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Regular form perturbations

Abstract. We present abstract results about the space regularity of solutions to elliptic and parabolic equations on L^p-spaces which are associated to perturbed sectorial forms $\mathfrak{a} + \mathfrak{b}$. As applications of our results, we introduce deGiorgi-Nash forms, which define quite general second order elliptic operators in divergence form. We give a wide class of examples of perturbations of such forms, such that the solutions of elliptic and parabolic equations associated to the perturbed operator are continuous. Furthermore, we prove that given any open subset Ω of $\mathbb{R}^N$ and any deGiorgi-Nash form $\mathfrak{a}$ with principal coefficients in $W^{1,\infty}$, there exists a potential $V \in L^\infty_{\text{loc}}$ such that the operator associated to $\mathfrak{a} + V$ generates a strongly continuous semigroup on $C_0(\Omega)$.

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1 - Preliminaries

1.1 - *Introduction and summary*

Form methods provide excellent means to define realizations of second order differential operators on L^2-spaces and obtain generator properties of such operators. Using extrapolation techniques, it is also possible to extend the semigroup associated to such operators to other L^p-spaces, in particular to the space L^∞. However, often – in particular in connection with stochastic processes – one is in-

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terested in semigroups on spaces of continuous functions, such as C_b or C_0. In particular, it is interesting to perturb such "regular forms", i.e. forms where one has a semigroup on a "regularity space" as C_b or C_0, and obtain a "regular form" again.

This problem is related to the Kato class, which was introduced by Aizenman and Simon in [**2**] in connection with Schrödinger operators, i.e. perturbations of the Laplacian by a potential $V \in L^1_{\mathrm{loc}}$. There, the (local) Kato class is defined as the set of all $V \in L^1_{\mathrm{loc}}$ satisfying a certain integrability condition (which itself goes back to Kato [**12**]). It is then proved that V belongs to the local Kato class if and only if $R(\lambda, \Delta)Vg$ is a continuous function for any bounded and measurable g (see [**2** Theorem 1.5]). Thus the Kato class is related to the continuity of solutions to elliptic problems.

Later, Stollman and Voigt replaced the Laplacian by a general regular, symmetric Dirichlet form and considered also measures instead of locally integrable functions as perturbations, see [**19, 21**]. Consequences for the semigroups generated by such perturbed operators were investigated in [**17, 7**] using a probabilistic approach. We also mention the connection of the Kato class with Miyadera perturbation [**20, 15**].

In this paper, we will replace the space of continuous functions by some abstract regularity space X. This allows for greater flexibility in the regularity looked for, e.g. when working on some domain $\Omega \subset \mathbb{R}^N$, one can require regularity also on the boundary by choosing $X = C(\overline{\Omega})$. Also, we consider general sub-Markovian forms $\mathfrak{a}$, dropping the requirement that $\mathfrak{a}$ be symmetric. We then define the abstract Kato class (associated with X and λ) as the set of all $\varphi \in D(\mathfrak{a})'$ such that $R(\lambda, \mathcal{A})\varphi \in X$. Here, $\mathcal{A} : D(\mathfrak{a}) \to D(\mathfrak{a})'$ is the operator associated to the form $\mathfrak{a}$, see Section 1.2.

We note that we do *not* seek to describe the elements of the Kato class by some integrability condition. We rather assume that already sufficiently many elements of the abstract Kato class are known.

In Section 2.1, we introduce local versions of the spaces $D(\mathfrak{a})$ and $D(\mathfrak{a})'$ and the operator $\mathcal{A}$. This is essential to define a local version of the Kato class in Section 2.2. There, we will also prove several properties of the Kato class and the local Kato class and in particular address the independence of the Kato class from the parameter λ. This does not always hold, see Section 3.1. Afterwards, we introduce Kato perturbations, which are the appropriate generalization of potentials and measures belonging to the classical Kato class. However, even in the classical situation, there are Kato perturbations which are not associated to a measure.

In Section 2.3, we consider the space X_0 of regular functions vanishing at infinity. As belonging to X_0 is in general not a local property, there is no local Kato class for X_0. To obtain semigroups on X_0, we present a theorem in the spirit of Lyapunov functions, cf. [**6** Theorem 4.3.2]. In order to prove the theorem, one needs a certain approximation result, which is equivalent to some abstract sort of Dirichlet boundary condition.

The last part of this paper is devoted to applications. We introduce deGiorgi-Nash forms, for which many elements of the Kato class for $X = C(\Omega)$ are known from the deGiorgi-Nash Theorem. In Section 3.3, we prove that for any deGiorgi-Nash form and any bounded $\Omega \subset \mathbb{R}^N$, there exists a potential $V \in L^\infty_{\rm loc}$ such that the semigroup associated to the perturbed form on $L^\infty(\Omega)$ leaves the space $C_0(\Omega)$ invariant.

1.2 - *Notation and setting*

Throughout this paper we will always work on the Hilbert space $L^2(M, dm)$, where M is a locally compact topological space which is countable at infinity and m is a positive Radon measure on M. We will often write L^p for $L^p(M, dm)$, $\|\cdot\|_p$ for the canonical norm in L^p and $\langle\cdot,\cdot\rangle_{p,q}$ for the canonical duality between L^p and L^q, where q is the conjugate index to p. For $p = 2$ we just write $\|\cdot\|$ for the canonical L^2-norm and $(\cdot,\cdot)$ for the scalar product in L^2. On L^2, we will consider densely defined sectorial forms. We briefly recall some notions and facts about sectorial forms. For more details we refer to [**11, 14**].

A *densely defined sesquilinear form* on L^2 is a mapping $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ which is linear in the first component and antilinear in the second; $D(\mathfrak{a})$ is a dense subspace of L^2 and is called *the domain* of $\mathfrak{a}$. The form $\mathfrak{a}$ is called *sectorial*, if its numerical range $\Theta(\mathfrak{a}) := \{\mathfrak{a}[u,u] : u \in D(\mathfrak{a}), \|u\| = 1\}$ is contained in some right open sector

$$\Sigma_{\gamma,\theta} := \{z \in \mathbb{C} \setminus \{\gamma\} \,:\, |\arg(z-\gamma)| \le \theta\}$$

for some $\gamma \in \mathbb{R}$ and $\theta \in \left[0, \frac{\pi}{2}\right)$. In this case, $(f,g)_{\mathfrak{a}} := (1+\gamma)(f,g) + \operatorname{Re}\mathfrak{a}[f,g]$ defines a scalar product on $D(\mathfrak{a})$. To simplify notation, we shall assume that $\gamma = 0$.

The norm induced by $(\cdot,\cdot)_{\mathfrak{a}}$ will be denoted by $\|\cdot\|_{\mathfrak{a}}$. Throughout this paper $D(\mathfrak{a})$ will be endowed with this norm. If $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is complete, the form $\mathfrak{a}$ is called *closed*.

We call the form $\mathfrak{a}$ *local*, if

(i) We have $\mathfrak{a}[u,v] = 0$, whenever u and v have disjoint support.

(ii) For every open subset ω of M, the space

$$D(a,\omega) := \{u \in D(\mathfrak{a}) \,:\, u = 0 \text{ a.e. on } M \setminus \omega\}$$

is dense in $L^2(\omega, dm)$.

We recall that the *support* $\operatorname{supp} f$ of a measurable function f is defined as G^c, where G is the union of all open sets ω such that $f = 0$ a.e. on ω.

We also consider the space $D(\mathfrak{a})'$ of bounded, antilinear functionals on $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$. However, we do not identify this space with $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ but we use

L^2 as a pivot space: $D(\mathfrak{a}) \hookrightarrow L^2 \hookrightarrow D(\mathfrak{a})'$. That is, we identify $f \in L^2$ with the bounded antilinear functional $\varphi_f : D(\mathfrak{a}) \ni g \mapsto (f, g)$. We denote the duality pairing between $D(\mathfrak{a})'$ and $D(\mathfrak{a})$ by $\langle \cdot, \cdot \rangle$.

Given a densely defined, closed, sectorial form $\mathfrak{a}$, we may associate an operator $\mathcal{A}$ on $D(\mathfrak{a})'$ with the form $\mathfrak{a}$ by defining

$$D(\mathcal{A}) := D(\mathfrak{a}), \qquad -\langle \mathcal{A}u, v \rangle := \mathfrak{a}[u, v] .$$

It is well known (cf. [**14**, Theorems 1.55 and 1.52]), that $\mathcal{A}$ defined in this way generates a holomorphic, strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $D(\mathfrak{a})'$. Furthermore, $\mathcal{T}$ leaves L^2 invariant and the restricted semigroup $T(t) := \mathcal{T}(t)|_{L^2}$ is also holomorphic and strongly continuous. The generator A_2 of T is the part of $\mathcal{A}$ in L^2.

A *sub-Markovian form* is a densely defined, closed, sectorial form $\mathfrak{a}$ on L^2, such that the associated semigroup T is real, positive and L^∞-contractive. The Beurling-Deny Criteria (cf. [**14**, Section 2.2]) give a useful characterization of sub-Markovian forms.

If $\mathfrak{a}$ is a sub-Markovian form with associated semigroup T on L^2, then, using the L^∞-contractivity of T, it is easy to see that $\|T(t)^* f\|_1 \leq \|f\|_1$ for all $f \in L^2 \cap L^1$ and $t \geq 0$. Hence $T(t)^*$ may be extended to a contraction operator $S(t)$ on L^1. Denoting the adjoint of $S(t)$ by $T_\infty(t)$, we obtain a semigroup on L^∞ satisfying $T_\infty(t)f = T(t)f$ for all $f \in L^2 \cap L^\infty$ and $t \geq 0$.

It is proved in [**14**, p. 56 ff.] that we obtain a consistent family $(T_p)_{2\leq p\leq\infty}$ of semigroups on L^p, i.e. for $f \in L^p \cap L^q$ we have $T_p(t)f = T_q(t)f$ for all $t \geq 0$. Here, $T_2 := T$. Furthermore, T_p is strongly continuous for $2 \leq p < \infty$ and T_∞ is an adjoint semigroup. In particular, T_∞ is $\sigma(L^\infty, L^1)$-continuous.

In what follows, we will denote by A_p the generator of T_p. This is the strong generator for $2 \leq p < \infty$ and the weak*-generator for $p = \infty$. It is known that the holomorphy of T_2 is inherited by the semigroups T_p for $2 \leq p < \infty$. For a proof of these facts and other properties of consistent families of semigroups we refer to [**3**, Chapter 7.2].

Since M is locally compact and countable at infinity, there exists a sequence $(\omega_n)_{n\geq 0}$ of open sets such that $\omega_n \Subset \omega_{n+1} \Subset M$ for any $n \geq 0$ (where $A \Subset B$ means $\overline{A}$ is compact and contained in B) and $\bigcup\limits_n \omega_n = M$. We fix – once and for all – such a sequence. It is easy to see that $D(\mathfrak{a}, \omega_n)$ as defined above is a closed subspace of $D(\mathfrak{a})$. Thus, if $\mathfrak{a}$ is local, then $(\mathfrak{a}_n, D(\mathfrak{a}, \omega_n))$ defined by $\mathfrak{a}_n[u, v] := \mathfrak{a}[u, v]$ for $u, v \in D(\mathfrak{a}, \omega_n)$, is a densely defined, closed, sectorial form on $L^2(\omega_n)$. We will denote by $\mathcal{A}_n : D(\mathfrak{a}, \omega_n) \to D(\mathfrak{a}, \omega_n)'$ the associated operator. Using the Beurling-Deny criteria, we see that $\mathfrak{a}_n$ is a sub-Markovian form if $\mathfrak{a}$ is. It is also possible to consider $\mathfrak{a}_n$ as a non-densely defined form on $L^2(M)$. For this we refer to [**14**, Chapter 2.6].

2 - Abstract results

2.1 - *Local forms*

In this section we are given a local, sub-Markovian form $\mathfrak{a}$ on $L^2(M, dm)$. We introduce local versions of the spaces $D(\mathfrak{a})$ and $D(\mathfrak{a})'$ and extend the operator $\mathcal{A}$ to an operator $\tilde{\mathcal{A}}$ defined on a local version of $D(\mathfrak{a})$ taking values in a local version of $D(\mathfrak{a})'$. Then we investigate the connection between the semigroup generators A_p and the extended operator $\tilde{\mathcal{A}}$.

As a local version of $D(\mathfrak{a})$, we will use the space

$$D(\mathfrak{a})_{\text{loc}} := \left\{ u \in L^2_{\text{loc}}(M) \,:\, \forall\, n \geq 0 \;\exists\, u_n \in D(\mathfrak{a}) \text{ s. t. } u = u_n \text{ a.e. on } \omega_n \right\}.$$

To define a local version of $D(\mathfrak{a})'$, we use the spaces $D(\mathfrak{a}, \omega_n)$ introduced in the previous section and then proceed similar to the definition of *distributions*. By $D(\mathfrak{a})_c$ we denote the vector space of all elements of $D(\mathfrak{a})$ having compact support in M. It is obviously $D(\mathfrak{a})_c = \bigcup D(\mathfrak{a}, \omega_n)$. Now we put

$$\begin{aligned} D(\mathfrak{a})'_{\text{loc}} := \{ \varphi : D(\mathfrak{a})_c \to \mathbb{C} \text{ antilinear } : \; \forall\, n \geq 0 \;\exists\, C_n \text{ such that} \\ |\varphi(u)| \leq C_n \cdot \|u\|_{\mathfrak{a}} \;\forall\, u \in D(\mathfrak{a}, \omega_n) \;\}. \end{aligned}$$

We note that if $\varphi \in D(\mathfrak{a})'_{\text{loc}}$, then $\varphi \in D(a, \omega_n)'$ for every $n \in \mathbb{N}$. Even more is true. If we endow $D(\mathfrak{a})_c$ with the inductive limit topology induced by the sequence $(D(\mathfrak{a}, \omega_n))_{n \in \mathbb{N}}$, then $D(\mathfrak{a})'_{\text{loc}}$ is exactly the dual space of $D(\mathfrak{a})_c$. However, we will not need this fact.

The reader should keep in mind that $D(\mathfrak{a})'_{\text{loc}}$ is a local version of $D(\mathfrak{a})'$ and not the dual of $D(\mathfrak{a})_{\text{loc}}$ — hence one should think of $[D(\mathfrak{a})']_{\text{loc}}$ rather than $[D(\mathfrak{a})_{\text{loc}}]'$.

Now we extend the operator $\mathcal{A}$ to an operator $\tilde{\mathcal{A}}$ defined on $D(\mathfrak{a})_{\text{loc}}$ and taking values in $D(\mathfrak{a})'_{\text{loc}}$.

Lemma 2.1. *Let $\mathfrak{a}$ be a local form on $L^2(M)$. Then the operator $\mathcal{A}$ has a unique extension to an operator $\tilde{\mathcal{A}}$ from $D(\mathfrak{a})_{\text{loc}}$ to $D(\mathfrak{a})'_{\text{loc}}$ satisfying the following condition:*

If $u \in D(\mathfrak{a})_{\text{loc}}$ and $u_n \in D(\mathfrak{a})$ satisfies $u = u_n$ a.e. on ω_n for some $n \in \mathbb{N}_0$, then

$$\langle \tilde{\mathcal{A}} u, v \rangle = \langle \mathcal{A} u_n, v \rangle, \tag{1}$$

for all $v \in D(\mathfrak{a})$ with $\operatorname{supp} v \Subset \omega_n$.

Proof. Let $u \in D(\mathfrak{a})_{\text{loc}}$. We have to give meaning to $\langle \tilde{\mathcal{A}} u, v \rangle$ for all $v \in D(\mathfrak{a})_c$. So let $\omega \Subset M$ and $v \in D(\mathfrak{a}, \omega)$ be given. There exists $n \geq 0$ such that $\omega \Subset \omega_n$. Moreover, since $u \in D(\mathfrak{a})_{\text{loc}}$ there exists $u_n \in D(\mathfrak{a})$ such that $u = u_n$ a.e. on ω_n. Define $\tilde{\mathcal{A}} u$ by equation (1). We only need to show that this is well defined. So suppose that $\omega \Subset \omega_n$

and $\omega \Subset \omega_m$ for some $n, m \in \mathbb{N}_0$. Further assume that u_n, u_m are two elements of $D(\mathfrak{a})$ coinciding a.e. with u on ω_n and ω_m respectively. We obtain

$$\langle \mathcal{A}u_n, v\rangle - \langle \mathcal{A}u_m, v\rangle = \mathfrak{a}[u_n - u_m, v] = 0$$

by locality, since $u_n - u_m$ vanishes on $\omega_n \cap \omega_m$ and hence its support is disjoint from $\operatorname{supp} v \subset \overline{\omega} \subset \omega_n \cap \omega_m$. □

Of course we expect some relation between the operator $\tilde{\mathcal{A}}$ and the operators A_p. We start with the following observation:

Proposition 2.1. *Let $2 \le p \le \infty$ and B_p be the part of A_2 in $X_p := L^2 \cap L^p$. Then, for $2 \le p < \infty$, A_p is the closure of B_p and A_∞ is the weak*-closure of B_∞. Furthermore, for $u \in D(A_\infty)$ there exists a sequence $u_n \in D(B_\infty)$ such that $u_n \rightharpoonup^* u$ and $B_\infty u_n \rightharpoonup^* A_\infty u$.*

Proof. Let $u \in D(B_p)$, i.e. $u \in D(A_2) \cap L^p$ and $A_2 u \in L^p$. By consistency we have

$$p\text{-}\int_0^t T_p(s)B_p u\, ds = 2\text{-}\int_0^t T_2(s)A_2 u\, ds = T_2(t)u - u = T_p(t)u - u\,, \tag{2}$$

where $p\text{-}\int$ denotes the Bochner integral in L^p for $2 \le p < \infty$ and the weak*-integral for $p = \infty$. It follows from [4, Proposition 3.1.9] ([**10**, Proposition 1.2.2] for the weak*-case) that $u \in D(A_p)$ and $A_p u = B_p u$.

Let us prove that A_p is in fact the closure of B_p. First consider the case $2 \le p < \infty$. By consistency, T_p and T_2 leave the Banach space X_p invariant. The restricted semigroup is strongly continuous and has generator B_p, which follows from a computation as in (2). In particular, $D(B_p)$ is dense in X_p and thus dense in L^p. Using the holomorphy of T_2 and consistency, we see that $D(B_p)$ is invariant under T_p. It is well known (cf. [8, Prop. II.1.7]), that this implies that $D(B_p)$ is a core for A_p.

For $p = \infty$ we choose a different approach. Given $u \in D(A_\infty)$, we put $v_n = \mathbb{1}_{\omega_n}(\lambda - A_\infty)u$. Then $v_n \in L^2 \cap L^\infty$, whence $u_n := R(\lambda, A_\infty)v_n \in D(B_\infty)$. Since $v_n \rightharpoonup^* (\lambda - A_\infty)u$ and since $R(\lambda, A_\infty)$ is weak*-continuous as an adjoint operator, we have $u_n \rightharpoonup^* u$. Furthermore

$$\begin{aligned} A_\infty u_n &= A_\infty R(\lambda, A_\infty)v_n \\ &= \lambda R(\lambda, A_\infty)v_n - v_n \\ &\rightharpoonup^* \lambda R(\lambda, A_\infty)(\lambda - A_\infty)u - (\lambda - A_\infty)u = A_\infty u\,. \end{aligned}$$

This proves the claim. □

Remark 2.1. *If we assume that not only T_2 but also the adjoint semigroup T_2^* is L^∞-contractive, then we obtain consistent semigroups T_p for $1 \le p \le \infty$, cf.* [**14**, p. 57]. *In this case, Proposition* 2.1 *also holds for* $1 \le p \le \infty$.

It follows from Proposition 2.1 that if M has finite measure so that $L^p \subset L^2$ for $p \ge 2$, then A_p is the part of A_2 in L^p. In particular, $\mathcal{A}$ is an extension of A_p. If $m(M) = \infty$, then L^p is not a subset of L^2 and hence we cannot expect $\mathcal{A}$ to be an extension of A_p. However, we may ask whether $\tilde{\mathcal{A}}$ is an extension of A_p, i.e. $D(A_p) \subset D(\mathfrak{a})_{\mathrm{loc}}$ and

$$\langle \tilde{\mathcal{A}} u, v\rangle = \int_M A_p u \cdot v \, dm\,,$$

for all $v \in D(\mathfrak{a})_c \cap L^q$, where q is the conjugate index to p. Theorem 2.1 shows that this is indeed the case under a somewhat technical assumption which can be verified in many examples.

Definition 2.1. *Let $\mathfrak{a}$ be a closed sectorial form. We say that $\mathfrak{a}$ has* rich domain *if there exist constants $(C_n)_{n\in\mathbb{N}}$ such that for every $u \in D(\mathfrak{a})$ and $n \in \mathbb{N}$ there exists $v \in D(\mathfrak{a})$ with the following properties:*

1. $v \in D(\mathfrak{a}, \omega_n)$ *and* $u = v$ *a.e. on* ω_{n-1};
2. $\|v\|_{L^2(\omega_n)} \le C_n \|u\|_{L^2(\omega_n)}$;
3. $\|\mathcal{A} v\|_{D(\mathfrak{a},\omega_n)'} \le C_n\big(\|u\|_{L^2(\omega_n)} + \|\mathcal{A} u\|_{D(\mathfrak{a},\omega_n)'}\big)$.

In the proof of the following theorem and also in what follows, we will treat the cases of norm convergence and weak*-convergence together. Given $f_n, f \in L^p$ we will write $p\text{-}\lim f_n = f$, which is to be understood as "f is the norm limit of f_n" for $p<\infty$, whereas for $p = \infty$ it stands for "f is the weak*-limit of f_n".

Theorem 2.1. *Let $\mathfrak{a}$ be a local sub-Markovian form with rich domain. Then $\tilde{\mathcal{A}}$ is an extension of A_p for any $2 \le p \le \infty$.*

Proof. Let $u \in D(A_p)$. By Proposition 2.1, there exists a sequence $u_n \in D(A_2|_{L^2\cap L^p}) \subset D(\mathfrak{a})$ such that $p\text{-}\lim u_n = u$ and $p\text{-}\lim A_p u_n = A_p u$. Furthermore, we have $A_p u_n \equiv \mathcal{A} u_n$. Note that the sequences u_n and $A_p u_n$ are bounded in L^p. Now fix $k \in \mathbb{N}$. Since $\mathfrak{a}$ has rich domain, there exists a sequence $v_n \in D(\mathfrak{a}, \omega_k) \cap L^p$ such that $v_n = u_n$ a.e. on ω_{k-1}. Furthermore,

$$\|v_n\|_{L^2(\omega_k)} \le C_k \|u_n\|_{L^2(\omega_k)} \le \tilde{C}_k \|u_n\|_{L^p(\omega_k)} \le M < \infty \tag{3}$$

and

$$
\begin{aligned}
\|\mathcal{A}v_n\|_{D(\mathfrak{a},\omega_k)'} &\leq C_k\left(\|u_n\|_{L^2(\omega_k)} + \|\mathcal{A}u_n\|_{D(\mathfrak{a},\omega_k)'}\right)\\
&\leq \tilde{C}_k\left(\|u_n\|_{L^p(\omega_k)} + \|A_p u_n\|_{L^p(\omega_k)}\right) \leq M < \infty\,, \qquad (4)
\end{aligned}
$$

for some constant M. Here we have used the inclusions $L^p(\omega_k) \hookrightarrow L^2(\omega_k) \hookrightarrow D(\mathfrak{a},\omega_k)'$ and the boundedness of the sequences u_n and $A_p u_n$ in L^p.

It follows from (3), that – after possibly passing to a subsequence – v_n converges weakly in $L^2(\omega_k)$ to some $v \in L^2(\omega_k)$. However, as a sequence in $D(\mathfrak{a},\omega_k)'$ it also converges weakly to (the same) v. Similarly, (4) and the reflexivity of $D(\mathfrak{a},\omega_k)'$ imply that – possibly passing to yet another subsequence – $\mathcal{A}v_n$ converges weakly to some $w \in D(\mathfrak{a},\omega_k)'$. Since $\mathcal{A}_k$ is a generator, its graph is closed and hence, by the Hahn-Banach theorem, also weakly closed. Thus $v \in D(\mathfrak{a},\omega_k)$ and $\mathcal{A}v = w$.

Now let $\omega \Subset \omega_{k-1}$ and $f \in D(A_2) \cap L^2(\omega) \subset D(\mathfrak{a},\omega_k) \cap L^q$. Here q is the conjugate index to p. We have

$$
\langle u, f\rangle_{p,q} = \lim_{n\to\infty} \int u_n \cdot f\, dm = \lim_{n\to\infty} \int v_n \cdot f\, dm = \langle v, f\rangle_{p,q}\,.
$$

By Proposition 2.1, $D(A_2) \cap L^2(\omega)$ is $\sigma(L^p, L^q)$-dense in $L^p(\omega)$. Hence, by density, it follows that $u = v$ a.e. on ω. Furthermore, we have

$$
\langle A_p u, f\rangle_{p,q} = \lim_{n\to\infty} \langle A_p u_n, f\rangle_{p,q} = \lim_{n\to\infty} \langle \mathcal{A}v_n, f\rangle = \langle \mathcal{A}v, f\rangle\,.
$$

Here the second equality follows from the fact that $u_n = v_n$ a.e. on ω_{k-1} and the locality of $\mathfrak{a}$. Since $D(A_2) \cap L^2(\omega)$ is the domain of the operator associated to the form $(\mathfrak{a}, D(\mathfrak{a},\omega))$, it is dense in $D(\mathfrak{a},\omega)$. It follows that $A_p u = \mathcal{A}v$ in $D(\mathfrak{a},\omega)'$. Since ω was arbitrary, it follows that $u \in D(\mathfrak{a})_{\mathrm{loc}}$ and $A_p u = \mathcal{A}u$. □

2.2 - *Kato perturbations*

In this section we consider again the Hilbert space $L^2(M, dm)$ as in the previous section and a local sub-Markovian form $\mathfrak{a}$ on $L^2(M, dm)$. In this whole section we fix $\lambda_0 \in -\Theta(\mathfrak{a})^c \subset \rho(\mathcal{A})$. We are interested in the elliptic equation

$$
\lambda_0 u - \tilde{\mathcal{A}}u = \varphi \qquad (5)
$$

where φ is an element of $D(\mathfrak{a})'_{\mathrm{loc}} \supset D(\mathfrak{a})'$. In particular, we want to investigate whether solutions to (5) have a certain regularity, i.e. whether u belongs to some function space X. If $\varphi \in D(\mathfrak{a})'$ then (5) has a unique solution $u \in D(\mathfrak{a})$. If $\varphi \in D(\mathfrak{a})'_{\mathrm{loc}}$, then we cannot expect solutions u of (5) in $D(\mathfrak{a})$. But there might be several solutions of the elliptic equation in $D(\mathfrak{a})_{\mathrm{loc}}$. We build our theory in such a way that we just need information about "local" solutions of (5), i.e. we consider $u_n = R(\lambda_0, \mathcal{A}_n)\varphi$. We call this a "local" solution, since u_n satisfies

$$\lambda_0(u_n, v) + \mathfrak{a}[u_n, v] = \langle \varphi, v \rangle ,$$

for all $v \in D(\mathfrak{a}, \omega_n)$, that is, $\lambda_0 u_n + \tilde{\mathcal{A}} u_n = \varphi$ on $D(\mathfrak{a}, \omega_n)$. For φ to belong to the local Kato class, we will require these "local" solutions of (5) to belong to X "locally".

Definition 2.2. *Let X and $(X(\omega_n))_{n\geq 0}$ be vector spaces of (equivalence classes of) measurable functions on M. We say that X* is localized by $(X(\omega_n))_{n\geq 0}$ *if*

1. $X(\omega_n) \downarrow X$, *i.e.* $X(\omega_{n+1}) \subset X(\omega_n)$ *for all* $n \geq 0$ *and* $X = \bigcap_n X(\omega_n)$*;*
2. *If $u \in X(\omega_n)$ and v is a measurable function such that $u = v$ a.e. on ω_n, then $v \in X(\omega_n)$.*

Here, in slight abuse of notation, we have identified a measurable function with its equivalence class. In the rest of this article, we will talk about measurable functions and tacitly identify them with their equivalence classes whenever necessary.

Definition 2.3. *Let X be a vector space of measurable functions and let $\mathfrak{a}$ be a local, sub-Markovian form on $L^2(M, dm)$.*

1. *The X*-Kato class $\operatorname{Kat}(\mathfrak{a}, \lambda_0, X)$ *of $\mathfrak{a}$ is defined as*

$$\operatorname{Kat}(\mathfrak{a}, \lambda_0, X) := \{\varphi \in D(\mathfrak{a})' \mid R(\lambda_0, \mathcal{A})\varphi \in X \} .$$

2. *Now assume that X is localized by $X(\omega_n)$. The* local X-Kato class *is defined by*

$$\operatorname{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X) := \bigcap_{n\in\mathbb{N}_0} \operatorname{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n)) ,$$

i.e. $\operatorname{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ *consists of those functionals $\varphi \in D(\mathfrak{a})'_{\mathrm{loc}}$ such that for all $n \in \mathbb{N}_0$ we have $R(\lambda_0, \mathcal{A}_n)\varphi \in X(\omega_n)$.*

Note that the local Kato class depends on the spaces $X(\omega_n)$ used to localize X. Clearly, $\operatorname{Kat}(\mathfrak{a}, \lambda_0, X)$ and $\operatorname{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ are vector spaces. We will see in Section 3.1, that the Kato class may depend on the parameter λ_0. We next characterize λ_0-independence of the Kato class. Note that this also characterizes λ_0-independence of the local Kato class, if we apply it to $\operatorname{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n))$.

Proposition 2.2. *Let $\mathfrak{a}$ be a local sub-Markovian form and X be a vector space of measurable functions.*

1. *Let $\lambda, \mu \in \rho(\mathcal{A})$ with $\lambda \neq \mu$. The following are equivalent:*
   (a) $\mathrm{Kat}(\mathfrak{a}, \lambda, X) \subset \mathrm{Kat}(\mathfrak{a}, \mu, X)$.
   (b) $D(\mathfrak{a}) \cap X \subset \mathrm{Kat}(\mathfrak{a}, \mu, X)$.

2. *Let $\Lambda \subset \rho(\mathcal{A})$ be a set containing at least two elements. The following are equivalent:*
   (a) $\mathrm{Kat}(\mathfrak{a}, \lambda, X) = \mathrm{Kat}(\mathfrak{a}, \mu, X)$ *for all* $\lambda, \mu \in \Lambda$.
   (b) $D(\mathfrak{a}) \cap X \subset \bigcap_{\lambda \in \Lambda} \mathrm{Kat}(\mathfrak{a}, \lambda, X)$.

Proof. 1. Assume (a) and let $u \in D(\mathfrak{a}) \cap X$. Then $\varphi := \lambda u - \mathcal{A}u \in \mathrm{Kat}(\mathfrak{a}, \lambda, X) \subset \mathrm{Kat}(\mathfrak{a}, \mu, X)$. The resolvent equation implies

$$R(\mu, \mathcal{A})\varphi - u = (\lambda - \mu)R(\mu, \mathcal{A})u\,.$$

By assumption, the lefthand side belongs to X. Since X is a vector space and $\lambda \neq \mu$ it follows that $R(\mu, \mathcal{A})u \in X$, proving (b). Now assume (b) and let $\varphi \in \mathrm{Kat}(\mathfrak{a}, \lambda, \mathcal{A})$. Then $u := R(\lambda, \mathcal{A})\varphi \in D(\mathfrak{a}) \cap X$, whence $R(\mu, \mathcal{A})\varphi = u + (\lambda - \mu)R(\mu)u \in X$, i.e. $\varphi \in \mathrm{Kat}(\mathfrak{a}, \mu, X)$. 2. follows from 1. since Λ contains at least two elements. □

Let us consider the classical situation where $\mathcal{A} = \Delta$ is the Laplacian on an open subset Ω of $\mathbb{R}^N$ and $X = C(\Omega)$. It is well known that $\Delta u = 0$ on an open subset ω of Ω implies that u is continuous on ω. Also in our abstract setting we require some connection between the operator $\mathcal{A}$ and the local spaces $X(\omega_n)$.

Definition 2.4. *We say that a local sub-Markovian form* $\mathfrak{a}$ *has* local kernel belonging to X, *if for all ω_n and $u \in D(\mathfrak{a})$ the relation $\lambda_0 u - \mathcal{A}u = 0$ on $D(\mathfrak{a}, \omega_n)$ implies that $u \in X(\omega_n)$.*

Proposition 2.3. *Let $\mathfrak{a}$ be a local sub-Markovian form and X be a vector space localized by $X(\omega_n)$. Assume that $\mathfrak{a}$ has local kernel belonging to X. Then*

1. $\mathrm{Kat}(\mathfrak{a}, \lambda_0, X) \subset \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$. *Furthermore, for all $n \geq 0$ we have* $\mathrm{Kat}(\mathfrak{a}_{n+1}, \lambda_0, X(\omega_{n+1})) \subset \mathrm{Kat}(\mathfrak{a}_n, \lambda_0, X(\omega_n))$.
2. *If $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$, $u \in D(\mathfrak{a})_{\mathrm{loc}}$ and $\lambda_0 u - \tilde{\mathcal{A}}u = \varphi$, then $u \in X$.*
   Conversely, if $\varphi \in D(\mathfrak{a})'_{\mathrm{loc}}$ and $\lambda_n u - \tilde{\mathcal{A}}u = \varphi$ for some $u \in D(a)_{\mathrm{loc}} \cap X$, then $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$.

Proof. 1. Let $\varphi \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ and $n \geq 0$. Then $u := R(\lambda_0, \mathcal{A})\varphi \in X \subset X(\omega_n)$. If we put $u_n := R(\lambda_0, \mathcal{A}_n)\varphi$, then $\lambda_0(u - u_n) - \mathcal{A}(u - u_n) = 0$ on $D(\mathfrak{a}, \omega_n)$. Since $\mathfrak{a}$

has local kernel belonging to X, we obtain $u - u_n \in X(\omega_n)$. But then also $u_n = u - (u - u_n) \in X(\omega_n)$, hence $\varphi \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X(\omega_n))$. Since n was arbitrary, $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$. The proof of the second statement is similar.

2. Fix $n \in \mathbb{N}_0$. By definition of $D(\mathfrak{a})_{\mathrm{loc}}$ there exists $v \in D(\mathfrak{a})$ such that $u = v$ a.e. on ω_{n+1}. By the definition of $\tilde{\mathcal{A}}$ we have $\lambda_0 v - \mathcal{A}v = \varphi$ on $D(\mathfrak{a}, \omega_n)$. Since $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ we have $u_n := R(\lambda_0, \mathcal{A}_n)\varphi \in X(\omega_n)$. Now the relation $\lambda_0(v - u_n) - \mathcal{A}(v - u_n) = 0$ on $D(\mathfrak{a}, \omega_n)$ implies that $v - u_n$ and hence also v and u belong to $X(\omega_n)$. Since n was arbitrary, $u \in X$.

For the converse, assume that $\lambda_0 u - \tilde{\mathcal{A}}u = \varphi \in D(\mathfrak{a})'_{\mathrm{loc}}$ for some $u \in D(\mathfrak{a})_{\mathrm{loc}} \cap X$. Let v be as above and put $w = R(\lambda_0, \mathcal{A})\varphi$. Then $(\lambda_0 - \mathcal{A})(v - w)$ vanishes on $D(\mathfrak{a}, \omega_n)$ and hence $v - w \in X(\omega_n)$. But since also $v \in X(\omega_n)$, it follows that $w \in X(\omega_n)$. This proves that $\varphi \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$. □

Definition 2.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be sesquilinear forms such that $\mathfrak{a}$ is a local, sub-Markovian form. Note that there are no further assumptions on $\mathfrak{b}$, in particular, $\mathfrak{b}$ is not assumed to be closed. The form $\mathfrak{b}$ is called a* sub-Markovian perturbation *of $\mathfrak{a}$ if $\mathfrak{a} + \mathfrak{b}$, defined by $D(\mathfrak{a} + \mathfrak{b}) := D(\mathfrak{a}) \cap D(\mathfrak{b})$, $(\mathfrak{a} + \mathfrak{b})[u, v] := \mathfrak{a}[u, v] + \mathfrak{b}[u, v]$, is a closed, sectorial form which is sub-Markovian. Such a perturbation will be called* local *if $\mathfrak{a} + \mathfrak{b}$ is local.*

We are particularly interested in local, sub-Markovian perturbations $\mathfrak{b}$ of a "regular" form $\mathfrak{a}$ such that the perturbed form $\mathfrak{a} + \mathfrak{b}$ is regular again. To that end, we introduce *Kato perturbations*:

Definition 2.6. *Let $\mathfrak{a}$ be a local, sub-Markovian form on $L^2(M, dm)$, $2 \leq p \leq \infty$ and $\mathfrak{b}$ be a local, sub-Markovian perturbation of $\mathfrak{a}$ such that $D(\mathfrak{a})_c \subset D(\mathfrak{b})$. For $u \in D(\mathfrak{b})$ we denote by $\mathcal{B}u$ the antilinear functional*

$$D(\mathfrak{b}) \ni v \mapsto \langle \mathcal{B}u, v \rangle := -\mathfrak{b}[u, v] \; .$$

1. *$\mathfrak{b}$ is called a* (p, X)-Kato perturbation *of $\mathfrak{a}$, if $D(\mathfrak{a}) \subset D(\mathfrak{b})$ and $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}, \lambda_0, X)$ for all $u \in D(\mathfrak{a}) \cap L^p(M)$.*
2. *Now let X be localized by $X(\omega_n)$. Then $\mathfrak{b}$ is called a* local (p, X)-Kato perturbation *of $\mathfrak{a}$ if $\mathcal{B}u \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda_0, X)$ for all $u \in D(\mathfrak{a})_c \cap L^p(M)$.*

Lemma 2.2. *Let X be a vector space localized by $X(\omega_n)$ and $\mathfrak{a}$ be a local, sub-Markovian form on $L^2(M)$ having local kernel belonging to X. Then $\mathfrak{b}$ is a local (p, X)-Kato perturbation of $\mathfrak{a}$ if and only if $\mathfrak{b}$ is a $(p, X(\omega_n))$-Kato perturbation of $\mathfrak{a}_n$ for all $n \geq 0$.*

Proof. Let $\mathfrak{b}$ be a local (p,X)-Kato perturbation of $\mathfrak{a}$ and $u \in D(\mathfrak{a},\omega_n) \cap L^p$. Then $u \in D(\mathfrak{a})_c \cap L^p$ whence $\mathcal{B}u \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a},\lambda_0,X) \subset \mathrm{Kat}(\mathfrak{a}_n,\lambda_0,X(\omega_n))$. That is, $\mathfrak{b}$ is a $(p,X(\omega_n))$-Kato perturbation of $\mathfrak{a}_n$.

Conversely, assume that $\mathfrak{b}$ is a $(p,X(\omega_n))$-Kato perturbation of $\mathfrak{a}_n$ for every $n \geq 0$. Let $u \in D(\mathfrak{a})_c$. Then there exists n_0, such that $u \in D(\mathfrak{a},\omega_n)$ for all $n \geq n_0$. By hypothesis, $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}_n,\lambda_0,X(\omega_n)$ for all $n \geq n_0$. However, by Proposition 2.3, we see $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a}_n,\lambda_0,X(\omega_n))$ for all $n \geq 0$. □

Theorem 2.2. *Let $2 \leq p \leq \infty$, $\mathfrak{a}$ be a local sub-Markovian form on $L^2(M)$, and $\mathfrak{b}$ be a local sub-Markovian perturbation of $\mathfrak{a}$. Denote by $\mathcal{S}$ and S_2 the operators associated to $\mathfrak{s} := \mathfrak{a}+\mathfrak{b}$ on $D(\mathfrak{s})'$ and L^2 respectively and by S_p the (if $p=\infty$: weak*-) generator of the extrapolated semigroup on L^p. Further suppose that Y is a vector space of measurable functions and that $R(\lambda_0,A_p)(L^p \cap Y) \subset X$.*

1. *If $\mathfrak{b}$ is a (p,X)-Kato perturbation of $\mathfrak{a}$, then $R(\lambda_0,S_p)(L^2 \cap L^p \cap Y) \subset X \cap L^p$.*
2. *Additionally assume that X is localized by $X(\omega_n)$, that $\mathfrak{a}$ has local kernel belonging to X, that, given $u \in D(\mathfrak{a})$ and $n \in \mathbb{N}$, we find $v \in D(\mathfrak{a},\omega_{n+1})$ such that $u = v$ a.e. on ω_n and that $\tilde{\mathcal{S}}$ is an extension of S_p. Then, if $\mathfrak{b}$ is a local (p,X)-Kato perturbation of $\mathfrak{a}$, then $R(\lambda_0,S_p)(L^p \cap Y) \subset X \cap L^p$.*

Proof. Let $f \in L^p(M) \cap Y$. Then $u = R(\lambda_0,S_p)f \in L^p$. We have to show that $u \in X$.

1. If $f \in L^2 \cap L^p$, then $u \in D(S_2) \cap L^p \subset D(\mathfrak{a}) \cap L^p$ and $S_p u = \mathcal{A}u + \mathcal{B}u$ by Proposition 2.1. Hence $u = R(\lambda_0,\mathcal{A})(f+\mathcal{B}u)$. By assumption $f \in \mathrm{Kat}(\mathfrak{a},\lambda_0,X)$ and also $\mathcal{B}u \in \mathrm{Kat}(\mathfrak{a},\lambda_0,X)$, since $u \in D(\mathfrak{a}) \cap L^p$. Thus, $u \in X$.

2. Since $\tilde{\mathcal{S}}$ is an extension of S_p, we have $u \in D(\mathfrak{s})_{\mathrm{loc}}$ and $(\lambda_0 - \tilde{\mathcal{A}})u = f + \tilde{\mathcal{B}}u$. By Proposition 2.3 2., it suffices to prove $f + \tilde{\mathcal{B}}u \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a},\lambda_0,X)$. Let $n \in \mathbb{N}_0$ be given. By hypothesis, there exists $v \in D(\mathfrak{a},\omega_{n+1})$ such that $u = v$ a.e. on ω_n. We may assume that $v \in L^p$. Otherwise we replace v by $w := u^+ \wedge v^+ - u^- \wedge v^-$ which is an element of $D(\mathfrak{a})_c$ (since $\mathfrak{a}$ is sub-Markovian) and satisfies $|w| \leq |u|$ whence it is an element of L^p. By definition, $\tilde{\mathcal{B}}u = \mathcal{B}v$ on $D(\mathfrak{a},\omega_n)$ and $\mathcal{B}v \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a},\lambda_0,X)$. It follows that $\tilde{\mathcal{B}}u \in \mathrm{Kat}(\mathfrak{a}_n,\lambda_0,X(\omega_n))$. Since n was arbitrary, the claim follows. □

The previous theorem gives sufficient conditions for $R(\lambda_0,S_p)$ to map L^p into $L^p \cap X$ and hence – in particular – for the domain of S_p to be a subset of X. It is also interesting to know whether also the semigroup T_p generated by S_p maps L^p to $L^p \cap X$.

For $2 \leq p < \infty$ there is no problem, since the holomorphy of the semigroup T_2 is inherited by the semigroup T_p for such p, see [3, Chapter 7.2]. However for $p = \infty$

holomorphy and not even differentiability of the semigroup T_∞ can be expected. Indeed , it follows from [**13**] that there exists an open bounded set $\Omega \subset \mathbb{R}^N$ such that the spectrum of the Neumann Laplacian on $L^\infty(\Omega)$ contains a vertical line. Thus, the semigroup generated by it cannot be holomorphic or differentiable and hence does *not* map $L^\infty(\Omega)$ into the domain of its generator.

Theorem 2.3. *Let Y be a closed subspace of $L^\infty(M)$ such that $D(S_\infty) \cap Y$ is norm dense in Y and assume that $R(\lambda, S_\infty)Y \subset Y$ for all $\lambda > 0$. Then Y is invariant under the semigroup T_∞ and the restricted semigroup $T_\infty|_Y$ is strongly continuous.*

Proof. For $u \in D(S_\infty)$ the map $t \mapsto T_\infty(t)u$ is strongly continuous. Since $D(S_\infty) \cap Y$ is norm dense in Y, the same is true for arbitrary $u \in Y$. In particular, for $u \in Y$ we have

$$R(\lambda, S_\infty)u = \int_0^\infty e^{-\lambda t} T_\infty(t)u \, dt$$

as a *Bochner* integral, not just as a weak* integral. Now consider the quotient map $Q : L^\infty(M) \to L^\infty(M)/Y$. It is a bounded operator, even though not necessarily weak* continuous. We obtain:

$$0 = QR(\lambda, S_\infty)u = Q \int_0^\infty e^{-\lambda t} T_\infty(t)u \, dt = \int_0^\infty e^{-\lambda t} Q T_\infty(t)u \, dt \, .$$

By [**4**, Theorem 1.7.3] we have $QT_\infty(t)u = 0$ a.e., that is, $T_\infty(t)u \in Y$ for almost every t. Since $t \mapsto T_\infty(t)u$ is strongly continuous, we have $T_\infty(t)u \in Y$ for every $t \geq 0$. □

2.3 - *Invariance of X_0*

Let us again consider a local, sub-Markovian form $\mathfrak{a}$. We are interested in the subspace X_0 of X consisting of elements of X vanishing at infinity, i.e.

$$X_0 := \{ f \in X \, : \, \forall \, \varepsilon > 0 \; \exists K \Subset M \text{ s. t. } |f(x)| \leq \varepsilon \;\; \forall \, x \in M \setminus K \}.$$

In particular, we want to know, whether X_0 is invariant under $R(\lambda, A_\infty)$. However, belonging to X_0 is usually *not* a local property:

Example 2.1. *The space $X = C_0(\mathbb{R}^N) := \{ u \in C(\mathbb{R}^N) \, : \, u(x) \to 0 \text{ as } x \to \infty \}$ cannot be localized. Indeed, consider the constant function $\mathbb{1} : x \mapsto 1$. If X was localized by some spaces $X(\omega_n)$, then for every $k \geq 0$ there exists a function*

$f_k \in C_0(\mathbb{R}^N)$ such that $f_k = \mathbb{1}$ on ω_k. It follows from the definition of "localized" that f_k and hence $\mathbb{1}$ is an element of $X(\omega_k)$. Since k was arbitrary, it would follow that $\mathbb{1} \in C_0(\mathbb{R}^N)$ — a contradiction.

Thus, to obtain semigroups on X_0, one has to use different techniques. One possibility is to use domination and we will use this approach of Section 3.2.

In this section, we introduce a second approach which makes use of Lyapunov functions and will be applied in Section 3.3.

Definition 2.7. *Let $\mathfrak{a}$ be a local form. We say that $\mathfrak{a}$* satisfies the local maximum principle *if the following holds:*

If $\lambda > 0$, $0 \le \varphi \in D(\mathfrak{a})'_{\text{loc}}$ and $v \in D(\mathfrak{a})^+_{\text{loc}}$ satisfies $\lambda v - \tilde{A}v = \varphi$, then $u_n \le v$, where $u_n = R(\lambda, \mathcal{A}_n)\varphi$. In other words, for any nonnegative $\varphi \in D(\mathfrak{a})'_{\text{loc}}$ the smallest nonnegative solution of $\lambda u - \tilde{A}u = \varphi$ on $D(\mathfrak{a}, \omega_n)$ is the one belonging to $D(\mathfrak{a}, \omega_n)$.

Here, we call an element $\varphi \in D(\mathfrak{a})_{\text{loc}}$ positive, if $\langle \varphi, u \rangle \ge 0$ for all $u \in D(\mathfrak{a})^+_c$.

Theorem 2.4. *Let $\mathfrak{a}$ be a local sub-Markovian form satisfying the local maximum principle and assume that $\tilde{A}$ is an extension of A_p for every $p \in [2, \infty]$. The following are equivalent:*

1. *$D(\mathfrak{a})_c$ is dense in $D(\mathfrak{a})$.*
2. *For some (equivalently all) $p \in [2, \infty]$ we have $p\text{-}\lim_{n\to\infty} R(\lambda, \mathcal{A}_n)f = R(\lambda, A_p)f$ for all $f \in L^p$.*
3. *For some (equivalently all) $p \in [2, \infty]$ we have that if $f \in L^p_+$ and $v \in D(\mathfrak{a})^+_{\text{loc}}$ satisfies $\lambda v - \tilde{A}v = f$ then $R(\lambda, A_p)f \le v$.*

Proof. 1. ⇒ 2. for $p = 2$: We have $D(\mathfrak{a}_n) \subset D(\mathfrak{a})$ and $\mathfrak{a}_n - \mathfrak{a} = 0$ is uniformly sectorial. Condition (1) states that $D := D(\mathfrak{a})_c$ is a core for $\mathfrak{a}$. Clearly, $D \subset \liminf D(\mathfrak{a}_n)$ and $\mathfrak{a}_n[u] \to \mathfrak{a}[u]$ for all $u \in D$. Now 2. for $p = 2$ follows directly from a version of the convergence theorem "from above" (cf. [**11**, Theorem VIII.3.6]) for nondensely defined forms.

Now assume that 2. is true for some $p \in [2, \infty]$. We show that 2. holds for any $q \in [2, \infty]$. It suffices to prove this for nonnegative $f \in L^q$. Since $(\lambda - A_q)R(\lambda, A_q)f = (\lambda - \tilde{A})R(\lambda, A_q)f = f$, the local maximum principle yields $R(\lambda, \mathcal{A}_n)f \le R(\lambda, \mathcal{A}_{n+1})f \le R(\lambda, A_q)f$ for all $n \ge 0$. Hence $R(\lambda, \mathcal{A}_n)f$ converges pointwise a.e. to some function $g \in L^q$.

If $f \in L^p \cap L^q$, then, by consistency, $R(\lambda, A_p)f = R(\lambda, A_q)f$. By our assumption we have $p\text{-}\lim R(\lambda, \mathcal{A}_n)f = R(\lambda, A_q)f$ and hence $g = R(\lambda, A_q)f$. The dominated convergence theorem implies $q\text{-}\lim R(\lambda, \mathcal{A}_n)f = R(\lambda, A_q)f$. Since the forms $\mathfrak{a}_m$ are uniformly sectorial, the operators $R(\lambda, \mathcal{A}_n)$ are uniformly bounded. Now the result for general $f \in L^p$ follows by approximation.

2. $\Rightarrow$ 3.: Let $v \in D(\mathfrak{a})^+_{\mathrm{loc}}$ be given such that $\lambda v - \tilde{A}v = f$ for some $f \in L^p_+$. By the local maximum principle we have $R(\lambda, \mathcal{A}_n)f \leq v$ for all n. But now 2. implies $R(\lambda, A_p)f = \lim R(\lambda, \mathcal{A}_n)f \leq v$.

Now assume 3. holds for some p. We prove that it holds for any $q \in [2, \infty]$. By density, there exists an increasing sequence $f_n \in L^p \cap L^q$, such that q-$\lim f_n = f$. Using consistency and positivity we obtain

$$R(\lambda, A_q)f_n = R(\lambda, A_p)f_n \leq R(\lambda, A_p)f \leq v\,,$$

by assumption. Hence $R(\lambda, A_q)f = q\text{-}\lim R(\lambda, A_q)f_n \leq v$, by the continuity of $R(\lambda, A_q)$.

3. $\Rightarrow$ 1.: Define the form $\mathfrak{b}$ by $\mathfrak{b}[u, v] = \mathfrak{a}[u, v]$ and $D(\mathfrak{b}) = \overline{D(\mathfrak{a})_c}^{D(\mathfrak{a})}$. Then $\mathfrak{b}$ is a closed sectorial form and the continuity of the lattice operations imply that it is also sub-Markovian. Furthermore, the local spaces and operators associated to the forms $\mathfrak{a}$ and $\mathfrak{b}$ agree, in particular, $\mathfrak{b}$ satisfies the local maximum principle. However, $\mathfrak{b}$ satisfies condition 1. of this theorem and therefore 3. of this theorem holds true for $\mathfrak{b}$. We obtain $R(\lambda, B_2)f \leq R(\lambda, A_2)f$ for all $f \in L^2_+$. Since we assumed that 3. holds also for $\mathfrak{a}$ we obtain the reversed inequality and thus $R(\lambda, A_2) = R(\lambda, B_2)$. In particular $D(A_2) = D(B_2)$. However, by general theory, $D(A_2)$ and $D(B_2)$ are cores of the forms $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Hence $\mathfrak{a}$ and $\mathfrak{b}$ coincide on a common core and thus have to be equal. □

Definition 2.8. *Let $\mathfrak{a}$ be a local sub-Markovian form. We say that $\mathfrak{a}$ has abstract* Dirichlet boundary conditions *if $\mathfrak{a}$ satisfies the local maximum principle and $D(\mathfrak{a})_c$ is dense in $D(\mathfrak{a})$.*

Lemma 2.3. *Let $p \in [2, \infty]$, $\lambda > 0$ and $\mathfrak{a}$ be a local sub-Markovian form which has abstract Dirichlet boundary conditions. Further suppose that $\tilde{A}$ is an extension of A_p. If $f, g \in D(\mathfrak{a})^+_{\mathrm{loc}}$ satisfy $g \leq \lambda f - \tilde{A}f$ and $g \in L^p$, then $R(\lambda, A_p)g \leq f$.*

Proof. First note that if $\mathfrak{a}$ is any sub-Markovian form, then, for $\lambda > 0$, also the resolvent $R(\lambda, \mathcal{A})$ of $\mathcal{A}$ is positive on $D(\mathfrak{a})'$. To see this, let $\varphi \in D(\mathfrak{a})'_+$ and define $u = R(\lambda, \mathcal{A})\varphi$. Since $\mathfrak{a}$ is submarkovian, $u^- \in D(\mathfrak{a})$ and $\mathfrak{a}[u^+, u^-] \leq 0$. Thus

$$0 \leq \langle \varphi, u^- \rangle = \lambda(u, u^-) + \mathfrak{a}[u, u^-] \leq -\lambda\|u^-\|^2 - \mathfrak{a}[u^-, u^-] \leq -\lambda\|u^-\|^2\,.$$

It follows that $u^- = 0$.

From this observation we obtain $R(\lambda, \mathcal{A}_n)g \leq R(\lambda, \mathcal{A}_n)(\lambda f - \tilde{A}f)$ for any $n \geq 0$. By the local maximum principle we have $R(\lambda, \mathcal{A}_n)(\lambda f - \tilde{A}f) \leq f$, whence $R(\lambda, \mathcal{A}_n)g \leq f$, for all $n \geq 0$. Since $\mathfrak{a}$ has abstract Dirichlet boundary conditions, Theorem 2.4 implies $R(\lambda, \mathcal{A}_n)g \to R(\lambda, A_p)g$ and the statement follows. $\square$

We are now prepared to tackle the invariance of X_0. We shall consider the space $X_b := X \cap L^\infty$ and assume that X_b is closed in L^∞. Clearly, X_0 is a closed subspace of X_b. By X_c we denote the vector space of all elements of X_b having compact support.

Theorem 2.5. *Let $\mathfrak{a}$ be a local sub-Markovian form which has abstract Dirichlet boundary conditions. Assume that X_b is a closed subspace of L^∞, that X_c is dense in X_0 and that for some $\lambda_1 > 0$ we have $R(\lambda, A_\infty)X_c \subset X_b$ for all $\lambda > \lambda_1$. If there exists $\lambda_0 > 0$ and a strictly positive function $\varphi \in X_0 \cap D(\mathfrak{a})_{\mathrm{loc}}$ such that*

$$\lambda_0\varphi - \tilde{\mathcal{A}}\varphi \geq 0\,, \tag{6}$$

then for $\lambda > \max\{\lambda_0, \lambda_1\}$ we have $R(\lambda, A_\infty)X_0 \subset X_0$. If $\tilde{\mathcal{A}}\varphi \in L^\infty_{\mathrm{loc}}$ then it suffices to check $\lambda_0\varphi - \tilde{\mathcal{A}}\varphi \geq 0$ outside a compact set $K \subset M$.

Proof. It follows from (6) that for $\lambda > \lambda_0$ we have $\lambda\varphi - \tilde{\mathcal{A}}\varphi \geq (\lambda - \lambda_0)\varphi$. Hence, by Lemma 2.3, $(\lambda - \lambda_0)R(\lambda, A_\infty)\varphi \leq \varphi$. For $f \in X_c$ we may find $c > 0$ such that $|f| \leq c\varphi$ since φ is strictly positive. It follows that

$$0 \leq R(\lambda, A_\infty)|f| \leq R(\lambda, A_\infty)c\varphi \leq \frac{c}{\lambda - \lambda_0}\varphi\,.$$

This implies $R(\lambda, A_\infty)X_c \subset X_0$. The general case follows by approximation, using that $R(\lambda, A_\infty)X_c \subset X_b$ for $\lambda > \lambda_1$.

For the addendum observe that if $\tilde{\mathcal{A}}\varphi \in L^\infty_{\mathrm{loc}}$, then $\lambda\varphi - \tilde{\mathcal{A}}\varphi \geq 0$, whenever $\lambda - \lambda_0 > \|\lambda_0\varphi - \tilde{\mathcal{A}}\varphi\|_{L^\infty(K)}$. $\square$

3 - Applications and examples

3.1 - *The $C(\overline{\Omega})$-Kato class for multiplication operators*

In this section we consider the simple case where $\mathcal{A}$ is a multiplication operator. The purpose of this section is to give an example that the (local) Kato class may depend heavily on the parameter λ.

We work on the space $L^2(\Omega, dx)$, where Ω is an open set in $\mathbb{R}^N$. We consider the sub-Markovian form $\mathfrak{a}$ defined by

$$\mathfrak{a}[u, v] := \int_\Omega u(x)\overline{v(x)}m(x)\,dx\,, \quad D(\mathfrak{a}) = L^2(\Omega)\,,$$

where $0 \le m \in L^\infty(\Omega)$. In this case, $D(\mathfrak{a}) = D(\mathfrak{a})' = L^2(\Omega)$. Furthermore, the associated operator $\mathcal{A}$ is the multiplication operator given by $\mathcal{A}u = -mu$. In particular, $\rho(\mathcal{A}) = \{\lambda : (\lambda + m)^{-1} \in L^\infty\}$ and for $\lambda \in \rho(\mathcal{A})$ we have $R(\lambda, \mathcal{A})f = (\lambda + m)^{-1} f$. We shall consider the regularity space $X = \{u \in L^2(\Omega) : \exists\, \tilde{u} \in C(\overline{\Omega})$ such that $\tilde{u} = u$ a.e. on $\Omega\}$. For $u \in X$ we denote its unique continuous version by $\tilde{u}$.

Proposition 3.1. *With the above definitions we have:*

1. *For* $\lambda \in \rho(\mathcal{A})$ *we have* $\mathrm{Kat}(\mathfrak{a}, \lambda, X) = \{u(\lambda + m) : u \in X\}$. *In particular,* $\mathrm{Kat}(\mathfrak{a}, \lambda, X)$ *is dense in* $L^2(\Omega)$.
2. *If* $\lambda, \mu \in \rho(\mathcal{A})$ *with* $\lambda \neq \mu$, *then*

$$\mathrm{Kat}(\mathfrak{a}, \lambda, X) \cap \mathrm{Kat}(\mathfrak{a}, \mu, X) = \{u(\lambda + m) \ : \ u \in X\,,\ \tilde{u}|_U \equiv 0\}\,,$$

where U *is the set of all points* $x \in \Omega$ *such that no version of* m *is continuous at* x.

Proof. 1. is clear. For 2., define m_0 by

$$m_0(x) := \limsup_{r \to 0+} \fint_{B(x,r) \cap \Omega} m(y)\, dy\,.$$

Since almost every $x \in \Omega$ is a Lebesgue point of m, m_0 is a version of m. It has the following additional property:

If m has a version $\bar{m}$ which is continuous at x_0, then $\bar{m}(x_0) = m_0(x_0)$ and m_0 is continuous at x_0. This means, m_0 is continuous at every point $x \in \Omega \setminus U$. Now let $f \in \mathrm{Kat}(\mathfrak{a}, \lambda, X) \cap \mathrm{Kat}(\mathfrak{a}, \mu, X)$. Then there exist $u, v \in X$ with

$$f = \tilde{u}(\lambda + m_0) = \tilde{v}(\mu + m_0) \quad a.e..$$

We see that $(\tilde{u} - \tilde{v})(\lambda + m_0) = \tilde{v}(\mu - \lambda)$ a.e. This implies that m_0 is continuous on the open set $\mathcal{O} := \{x \in \Omega : \tilde{u}(x) \neq \tilde{v}(x)\}$. Indeed, m_0 agrees almost everywhere on $\mathcal{O}$ with the continuous function $(\tilde{u} - \tilde{v})^{-1}\tilde{v}(\mu - \lambda) - \lambda$. Since $\mathcal{O}$ is open, it follows that m has a version which is continuous at every point in $\mathcal{O}$. But the properties of m_0 imply that in fact m agrees with this continuous version everywhere on $\mathcal{O}$. Now define $w := (\tilde{u} - \tilde{v})(\lambda + m_0)$. Clearly, w is continuous at every point $x \in \mathcal{O}$. If $x \in \Omega \setminus \mathcal{O}$, then $\tilde{u}(y) - \tilde{v}(y) \to \tilde{u}(x) - \tilde{v}(x) = 0$ for $y \to x$. Since $(\lambda + m_0)$ is bounded, it follows that w is continuous at x. Altogether, w is continuous.

We have seen that w and $\tilde{v}(\mu - \lambda)$ are two continuous functions which are equal a.e.. Hence they are equal everywhere; in particular, $\tilde{v} = 0$ on $\mathcal{O}^c$. It follows that $\tilde{v}(\lambda + m_0)$ is a continuous function, whence f has a continuous version which vanishes on $\mathcal{O}^c$ and thus in particular on U. This proves one inclusion in the statement, the other inclusion is obvious. □

Lemma 3.1. *There exists a measurable function* $m : [0,1] \to [0,1]$ *such that the following holds. If* $\bar{m}$ *is a measurable function such that the set* $N := \{x \in [0,1] : \bar{m}(x) \neq m(x)\}$ *is a Lebesgue null set, then* $\bar{m}$ *is not continuous in every* $x \in [0,1] \setminus N$.

Proof. Let O_n be a sequence of open sets which are dense in $[0,1]$, such that $|O_n| \le \frac{1}{n}$ and $\bigcap O_n = \emptyset$. Such a sequence may be obtained as follows:

Let $\{q_k : k \in \mathbb{N}\} = \mathbb{Q} \cap [0,1]$ and $\{r_k : k \in \mathbb{N}\} = (\mathbb{Q} + \pi) \cap [0,1]$. Then define

$$O_n := \begin{cases} \bigcup_{k\in\mathbb{N}} B\left(q_k, \frac{1}{n2^{-k}}\right), & n \text{ even} \\ \bigcup_{k\in\mathbb{N}} B\left(r_k, \frac{1}{n2^{-k}}\right), & n \text{ odd} \end{cases}.$$

Now we define

$$m(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{1}_{O_n}(t).$$

Clearly, m is a bounded, measurable function with values in $[0,1]$. Let $\bar{m}$ be a version of m, say $\bar{m} = m$ on $[0,1] \setminus N$ for a null set N, and x_0 be a continuity point of $\bar{m}$. Given $j \in \mathbb{N}$, we find δ_j such that

$$|\bar{m}(x_0) - \bar{m}(y)| < \frac{1}{2^{j+1}} \quad \text{for all } y \in B(x_0, \delta_j).$$

It follows from the triangle inequality that

$$|m(x) - m(y)| < \frac{1}{2^j} \quad \text{for all } x, y \in B(x_0, \delta_j) \setminus N. \tag{7}$$

But now we see that for $n = 1, \ldots, j-1$ and $x, y \in B(x_0, \delta_j) \setminus N$ we have $x \in O_n$ if and only if $y \in O_n$.

Indeed, if $x \in O_1$ whereas $y \notin O_1$, then $|m(x) - m(y)| \ge 2^{-1} - 2^{-2} = 2^{-2}$ by the definition of m and the reverse triangle inequality. This contradicts (7). Now assume that we know that $x \in O_n$ iff $y \in O_n$ for $1 \le n \le k-1$. If $x \in O_k$ whereas $y \notin O_k$, then $|m(x) - m(y)| \ge 2^{-(k+1)}$. This contradicts (7) whenever $k+1 < j$. Thus the statement follows by induction.

It follows that $B(x_0, \delta_j) \setminus N$ is either a subset of O_n or of O_n^c for $1 \le n \le j-1$. However, if $B(x_0, \delta_j) \setminus N \subset O_n^c$, then we have $B(x_0, \delta_j) \subset O_n^c$, since O_n^c is closed and hence contains the closure of every set it contains. But we cannot have $B(x_0, \delta_j) \subset O_n^c$, since O_n is dense. Thus for any $j \in \mathbb{N}$ we have $B(x_0, \delta_j) \setminus N \subset O_n$ for $n = 1, \ldots, j-1$. Since j was arbitrary, $x_0 \notin N$ implies $x_0 \in \bigcap O_n = \emptyset$. Thus x_0 can only lie in N. $\square$

The following corollary shows that the Kato class may depend heavily on the parameter λ.

Corollary 3.1. *There exists a local sub-Markovian form* $\mathfrak{a}$ *and a regularity space* X *such that* $\mathrm{Kat}(\mathfrak{a},\lambda,X)$ *is dense in* L^2 *for every* $\lambda \in \rho(\mathcal{A})$ *whereas for* $\lambda,\mu \in \rho(\mathcal{A})$ *with* $\lambda \neq \mu$ *we have* $\mathrm{Kat}(\mathfrak{a},\lambda,X) \cap \mathrm{Kat}(\mathfrak{a},\mu,X) = \{0\}$.

Proof. Take $\mathfrak{a}$ as above with the function m from Lemma 3.1. If m_0 is defined as in the proof of Proposition 3.1, then it follows that m_0 is a version of m which is continuous in every point such that m has a version being continuous in that point. Lemma 3.1 implies that m_0 is only continuous on a null set. Now the claim follows from Proposition 3.1. □

3.2 - *Regular perturbations of deGiorgi-Nash forms*

In this section we introduce a special class of sub-Markovian forms on the Hilbert space $L^2(\Omega, dx)$, where Ω is a domain in $\mathbb{R}^N$. For these forms, many elements of the local $C(\Omega)$-Kato class are known as a consequence of the deGiorgi-Nash theorem.

Definition 3.1. *Let* $\Omega \subset \mathbb{R}^N$ *be a domain and let* a_{ij}, b_i, c *belong to* $L^\infty(\Omega, dx, \mathbb{R})$ *for* $1 \leq i,j \leq N$. *Further suppose that* $c \geq 0$. *Assume that there exist constants* $\eta > 0$ *and* $M \geq 0$ *such that the inequalities*

$$\mathrm{Re} \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\overline{\xi_i} \geq \eta|\xi|^2 \quad \textit{and} \quad \left|\mathrm{Im} \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\overline{\xi_j}\right| \leq M \mathrm{Re} \sum a_{ij}(x)\xi_i\overline{\xi_j}$$

hold for all $\xi \in \mathbb{C}^N$ *and almost every* x. *A* deGiorgi-Nash form *is a form* $(\mathfrak{a}, D(\mathfrak{a}))$ *satisfying the following conditions:*

1. $D(\mathfrak{a})$ *is a closed subspace of* $H^1(\Omega)$ *containing* $H^1_0(\Omega)$ *such that if* $f \in D(\mathfrak{a})$ *then also* $\mathrm{Re} f$ *and, for real-valued* f, *also* f^+, $\mathrm{sgn} f \cdot (1 \wedge f) \in D(\mathfrak{a})$.
2. *For all* $f, g \in D(\mathfrak{a})$ *we have*

$$\mathfrak{a}[f,g] = \int_\Omega \sum_{i,j=1}^{N} a_{ij} D_i f \overline{D_j g} + \sum_{i=1}^{N} b_i (D_i f)\overline{g} + c f \overline{g}\, dx\,.$$

Clearly, deGiorgi-Nash forms are densely defined and local. It is not hard to see that they are also sectorial and closed, in fact, $\|\cdot\|_{\mathfrak{a}}$ is equivalent to the Sobolev norm $\|\cdot\|_{H^1}$. It follows from [**14**, 4.1, 4.2 and 4.9], that deGiorgi-Nash forms are sub-Markovian.

We note that $D(\mathfrak{a})_{\mathrm{loc}} = H^1_{\mathrm{loc}}(\Omega)$ and that if $D(\mathfrak{a}) = H^1_0(\Omega)$ then $D(\mathfrak{a})' = H^{-1}(\Omega)$. Otherwise $D(\mathfrak{a})'$ is a subspace of $H^{-1}(\Omega)$. It follows from Hölder's inequality that for bounded Ω we have $W^{-1,p}(\Omega) \hookrightarrow H^{-1}(\Omega)$ for $p \geq 2$. Recall that any $\varphi \in W^{-1,p}$ may be represented as $g + \sum_{i=1}^{N} D_i f_i$, where $g, f_i \in L^p$, see [**1**, Chapter III]. Thus in the injection $W^{-1,p} \hookrightarrow H^{-1}(\Omega)$ we identify φ with the functional

$$H^1_0(\Omega) \ni u \mapsto \int_{\Omega} \left(gu - \sum_{i=1}^{N} f_i D_i u \right) dx \, .$$

We will be interested in the regularity space $X = C(\Omega)$, more precisely

$$X = \{ u \in L^1_{\mathrm{loc}}(\Omega) \, : \, u \text{ has a version which is continuous on } \Omega \} \, .$$

For localization we will use a sequence ω_n of open, bounded sets such that $\overline{\omega}_n \subset \omega_{n+1} \subset \Omega$. This corresponds to choosing $M = \Omega$ in the previous sections. We will discuss an application of choosing M differently in the next section. In this case, $D(\mathfrak{a}, \omega_n) = \tilde{H}^1_0(\omega) := \{ u \in H^1(\mathbb{R}^N) : u = 0 \text{ a.e. on } \omega_n^c \}$. However, $\tilde{H}^1_0(\omega_n) = H^1_0(\omega_n)$ if ω_n satisfies a mild regularity assumption, e.g. if ω_n has Lipschitz boundary. It is no loss of generality to assume that $D(\mathfrak{a}, \omega_n) = H^1_0(\omega_n)$ since every domain may be exhausted by an increasing sequence of open sets having Lipschitz boundary.

We localize our regularity space X by the spaces

$$X(\omega_n) := \{ u \in L^1_{\mathrm{loc}}(\Omega) \, : \, u \text{ has a version which is continuous on } \omega_n \} \, .$$

Now elements of the Kato class for $X = C(\Omega)$ are easily available from the deGiorgi-Nash theorem [**9**, Theorems 8.22 and 8.24], which we restate in our terminology:

Theorem 3.1 (deGiorgi-Nash). *Assume $N \geq 2$ and let $\mathfrak{a}$ be a deGiorgi-Nash form on $L^2(\Omega)$, ω be an open subset of Ω and $\lambda \in \mathbb{C}$. Furthermore, let $f_1, \ldots, f_N \in L^p(\Omega, dx)$, $g \in L^{\frac{p}{2}}(\Omega, dx)$ for some $p > N$ and $\psi \in H^1(\Omega)$ be given. If $u \in D(\mathfrak{a})$ is a solution of the generalized Dirichlet Problem*

$$\mathbf{D}_{\mathfrak{a},\lambda,\omega} \begin{cases} \lambda u - \mathcal{A}u &= g + \sum D_i f_i \quad \text{on } H^1_0(\omega) \\ \qquad u &= \psi \qquad\qquad\quad \text{on } \partial\omega \end{cases}$$

then u is locally Hölder continuous on ω.

It follows that appropriate (local) L^p and Sobolev spaces belong to the (local) Kato class associated to $\mathfrak{a}$.

Corollary 3.2. *Let* $(\mathfrak{a}, D(\mathfrak{a}))$ *be a deGiorgi-Nash form,* $\lambda \in \rho(\mathcal{A})$ *and* $N \geq 2$.

1. *If* Ω *is bounded and* $D(\mathfrak{a}) = H_0^1(\Omega)$, *then* $L^p(\Omega) \subset \mathrm{Kat}(\mathfrak{a}, \lambda, X)$ *for* $p \in \left(\frac{N}{2}, \infty\right]$ *and* $W^{-1,p}(\Omega) \subset \mathrm{Kat}(\mathfrak{a}, \lambda, X)$ *for* $p \in (N, \infty]$.
2. *If* $p \in \left(\frac{N}{2}, \infty\right]$ *then* $L^p_{\mathrm{loc}}(\Omega) \subset \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$. *If* $p \in (N, \infty]$ *then* $W^{-1,p}_{\mathrm{loc}}(\Omega) := \bigcap W^{-1,p}(\omega_n) \subset \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$.
3. $\mathfrak{a}$ *has local kernel belonging to* X (*see Definition* 2.4) *and* $\mathrm{Kat}(\mathfrak{a}, \lambda, X)$ *and* $\mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$ *are independent of* $\lambda \in \rho(\mathcal{A})$.

Proof. 1. If Ω is bounded, then $L^p(\Omega)$, $W^{-1,p}(\Omega) \subset H^{-1}(\Omega)$ for the values of p given in the statement. Since $D(\mathfrak{a}) = H_0^1(\Omega)$ we have $H^{-1}(\Omega) = D(\mathfrak{a})'$. The assertion now follows immediately from Theorem 3.1 noting that $u = R(\lambda, \mathcal{A})\varphi$, is a solution of $\mathbf{D}_{\mathfrak{a},\lambda,\Omega}$ for $\psi = 0$ and right hand side φ.

2. Follows from 1. and the definition of the local Kato class, observing that $\mathfrak{a}_n$ is merely the form $\mathfrak{a}$ restricted to $H_0^1(\omega_n)$.

3. If $u \in D(\mathfrak{a})$ satisfies $\lambda_0 u - \mathcal{A}u = 0$ on $D(\mathfrak{a}, \omega_n)$, then u is a solution of $\mathbf{D}_{\mathfrak{a},\lambda_0,\omega_n}$ with right hand side $0 \in L^\infty(\omega_n)$ and boundary values $\psi = u$. It follows from Theorem 3.1 that $u \in X(\omega_n)$. To see that the Kato classes are independent of λ observe that since $D(\mathfrak{a}) \cap X \subset L^\infty_{\mathrm{loc}}(\Omega)$ we have $D(\mathfrak{a}) \cap X \in \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$ by part 2. Since λ was arbitrary, $D(\mathfrak{a}) \cap X \subset \bigcap_{\lambda \in \rho(\mathcal{A})} \mathrm{Kat}_{\mathrm{loc}}(\mathfrak{a}, \lambda, X)$. Taking into account that $\mathfrak{a}$ has local kernel belonging to X, it follows from Proposition 2.2 that $D(\mathfrak{a}) \cap X \subset \bigcap_{\lambda \in \rho(\mathcal{A})} \mathrm{Kat}(\mathfrak{a}, \lambda, X)$. Proposition 2.3 yields the claim. □

We now turn to Kato perturbations of deGiorgi-Nash forms. We will focus on perturbing a deGiorgi-Nash form by a measure. Viewed as an operator, a measure μ should be associated with the form $\mathfrak{m}[u, v] = \int_\Omega u\overline{v}\, d\mu$. However, if μ is not absolutely continuous with respect to Lebesgue measure, then the meaning of the latter integral is not clear. This leads to the following

Definition 3.2. *Let* $(\mathfrak{a}, D(\mathfrak{a}))$ *be a deGiorgi-Nash form on* $L^2(\Omega, dx)$. *A positive measure* μ *on* Ω *is called* admissible for $\mathfrak{a}$ *if there is a continuous linear mapping*

$$J : D(\mathfrak{a}) \to L^2_{\mathrm{loc}}(\Omega, d\mu), \quad u \mapsto \tilde{u}$$

such that the following hold:

(A1) *J preserves positivity, i.e.* $u \geq 0$ *dx-a.e. implies* $Ju \geq 0$ *dμ-a.e.*

(A2) *J is multiplicative, i.e. if* $u, v \in D(\mathfrak{a})$ *are such that* $u \cdot v \in D(\mathfrak{a})$ *then* $J(u \cdot v) = J(u)J(v)$.

(A3) *If* $u \in D(\mathfrak{a})$ *satisfies* $u \leq 1$ *then* $Ju \leq 1$.

(A4) *For* $\omega \Subset \Omega$ *there exists a constant* C_ω *such that* $\|Ju\|_{L^2(\mu)} \leq C_\omega \cdot \|u\|_{\mathfrak{a}}$ *for all* $u \in D(\mathfrak{a}, \omega)$.

Lemma 3.2. *Let* $(\mathfrak{a}, D(\mathfrak{a}))$ *be a deGiorgi-Nash form,* $N \geq 2$ *and* $q > \dfrac{N}{2}$. *If* $V \in L^q_{\text{loc}}(\Omega, dx)$ *is positive, then* $\mu = Vdx$ *is admissible for* $\mathfrak{a}$. *One can choose* J *as the identity map.*

Proof. We first note that for $Ju = u$ the conditions (A1)-(A3) are obvious. Let $u \in D(\mathfrak{a}) \subset H^1(\Omega)$. By Sobolev embeddings, $u \in L^{\frac{2N}{N-2}}_{\text{loc}}(\Omega)$. Hence, by Hölder's inequality, $|u|^2 V \in L^r_{\text{loc}}(\Omega)$ for

$$\frac{1}{r} = \frac{1}{q} + 2\,\frac{N-2}{2N} < \frac{2}{N} + \frac{N-2}{N} = 1\,.$$

This implies that $Ju = u \in L^2_{\text{loc}}(\Omega, Vdx)$. Now let $\omega \Subset \Omega$. Possibly embedding ω into a larger set with Lipschitz boundary, we may assume that $D(\mathfrak{a}, \omega) = H^1_0(\omega)$. Hence, for $u \in D(\mathfrak{a}, \omega)$ we have

$$\int_\omega |u|^2 V\, dx \leq \|V\|_{L^q(\omega,dx)} \|u\|^2_{L^{\frac{2N}{N-2}}(\omega,dx)} \leq C^2 \|V\|_{L^q(\omega,dx)} \|u\|^2_{H^1_0(\omega)}\,.$$

Taking square roots, it follows that condition (A4) is satisfied. □

EXAMPLE 3.1. *If* $N = 1$, *then* $D(\mathfrak{a}) \subset H^1(\Omega) \hookrightarrow C(\Omega)$. *Thus if we choose* J *as this injection restricted to* $D(\mathfrak{a})$, *we see that any locally finite measure on* Ω *is admissible for* $\mathfrak{a}$.

Given a deGiorgi-Nash form $\mathfrak{a}$ and μ admissible for $\mathfrak{a}$, we define the form $\mathfrak{m}$ by

$$\mathfrak{m}[u,v] = \int_\Omega \tilde{u}\overline{\tilde{v}}\, d\mu, \quad D(\mathfrak{m}) = \left\{u \in D(\mathfrak{a}) \,:\, \tilde{u} \in L^2(\Omega, d\mu)\right\}, \tag{8}$$

where we wrote $\tilde{u} := J(u)$. For $u \in D(\mathfrak{m})$ we will write $\mathcal{M}u$ for the antilinear functional $-\mathfrak{m}[u,\cdot]$.

Remark 3.1. *We note that the form* $\mathfrak{m}$ *depends not only on* μ *but also on the mapping* J. *However, for certain measures* μ *there are canonical choices for* J. *If* μ *is absolutely continuous with respect to Lebesgue measure, then* $J = id$ *is the canonical choice. At the end of this section we will show that, under additional assumptions on* $\mathfrak{a}$, *certain measures which are absolutely continuous with respect to the Choquet capacity associated to* $\mathfrak{a}$ *are admissible for* $\mathfrak{a}$. *Here the canonical choice for* Ju *is the quasi-continuous representative of* u.

We will prove in Theorem 3.3 below that if μ is admissible for $\mathfrak{a}$, then $\mathfrak{m}$ is a sub-Markovian perturbation of $\mathfrak{a}$. Accepting this for the moment, we infer from Corollary 3.2 that $\mathfrak{m}$ is a Kato perturbation of $\mathfrak{a}$.

Theorem 3.2. *Let $\mathfrak{a}$ be a deGiorgi-Nash form, $N \geq 2$ and $\mathfrak{m}$ be defined by* (8) *for some measure $\mu \geq 0$, admissible for $\mathfrak{a}$. Then $\mathfrak{m}$ is a local $(p, C(\Omega))$-Kato perturbation of $\mathfrak{a}$ for every $p \geq 2$.*

Proof. We first note that by condition (A4) we have $D(\mathfrak{a})_c \subset D(\mathfrak{m})$. Now let $\omega \Subset \Omega$, $v \in W_0^{1,q}(\omega) \subset H_0^1(\omega) \subset D(\mathfrak{a})$ and $u \in D(\mathfrak{m})$. We have

$$|\mathfrak{m}[u, v]| \leq \int_\Omega |\tilde{u}\overline{\tilde{v}}| \, d\mu \leq \|\tilde{u}\|_{L^2(\Omega, d\mu)} \|\tilde{v}\|_{L^2(\Omega, d\mu)} \leq C_1 \|v\|_{H_0^1(\omega)} \leq C_2 \|v\|_{W^{1,q}(\omega)} .$$

Here we have used the Cauchy-Schwarz inequality, (A4) and the continuity of the embedding $W_0^{1,q}(\omega)$ into $H_0^1(\Omega)$. Since ω was arbitrary, it follows that $\mathcal{M}u \in W_{\mathrm{loc}}^{-1,q}(\Omega)$. Note that this is true for all $u \in D(\mathfrak{m})$, without any L^p-condition. It thus follows from Corollary 3.2 that $\mathfrak{m}$ is a $(p, C(\Omega))$-Kato perturbation of $\mathfrak{a}$ for every $p \geq 2$. □

We note that Kato perturbations of deGiorgi-Nash forms need not be associated to a function or a measure:

EXAMPLE 3.2. *Consider the form*

$$\mathfrak{b}[u, v] := \int_\Omega \sum_{i=1}^N d_i D_i u \cdot \overline{v} \, dx, \quad D(\mathfrak{b}) := H_0^1(\Omega) .$$

Using Sobolev embeddings and a perturbation result for forms (*see* [**11**, Theorem VI.1.33]) *it can be shown that for $d_i \in L^q(\Omega)$ the form $\mathfrak{b}$ is well defined and a sub-Markovian perturbation of any deGiorgi-Nash form $\mathfrak{a}$, provided that $q > \max\{2, N\}$ and $D(\mathfrak{a}) = H_0^1(\Omega)$. In this case, $\mathcal{B}u \in L^r$ where $\frac{1}{r} = \frac{1}{2} + \frac{1}{q}$. Thus, if $r > \frac{N}{2}$, then $\mathfrak{b}$ is a $(p, C(\Omega))$-Kato perturbation of $\mathfrak{a}$ for any $p \in [2, \infty]$.*

We now verify the abstract assumptions made in the previous section for the perturbed form $\mathfrak{a} + \mathfrak{m}$.

Theorem 3.3. *Let $\mathfrak{a}$ be a deGiorgi-Nash form and $\mu \geq 0$ be an admissible measure for $\mathfrak{a}$. Define $\mathfrak{m}$ by* (8). *Then we have:*

1. $\mathfrak{m}$ *is a local sub-Markovian perturbation of* $\mathfrak{a}$.
2. *If the coefficients* a_{ij} *belong to* $W^{1,\infty}(\Omega)$, *then* $\mathfrak{a}+\mathfrak{m}$ *has rich domain.*
3. *If the sets* ω_n *are chosen such that* $H^1_0(\omega_n)=\tilde{H}^1_0(\omega_n)$, *then* $\mathfrak{a}+\mathfrak{m}$ *satisfies the local maximum principle* (*see Definition* 2.8).

Proof. 1. We prove that $\mathfrak{a}+\mathfrak{m}$ is a closed sectorial form. Since J is positivity preserving, the numerical range of $\mathfrak{m}$ is a subinterval of the positive real axis, whence $\mathfrak{a}+\mathfrak{m}$ is sectorial. To see that $\mathfrak{a}+\mathfrak{m}$ is closed, first observe that $\|u\|_{\mathfrak{a}+\mathfrak{m}} \simeq \|u\|_{H^1}+\|\tilde{u}\|_{L^2(d\mu)}$. Hence, given a $\|\cdot\|_{\mathfrak{a}+\mathfrak{m}}$ Cauchy sequence, we see that it is a Cauchy sequence in $(D(\mathfrak{a}),\|\cdot\|_{H^1})$ and in $L^2(\Omega,d\mu)$. By completeness of these spaces, there exist $u\in D(\mathfrak{a})$ and $v\in L^2(\Omega,d\mu)$ such that $u_n\to u$ with respect to $\|\cdot\|_{\mathfrak{a}}$ and $\tilde{u}_n\to v$ with respect to $\|\cdot\|_{L^2(d\mu)}$. Since $\tilde{u}_n\to\tilde{u}$ in $L^2_{\mathrm{loc}}(d\mu)$, we have $\tilde{u}=v$. This proves $u\in D(\mathfrak{a}+\mathfrak{m})$. Clearly, $u_n\to u$ with respect to $\|\cdot\|_{\mathfrak{a}+\mathfrak{m}}$. By condition (A4), $C_c^\infty(\Omega)\subset D(\mathfrak{a})\cap D(\mathfrak{m})$. Hence $\mathfrak{a}+\mathfrak{m}$ is densely defined. That $\mathfrak{a}+\mathfrak{m}$ is sub-Markovian follows from checking the Beurling-Deny criteria. Locality of $\mathfrak{a}+\mathfrak{m}$ follows from that of $\mathfrak{a}$ and (A2).

2. Let $n\in\mathbb{N}$ and choose $\varphi\in C_c^\infty(\omega_n)$ such that $0\le\varphi\le1$ and $\varphi\equiv1$ on ω_{n-1}. Using (A2) – (A4), it is easily seen that multiplication with such a function is a bounded operator on $D(\mathfrak{a}+\mathfrak{m})$. Conditions 1. and 2. in the definition of "rich domain" (see Definition 2.1) for $v=\varphi u$ are obvious. It remains to show that there exists a constant $\tilde{C}_n$, independent of u, such that

$$\|(\mathcal{A}+\mathcal{M})\varphi u\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)'}\le\tilde{C}_n\big(\|u\|_{L^2(\omega_n)}+\|(\mathcal{A}+\mathcal{M})u\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)'}\big)\,. \tag{9}$$

To that end, first observe that

$$(\mathfrak{a}+\mathfrak{m})[\varphi u,w]=(\mathfrak{a}+\mathfrak{m})[u,\varphi w] \tag{10}$$

$$+\int_{\omega_n}u\cdot\left(\sum_{i,j=1}^N a_{ij}D_i\varphi\overline{D_jw}+\sum_{i=1}^N b_i\overline{w}D_i\varphi\right)dx \tag{11}$$

$$+\int_{\omega_n}u\cdot\sum_{i,j=1}^N D_i\big(\overline{w}a_{ij}D_j\varphi\big)\,dx\,. \tag{12}$$

Now let $B:=\{v\in D(\mathfrak{a}+\mathfrak{m},\omega_n):\|v\|_{\mathfrak{a}+\mathfrak{m}}\le1\}$. By definition, we have $\|(\mathcal{A}+\mathcal{M})\varphi u\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)'}=\sup_{w\in B}|(\mathfrak{a}+\mathfrak{m})[\varphi u,w]|$. To estimate this norm, it thus suffices to estimate the absolute value of the terms in (10), (11) and (12). Since multiplication with φ is a bounded operation on $D(\mathfrak{a}+\mathfrak{m})$, there exists a constant $C_{n,1}$ such that

$$\sup_{w\in B}|(\mathfrak{a}+\mathfrak{m})[u,\varphi w]|\le C_{n,1}\|(\mathcal{A}+\mathcal{M})u\|_{D(\mathfrak{a}+\mathfrak{m},\omega_n)}\,.$$

Using the Cauchy-Schwarz inequality, the absolute values of the terms in (11) and (12) may be estimated by $C_{n,2} \cdot \|u\|_{L^2(\omega_n,dx)}$, where $C_{n,2}$ is a constant depending only on the coefficients a_{ij}, b_i and φ. Together, estimate (9) follows.

3. Let $0 \leq \varphi \in D(\mathfrak{a}+\mathfrak{m})'_{\text{loc}}$ and $0 \leq v \in D(\mathfrak{a}+\mathfrak{m})_{\text{loc}}$ with $\lambda v - (\tilde{\mathcal{A}} + \tilde{\mathcal{M}})v = \varphi$ be given. Fix $n \geq 0$ and put $u_n = R(\lambda, (\mathcal{A}+\mathcal{M})_n)\varphi$. By the definition of $D(\mathfrak{a}+\mathfrak{m})_{\text{loc}}$ there exists $v_{n+1} \in D(\mathfrak{a}+\mathfrak{m})$ such that $v = v_{n+1}$ a.e. on ω_{n+1}. We obtain

$$\lambda(u_n - v_{n+1}, w) + (\mathfrak{a}+\mathfrak{m})[u_n - v_{n+1}, w] = \langle \varphi, w\rangle - \langle \varphi, w\rangle = 0\,, \tag{13}$$

for all $w \in D(\mathfrak{a}+\mathfrak{m}, \omega_n)$. Arguing as in the proof of the weak maximum principle (cf [**9**, Theorem 8.1]), this implies

$$\sup_{\omega_n} (u_n - v_{n+1}) \leq \sup_{\partial\omega_n} (u_n - v_{n+1})^+ \;. \tag{14}$$

However, u_n vanishes on the boundary of ω_n, whereas v_{n+1} is positive there, whence $(u_n - v_{n+1})^+ = 0$. Thus, (14) implies that $u_n \leq v_{n+1} = v$ on ω_n. But since u_n vanishes almost everywhere outside ω_n, we have $u_n \leq v$ a.e. on Ω. □

Remark 3.2. *If one drops the requirement that $c \geq 0$ in the definition of deGiorgi-Nash form, then one obtains quasi sub-Markovian forms, i.e. forms $\mathfrak{a}$ such that $\gamma + \mathfrak{a}$ is sub-Markovian for some $\gamma > 0$. All of our theory works also for quasi sub-Markovian forms. However, perturbing a quasi sub-Markovian form by a signed measure μ, i.e. allowing signed measures in (8), one cannot expect $\mathfrak{a}+\mathfrak{m}$ to be quasi sub-Markovian again, unless the negative part μ^- has an L^∞ density with respect to Lebesgue measure. Indeed, the form*

$$(\mathfrak{a}+\mathfrak{m})[u,v] := \int_0^1 u'\overline{v'}\,dx - f(1)\overline{v(1)}$$

is not quasi sub-Markovian.

We are now ready to return to our initial question when the perturbation of a "regular form" is regular again. We recall that an open subset $\Omega \subset \mathbb{R}^N$ is called *Dirichlet regular* if the semigroup generated by the Dirichlet Laplacian Δ_Ω^D on $L^\infty(\Omega)$ leaves the space $C_0(\Omega)$ invariant and if the restriction of the semigroup to $C_0(\Omega)$ is strongly continuous. Here the Dirichlet Laplacian is the operator associated with the deGiorgi-Nash form $\mathfrak{a}$ with $D(\mathfrak{a}) = H_0^1(\Omega)$ and coefficients $a_{ij} = \delta_{ij}$, $b_i = 0$, $c = 0$ for $i,j = 1,\dots,N$. However, this generalizes to much more deGiorgi-Nash forms. Indeed, it is proved in [**5**, Theorem 4.1] that if $\mathfrak{a}$ is any deGiorgi-Nashform with $D(\mathfrak{a}) = H_0^1(\Omega)$ and Ω is Dirichlet regular, then the semigroup T_∞ leaves $C_0(\Omega)$ invariant and the restriction to $C_0(\Omega)$ is strongly continuous, i.e. T_∞ is a *Feller semigroup.*

Theorem 3.4. *Let* $\mathfrak{a}$ *be a deGiorgi-Nash form. Denote the semigroup on* $L^\infty(\Omega)$ *associated to* $\mathfrak{a}$ *by* T_∞ *and by* A_∞ *its generator. If* Ω *has infinite measure, additionally assume that* $a_{ij} \in W^{1,\infty}(\Omega)$*. Let* $\mu \geq 0$ *be an admissible measure for* $\mathfrak{a}$ *and define* $\mathfrak{m}$ *by* (8)*. Now let* P_∞ *be the semigroup on* $L^\infty(\Omega)$ *associated to* $\mathfrak{a} + \mathfrak{m}$ *and denote its generator by* S_∞ (= "$A_\infty - \mu$")*. Then the following hold.*

1. $R(\lambda, S_\infty)L^\infty(\Omega) \subset C_b(\Omega)$ *for every* $\lambda \in \rho(S_\infty)$*.*
2. *If* Ω *is Dirichlet regular and* $D(\mathfrak{a}) = H_0^1(\Omega)$*, then* $R(\lambda, S_\infty)$ *leaves* $C_0(\Omega)$ *invariant for every* $\lambda \in \rho(S_\infty)$*.*

Proof. 1. By Theorem 3.3, $\mathfrak{m}$ is a local sub-Markovian perturbation of $\mathfrak{a}$. In particular, T_∞ exists. By Theorem 3.2, $\mathfrak{m}$ is a local $(\infty, C(\Omega))$-Kato perturbation of $\mathfrak{a}$.

It follows from Theorem 3.1 that $R(\lambda, A_\infty)L^\infty(\Omega) \subset C_b(\Omega)$. Note that for every $u \in D(\mathfrak{a})$ and every $n \in \mathbb{N}$ we find an element of $H_0^1(\omega_{n+1})$ which coincides with u on ω_n – just multiply u with a suitable function in $C_c^\infty(\omega_{n+1})$. Furthermore, if $\mathcal{S}$ denotes the operator associated with $\mathfrak{a} + \mathfrak{m}$ on $D(\mathfrak{a} + \mathfrak{m})'$, then its localized version $\tilde{\mathcal{S}}$ is an extension of S_∞. This is clear if Ω has finite measure, in the other case it follows from Theorem 2.1 since $\mathfrak{a} + \mathfrak{m}$ has rich domain by Theorem 3.3. We also note that by Corollary 3.2, $\mathfrak{a}$ has local kernel belonging to $C(\Omega)$. Hence the hypothesis of Theorem 2.2 is satisfied and we may conclude that $R(\lambda, S_\infty)L^\infty(\Omega) \subset C_b(\Omega)$ for every $\lambda \in \Theta(\mathfrak{a} + \mathfrak{m})^c$ the case for general λ follows from the resolvent equation.

2. By 1. we have $R(\lambda, S_\infty)f \in C_b(\Omega)$ for every $f \in L^\infty(\Omega)$. It remains to prove that $R(\lambda, S_\infty)f \in C_0(\Omega)$ for $f \in C_0(\Omega)$.

Using Propositions 2.20 and 2.21 of [**14**], it is easy to see that $|P_2(t)f| \leq T_2|f|$ for every $f \in L^2(\Omega)$. This relation clearly extends also to the semigroups P_∞ and T_∞. Now take Laplace transforms using the weak*-integral. It follows that

$$|R(\lambda, S_\infty)f| \leq R(\lambda, A_\infty)|f|$$

for all $\lambda > 0$ and $f \in L^\infty$. Let $f \in C_0(\Omega)$. By [**5**, Theorem 4.1], $R(\lambda, A_\infty)|f| \in C_0(\Omega)$ since Ω is Dirichlet regular and $D(\mathfrak{a}) = H_0^1(\Omega)$. Since $R(\lambda, S_\infty)f \in C_b(\Omega)$, the above inequality implies that $R(\lambda, S_\infty)f \in C_0(\Omega)$. □

We end this section by comparing our results with the results of [**15, 19**], where regular perturbations of Dirichlet forms were considered. This also gives wealth of admissible measures which define local $(\infty, C(\Omega))$-Kato perturbations.

In [**15, 19**], the authors consider a regular, symmetric Dirichlet form on

$L^2(M, dm)$, where M is a locally compact Hausdorff space and m is a Radon measure on X with full support. Recall that the assumption that $\mathfrak{a}$ be regular means that $D(\mathfrak{a}) \cap C_c(X)$ is a core for $\mathfrak{a}$ and that $D(\mathfrak{a}) \cap C_c(X)$ is dense in $C_c(X)$ with the supremum norm.

If $X = \Omega$ is a domain in $\mathbb{R}^N$ and m is the Lebesgue measure on Ω, then some deGiorgi-Nash forms fulfill these assumptions. Note that in this case we necessarily have $D(\mathfrak{a}) = H_0^1(\Omega)$.

Associated with a regular, symmetric Dirichlet form $\mathfrak{a}$, there is a Choquet capacity $\mathrm{Cap}_\mathfrak{a}$. Using this capacity, the authors of [**15, 19**] introduce several classes of measures, in particular the class M_0 of measures absolutely continuous with respect to $\mathrm{Cap}_\mathfrak{a}$ and the Kato class S_K.

The authors also consider a local Kato class $S_{K,\mathrm{loc}}$ which is defined by

$$S_{K,\mathrm{loc}} := \{\mu \in M_0 \,:\, \mathbb{1}_K\mu \in S_K \text{ for all compact sets } K \subset X\}\,.$$

Proposition 3.2. *Let $\mathfrak{a}$ be a deGiorgi-Nash form which is also a regular, symmetric Dirichlet form on $L^2(\Omega, dx)$. In particular, $D(\mathfrak{a}) = H_0^1(\Omega)$. Furthermore, let $\mu \in S_{K,\mathrm{loc}}$.*

1. *If $Ju = \tilde{u}$ is the quasi-continuous representative of u, then, with this map J, μ is admissible.*
2. *Define $\mathfrak{m}$ by (8). Then $\mathfrak{m}$ is a $(p, C(\Omega))$-Kato perturbation of $\mathfrak{a}$ for all $p \geq 2$.*

Proof. 1. By the properties of the capacity, J satisfies (A1), (A2) and (A3). This has nothing to do with the measure μ. Now let $\omega \Subset \Omega$ and put $K := \overline{\omega}$. Since $\mathbb{1}_K\mu \in S_K$, it follows from [**19**, Theorem 3.1] that there exists a constant $M = M_K$ such that

$$\int_K |\tilde{u}|^2 \, d\mu \leq M_K \|u\|_\mathfrak{a}^2$$

for all $u \in D(\mathfrak{a})$. This proves that $J(D(\mathfrak{a})) \subset L^2_{\mathrm{loc}}(\Omega, d\mu)$. Furthermore, $\|\tilde{u}\|_{L^2(\mu)} \leq \sqrt{M_K} \cdot \|u\|_\mathfrak{a}$ for all $u \in D(\mathfrak{a}, \omega)$, i.e. (A4) is satisfied.

2. This is immediate from Theorem 3.2. □

Using the notation of Theorem 3.4, the authors of [**15**] prove that if T_∞ is a Feller semigroup and $\mu \in S_{K,\mathrm{loc}}$ then also the perturbed semigroup P_∞ is a Feller semigroup.

In our approach, we obtain from Theorem 3.4 that $R(\lambda, S_\infty)C_0(\Omega) \subset C_0(\Omega)$ whereas consequences for the semigroup have to be proved separately in a second step. On the other hand, we cannot only consider more general forms than in [**15, 19**], but also more general perturbations such as forms which are not associated to a measure (Example 3.2). Furthermore, in some cases we can also prove that P_∞ is a Feller semigroup even though T_∞ is not. This will be done in the next section.

3.3 - *Perturbation by a Potential and Semigroups on C_0*

In this section we address the question whether it is possible to perturb a deGiorgi-Nash form such that associated to the perturbed form there is a strongly continuous semigroup on $C_0(\Omega)$ even though this is not necessarily the case for the unperturbed form. In this section we do not assume that $\Omega \subset \mathbb{R}^N$ is Dirichlet regular.

Theorem 3.5. *Let $\mathfrak{a}$ be a deGiorgi-Nash form with $D(\mathfrak{a}) = H_0^1(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$. If $g \in C^2(\Omega) \cap C_0(\Omega)$ is strictly positive and satisfies*

$$|D^\alpha g| \leq C|g|^{1-|\alpha|} \quad \text{for } 1 \leq |\alpha| \leq 2 \tag{15}$$

for some constant $C \geq 0$, then $V = g^{-2}$ is a local $(\infty, C(\Omega))$-Kato perturbation of $\mathfrak{a}$ and the perturbed semigroup on $L^\infty(\Omega)$ leaves $C_0(\Omega)$ invariant. Furthermore, the restriction of the perturbed semigroup to $C_0(\Omega)$ is strongly continuous.

Proof. We have $V \in L^\infty_{\text{loc}}(\Omega)$. Define $\mathfrak{m}$ defined by (8) for $\mu = Vdx$ and denote by S_∞ the weak*-generator of the semigroup P_∞ associated to $\mathfrak{a} + \mathfrak{m}$ on L^∞. By Theorem 3.4, $R(\lambda, S_\infty)$ leaves the space $C_b(\Omega)$ invariant for every $\lambda > 0$. Theorem 3.3 yields that $\mathfrak{a} + \mathfrak{m}$ satisfies the local maximum principle. It is easy to see that $\mathfrak{a} + \mathfrak{m}$ has abstract Dirichlet boundary conditions. We may hence use Theorem 2.5 to prove invariance of $C_0(\Omega)$.

We try to use $\varphi = g^\gamma$ as a Lyapunov function. Here, γ is a positive constant to be specified later. Then $\varphi \in C^2(\Omega) \cap C_0(\Omega)$ is strictly positive. Using integration by parts, we see that

$$\tilde{\mathcal{A}}\varphi = \sum_{i,j=1}^N a_{ij} D_{ij}\varphi - \sum_{i=1}^N \tilde{b}_i D_i \varphi - c\varphi \,.$$

Here, $\tilde{b}_i$ are modified coefficients depending on b_i and partial derivatives of a_{ij} obtained from integration by parts. Rewriting this in terms of g we have

$$\tilde{\mathcal{A}}\varphi = \gamma g^{\gamma-1} \tilde{\mathcal{A}}_0 g + \gamma(\gamma-1) g^{\gamma-2} \langle C\nabla g, \nabla g\rangle - cg^\gamma \,,$$

where $\tilde{\mathcal{A}}_0 u := \tilde{\mathcal{A}}u + cu$ and C is the matrix containing the entries a_{ij}. Thus, we see that

$$\lambda\varphi - (\tilde{\mathcal{A}} - V)\varphi = g^{\gamma-2}\big((\lambda + c)g^2 - \gamma g \tilde{\mathcal{A}}_0 g - \gamma(\gamma-1)\langle C\nabla g, \nabla g\rangle + 1\big) \,.$$

It follows from the assumptions on g that $g\tilde{\mathcal{A}}_0 g$ is a bounded function, say

$|g\tilde{A}_0 g| \le M$. Choose $0 < \gamma < \min\left\{\frac{1}{2M}, 1\right\}$. Since $\langle C\nabla g, \nabla g\rangle \ge 0$, we obtain

$$\lambda\varphi - \tilde{A}\varphi + V\varphi \ge g^{\gamma-2}\left((\lambda + c)g^2 + \frac{1}{2}\right) \ge 0 .$$

It follows from Theorem 2.5 that $R(\lambda, S_\infty)C_0(\Omega) \subset C_0(\Omega)$ for $\lambda > 0$. Clearly, $C_c^\infty(\Omega) \subset D(\mathfrak{a} + \mathfrak{m})_c \subset D(S_\infty)$. Hence, by Theorem 2.3, P_∞ leaves $C_0(\Omega)$ invariant and the restricted semigroup $P_\infty|_{C_0(\Omega)}$ is strongly continuous. □

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\mathfrak{a}$ be a deGiorgi-Nash form on $H_0^1(\Omega)$ with $a_{ij} \in W^{1,\infty}(\Omega)$. Then there exists a potential $V \in L^\infty_{\mathrm{loc}}(\Omega)$ such that the semigroup P_∞ associated to $\mathfrak{a} + \mathfrak{m}$ leaves $C_0(\Omega)$ invariant and the restriction to that space is strongly continuous.*

Proof. Let $\rho(x) := \inf\{|x - y| : y \in \partial\Omega\}$. Then ρ is Lipschitz continuous, strictly positive and $\rho \in C_0(\Omega)$. It follows from [**18**, Theorem VI.2], that there exists $g \in C^\infty(\Omega)$ such that $c_1\rho \le g \le c_2\rho$ and the estimates (15) hold. Hence Theorem 3.5 yields the thesis for $V = g^{-2}$. □

Thus, if we perturb an operator associated to deGiorgi-Nash forms with $a_{ij} \in W^{1,\infty}$ with a potential which grows near the boundary as the square of the distance to the boundary, then a realization of the perturbed operator on $C_0(\Omega)$ generates a strongly continuous semigroup on $C_0(\Omega)$.

However, not every boundary point of an open set is "bad". Define the "good" boundary Γ_0 by

$$\Gamma_0 := \{x \in \partial\Omega : \exists\, g_x \in L^\infty(\Omega) \text{ strictly positive, such that } P_\infty(t)g_x(y) \to 0 \text{ as } y \to x\ \forall\, t \ge 0\}.$$

If $x \in \Gamma_0$, then we have $P_\infty(t)f(y) \to 0$ as $y \to x$ for all $f \in C_0(\Omega)$ and all $t \ge 0$. Indeed, if g_x strictly positive and $f \in C_c(\Omega)$, then there exists a constant c such that $|f| \le c \cdot g_x$. The positivity of $P_\infty(t)$ yields $|P_\infty(t)f| \le cP_\infty(t)g_x$, whence $P_\infty(t)f(y) \to 0$ as $y \to x$ if $P_\infty(t)g_x(y) \to 0$ as $y \to x$. Now the density of $C_c(\Omega)$ in $C_0(\Omega)$ proves the assertion.

Thus, in order to prove invariance of $C_0(\Omega)$, it remains to take care of the "bad boundary" $\Gamma_1 := \partial\Omega \setminus \Gamma_0$. The question arises, whether it suffices to perturb $\mathfrak{a}$ near Γ_1, or else, to perturb $\mathfrak{a}$ with a potential which grows near the "good boundary" Γ_0 slower than ρ^{-2}. Indeed this is possible as the following consideration shows:

We consider $\mathfrak{a}$ as a form on $M := \Omega \cup \Gamma_0$. Our regularity space X however is unchanged: $X := \{u \in L^1_{\mathrm{loc}}(\Omega) : \exists$ a version of u continuous on $\Omega\}$. The approximating sequence ω_n has to be chosen such that $\bigcup \omega_n = M$, i.e. ω_n has to contain some of the boundary of Ω. However, for $X(\omega_n)$ we still only demand a version continuous in the interior.

EXAMPLE 3.3. *We consider* $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^N$. *Then the "good boundary" is the sphere* $\partial B(0,1)$, *whereas the "bad" boundary consists of the point* 0. *Thus* $M := \{x \in \mathbb{R}^N : 0 < |x| \leq 1\}$. *For localization we consider* $\omega_n := \left\{x \in \mathbb{R}^N : \frac{1}{n} < |x| \leq 1\right\}$ *and*

$$X(\omega_n) = \left\{u \in L^1_{\text{loc}}(M) \,:\, u \text{ has a version continuous on } \frac{1}{n} < |x| < 1\right\}.$$

Thus we have changed what we consider a "compact subsetset of Ω" whereas our notions of continuity remain unchanged (we do *not* require continuity on the boundary). It should be noted, that concerning the Kato-class nothing has changed. Only "local" now means local with respect to M (e.g. $L^\infty_{\text{loc}}(M)$ is the space of functions bounded on compact subsets of M, they may not explode near the good boundary). This change in compact subsets now yields a different space X_0:

$$X_0 := \{u \in C(\Omega) \,:\, u(x) \to 0 \text{ as } x \to \Gamma_1\}.$$

The proofs of Theorem 3.5 remains unchanged when replacing $C_0(\Omega)$ by X_0. Using as g a regular version of $\rho_1(x) = \text{dist}(x, \Gamma_1)$ we see that perturbing with a potential exploding near the bad boundary implies that P_∞ leaves invariant the continuous functions vanishing on the bad boundary. Combining this with the domination result above, we see that $P_\infty C_0(\Omega) \subset C_0(\Omega)$ for such perturbations.

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