{m-1} + (n-1)h_N)_{|b^{n-1}_{m-1}}\Big) - Q\Big(v(t_{m-1} + nh_N)_{|b^n_{m-1}}\Big) \\ & \leq \frac{1}{N}\,\|v(t_{m-1})\|_\infty\,\frac{20\hat\lambda}{c}\end{aligned}$$

by Proposition 4.3. Now, fix (t,x) and (t,y) on b^n_{m-1} with $x<y$. Then, using together the usual decay estimate [5, Theorem 10.3] or [7, Theorem 1] on the region T^n_{m-1}, together with (28), we have:

$$(29)\qquad v_i(t,y) \leq v_i(t,x) + \frac{N}{L}\,\frac{y-x}{2^{m-2}}\,\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\,\frac{20\hat\lambda}{c}\,\|v(t_{m-1})\|_\infty\,.$$

Integrating (29) separately in y and in x, we obtain

$$(30)\qquad |v_i(t,x)| \leq \frac{1}{l}\left|\int_{y-l}^{y} v_i(t,y)\,dx\right| + \frac{N}{L}\,\frac{l}{2^{m-1}}\,\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\,\frac{20\hat\lambda}{c}\,\|v(t_{m-1})\|_\infty\,.$$

At this point we consider three different cases, depending on which coefficients in (5) vanish.

1. $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$. Hence by Proposition 6.1,

$$\left| \int_l v_i(t,x)dx \right| \leq C'\eta(l + C''t) \qquad \text{for } i = 1, 2\,. \tag{31}$$

(Note that it is this case that covers the situation considered in [**10**]).

2. $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$. Then, using Proposition 6.2

$$\left| \int_l v_i(t,x)dx \right| \leq C'\eta(l + C''t) + C\|v(t)\|_\infty^3 t \qquad \text{for } i = 1, 2\,.$$

3. $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ $\left(\text{or } \dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0 \text{ and } \dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0\right)$. Hence, by an application of Proposition 6.1 and Proposition 6.2:

$$\left| \int_l v_i(t,x)dx \right| \leq C'\eta(l + C''t) + C\|v(t)\|_\infty^3 t \qquad \text{for } i = 1, 2\,.$$

Using the (worst) estimate of cases 2. and 3., we have

$$|v_i(t,x)| \leq C'\eta\left(1 + C''\frac{t}{l}\right) + C\|v(t)\|_\infty^3 \frac{t}{l} + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\,\|v(t_{m-1})\|_\infty\,.$$

Setting $l/t = \sqrt{\eta + \|v(t)\|_\infty^3}$, using the fact that $t \leq t_m$ and, by Proposition 4.2 and the inductive assumption, $\|v(t)\|_\infty \leq MK\sqrt{\eta}$, we have

$$\begin{aligned}
\|v(t)\|_\infty &\leq C\left(\eta + \sqrt{\eta} + \|v(t)\|_\infty^{3/2}\right) + C\sqrt{\eta + \|v(t)\|_\infty^3}\,t + \frac{C}{N}\,\|v(t_{m-1})\|_\infty \\
&\leq C\sqrt{\eta} + \frac{CN}{c}\sqrt{\eta} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\,\|v(t_{m-1})\|_\infty \\
&\leq CN\sqrt{\eta} + \frac{C}{N}\|v(t_{m-1})\|_\infty\,.
\end{aligned}$$

Choosing $N = 4CM$ and $K = 4MNC$, by the inductive hypothesis, we get $\|v(t)\|_\infty \leq \dfrac{K}{2M}\sqrt{\eta}$. So, we can conclude:

$$\|v(t_m)\|_\infty \leq 2M\|v(t)\|_\infty \leq K\sqrt{\eta}$$

completing the proof. Obviously, the proof is exactly the same if, instead of $\triangle_m$, we consider a generic trapezoid $\triangle_m(x)$ for some $x \in \mathbb{R}$. □

Remark that in the previous proof, case 1 covers the situation treated in [**10**]. Indeed, in (31) the optimal choice for l/t is $l/t = \sqrt{\eta}$, exactly as in [**10**].

6 - The integral estimate

In the case that $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ (respectively $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$), there exists an invariant region for the variable u_1 (respectively u_2), see [**4**, Lemma 6.1]. We use this fact to prove the following result.

Proposition 6.1. *Let $v = v(t,x)$ be the solution to* (2) *constructed in the previous sections, with an initial data satisfying* (3) *and* (21). *If* $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ *and* $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, *then, for all segment l and for all $\bar{t} \geq 0$:*

$$\left| \int_l v_i(\bar{t}, x)dx \right| \leq C'\eta\big(l + C''\bar{t}\big). \tag{32}$$

The set of systems considered in [**10**] satisfies the hypothesis of the previous proposition, so that they always have an invariant zone. On the contrary, we consider also the other cases.

Proposition 6.2. *Let $v = v(t,x)$ be the solution to* (2) *constructed in the previous sections, with an initial data satisfying* (3) *and* (21). *If* $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ *and* $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$, *then, for all segment l and for all $\bar{t} \geq 0$:*

$$\left| \int_l v_i(\bar{t}, x)dx \right| \leq C'\eta\big(l + C''\bar{t}\big) + C\|v(\bar{t})\|_\infty^3 \bar{t}. \tag{33}$$

For a detailed proof of all these integral estimates see [**4**, Section 6].

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