Riv. Mat. Univ. Parma, Vol. **1** (2010), 307-319

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Γ-convergence of strongly local Dirichlet functionals

Abstract. We prove the Γ-convergence of strongly local Dirichlet functionals and the convergence of the related minima.

Keywords. Γ-convergence of functionals, convergence of the minima, relaxed Dirichlet problems, Dirichlet forms.

Mathematics Subject Classification (2000): 31C25, 35B65, 35J70.

1 - Introduction

Many authors dealed with the asymptotic behavior of the solutions of Dirichlet problems in perturbed domains (see [**14**] for an initial exhaustive bibliography. See also [**2**] for further developments in the subelliptic case).

More recently M. Biroli and S. Marchi [**3**] treated the case of equations related to Riemannian Dirichlet forms. The authors proved the weak convergence of the solutions of "relaxed" Dirichlet problems, with a potential, described by a suitable nonnegative measure. More generally the result was proved starting from a sequence of relaxed Dirichlet problems.

The aim of this paper is to give a variational motivation to the above result proving by the methods of the Γ-convergence of functionals of the Calculus of Variations (the same method applied in [**14**]), the convergence of the minima of the functionals associated to the related relaxed Dirichlet equations.

Received: September 18, 2009; accepted in revised form: June 18, 2010.

2 - Assumptions and preliminary results

We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that $\text{supp}[m] = X$. Let $\Phi: L^p(X,m) \to [0,+\infty]$, $p > 1$, be a l.s.c. strictly convex functional with domain D, i.e. $D = \{v : \Phi(v) < +\infty\}$, such that $\Phi(0) = 0$. We assume that D is a dense linear subspace of $L^p(X,m)$, which can be endowed with a norm $\|.\|_D$ with respect to which it is a Banach space and the following estimate holds

$$c_1 \|v\|_D^p \le \Phi_1(v) = \Phi(v) + \int_X |v|^p m(dx) \le c_2 \|v\|_D^p$$

for every $v \in D$, where $c_1,\ c_2$ are positive constants. $D \cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on X and D_0 will denote its closure.

A Choquet capacity can be defined, moreover every function in D_0 is quasi-continuous and is defined quasi-everywhere [**7**] and [**3**]. We assume that Φ is a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$ [**7**]. This means in particular that Φ has the following representation on D_0: $\Phi(u) = \int_X \alpha(u)(dx)$ where α is a non-negative bounded Radon measure depending on $u \in D_0$, said the energy measure of Φ, which does not charge sets of zero capacity. The energy $\alpha(u)$ is convex with respect to u in D_0 in the space of measures and it is homogeneous of degree $p > 1$, i.e. $\alpha(tu) = |t|^p \alpha(u)$, $\forall u \in D_0,\ \forall t \in \boldsymbol{R}$ and it is of Markov type.

Assume that for every $u, v \in D_0$ we have

$$\lim_{t \to 0} \frac{\alpha(u + tv) - \alpha(u)}{t} = \mu(u, v)$$

in the weak* topology of $\mathcal{M}$ (where $\mathcal{M}$ is the space of Radon measures on X) uniformly for u, v in a compact set of D_0, where $\mu(u, v)$ is defined on $D_0 \times D_0$. We assume that μ is a Riemannian Dirichlet form. In particular μ is homogeneous of degree $p - 1$ in u and linear in v; it satisfies the chain rule, the truncation property, the Leibniz rule and Young's inequality.

Assume that a distance d could be defined on X, such that $\alpha(d) \le m$ in the sense of the measures and

(i) The metric topology induced by d is equivalent to the original topology of X.

(ii) Denoting by $B(x, r)$ the ball of center x and radius r (for the distance d), for every fixed compact set K there exist positive constants c_0 and r_0 such that

$$m(B(x,r)) \le c_0 m(B(x,s)) \left(\frac{r}{s}\right)^{\nu} \qquad \forall x \in K \quad \text{and} \quad 0 < s < r < r_0. \tag{2.1}$$

We assume without loss of generality $p < \nu$.

From the properties of d the existence of cut-off functions can be proved. A scaled Poincaré inequality is assumed and Sobolev's inequality is proved. We refer the reader to [7] for any detail.

3 - The space of measures $\mathfrak{M}_0^p(\Omega)$

Let Ω be an open subset of X and let $D_o(\Omega)$ be the closure of $D \cap C^o(\Omega)$ in D.

We denote by $\mathfrak{M}_0^p(\Omega)$ the set of all non-negative Borel measures ζ such that

(i) $\zeta(B) = 0$ for every Borel set $B \subset \Omega$ with $p - cap(B, \Omega) = 0$;

(ii) $\zeta(B) = inf\{\zeta(U), U \text{quasi-open}, B \subset U\}$.

Property (ii) is a weak regularity property of the measure ζ. Since any quasi-open set differs from a Borel set by a set of p-capacity zero, then $\zeta(U)$ is well defined when U is quasi-open and ζ satisfies (i), so condition (ii) makes sense.

Let $L^r_\zeta(\Omega)$, $1 \leq r \leq +\infty$ be the usual Lebesgue space with respect to the measure ζ.

If $\zeta \in \mathfrak{M}_0^p(\Omega)$, then the space $D_0(\Omega) \cap L^p_\zeta(\Omega)$ is well defined because the functions in $D_0(\Omega)$ are defined ζ-almost everywhere in Ω. Moreover the space $D_0(\Omega) \cap L^p_\zeta(\Omega)$ is a Banach space for the norm

$$\|u\|^p_{D_0(\Omega)\cap L^p_\zeta(\Omega)} = \|u\|^p_{D_0(\Omega)} + \|u\|^p_{L^p_\zeta(\Omega)}.$$

A non-negative Borel measure which is finite on compact sets of Ω is a non-negative Radon measure on Ω (continuous linear functional on $C_0(\Omega)$). We say that a Radon measure σ belongs to $D^{-1}(\Omega)$ if there exists $f \in D^{-1}(\Omega)$ such that

$$\langle f, \varphi \rangle = \int_\Omega \varphi \, \sigma(dx) \tag{3.1}$$

for every $\varphi \in C_0^\infty(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between $D^{-1}(\Omega)$ and $D_0(\Omega)$. We identify σ and f. We observe that for every non-negative $f \in D^{-1}(\Omega)$ there exists a non-negative Radon measure σ such that (3.1) holds. Moreover every non-negative Radon measure in $D^{-1}(\Omega)$ belongs to $\mathfrak{M}_0^p(\Omega)$.

4 - Relaxed Dirichlet problems and convergence of the minima

Let Ω be a bounded, open subset of X. For any $\zeta_n \in \mathfrak{M}_0^p(\Omega)$, $n = 1, 2, \ldots$ and $\zeta \in \mathfrak{M}_0^p(\Omega)$ and for any $f \in D^{-1}(\Omega)$, let us denote by u_n, u_o the solutions of the relaxed

Dirichlet problems

$$\int_\Omega \mu(u_n, v)(dx) + \int_\Omega |u_n|^{p-2} u_n v \zeta_n(dx) = \langle f, v \rangle \tag{4.1}$$

$u_n \in D_0(\Omega) \cap L^p_{\zeta_n}(\Omega)$, $\forall v \in D_0(\Omega) \cap L^p_{\zeta_n}(\Omega)$,

$$\int_\Omega \mu(u_o, v)(dx) + \int_\Omega |u_o|^{p-2} u_o v \zeta(dx) = \langle f, v \rangle \tag{4.2}$$

$u_o \in D_0(\Omega) \cap L^p_\zeta(\Omega)$, $\forall v \in D_0(\Omega) \cap L^p_\zeta(\Omega)$.

Assuming suitable monotonicity properties for the form μ, and that, for any $u, v \in D_0(\Omega)$, $\mu(u, v)$ has a L^1 local density (in particular this holds for $\alpha(u)$), then the following convergence result holds true [3].

For every $f \in D^{-1}(\Omega)$ and for every sequence $(\zeta_n) \subseteq \mathfrak{M}_0^p(\Omega)$ there exists a subsequence denoted again by (ζ_n) and a measure $\zeta \in \mathfrak{M}_0^p(\Omega)$ such that the sequence of the related solutions (u_n) of (4.1) converges to the solution u_o of (4.2) weakly in $D_0(\Omega)$. Moreover for any open subset U of Ω this holds true also in U.

In this paper we want to give the variational motivation of the former result by proving in Theorem 6.1 that the sequence of the minima (m_n) given by

$$m_n = \min_{D_o(\Omega)} \left[\int_\Omega \alpha(u)(dx) + \int_\Omega |u|^p \zeta_n(dx) - \langle f, u \rangle \right] \tag{4.3}$$

converges to the minimum value m_o given by

$$m_o = \min_{D_o(\Omega)} \left[\int_\Omega \alpha(u)(dx) + \int_\Omega |u|^p \zeta(dx) - \langle f, u \rangle \right]. \tag{4.4}$$

Moreover we prove in Theorem 6.1 that this result is equivalent to the compactness of the set $\mathfrak{M}_0^p(\Omega)$ with respect to the γ-convergence according to the following definitions.

Definition 4.1. *Let Y be an arbitrary metric space. Let G_n, $n = 1, 2, \dots, G$ be functions from Y to $\bar{\mathbb{R}}$. We say that G_n* Γ-converges to G in Y, *and we write*

$$G = \Gamma - \lim_{n \to \infty} G_n$$

in Y, if the following conditions are satisfied:

(a) *for every $u \in Y$ and for every sequence (u_n) converging to u in Y*

$$G(u) \le \liminf_{n \to +\infty} G_n(u_n);$$

(b) *for every $u \in Y$ there exists a sequence (u_n) converging to u in Y such that*

$$G(u) = \lim_{n\to+\infty} G_n(u_n).$$

Let us observe that the Γ-convergence is preserved under addition to G_n, G of a continuous function from Y to $\bar{\mathbb{R}}$. In our context we refer in particular to $Y = L^p(\Omega, m)$ and G given by the following definition.

Definition 4.2. *Let $\zeta \in \mathfrak{M}_0^p(\Omega)$. Then*

$$G_\zeta(u) := \begin{cases} \int_\Omega \alpha(u)(dx) + \int_\Omega |u|^p \zeta(dx) & \text{if } u \in D_o(\Omega) \\ +\infty & \text{elsewhere in } L^p(\Omega, m). \end{cases}$$

Definition 4.3. *We say that $\zeta_n \in \mathfrak{M}_0^p(\Omega)$* γ-converges to $\zeta \in \mathfrak{M}_0^p(\Omega)$ *if the sequence (G_{ζ_n}) Γ-converges to G_ζ according to the Definitions* 4.1 *and* 4.2.

We will prove in Theorem 6.1 the compactness of $\mathfrak{M}_0^p(\Omega)$ under γ-convergence and in Lemma 6.5 that this is equivalent to the convergence of the minima (4.3) to the minimum (4.4).

Remark 4.4. *Let $\zeta \in \mathfrak{M}_0^p(\Omega)$ and let $f \in D^{-1}(\Omega)$. Let us define*

$$G_\zeta(u,f) := \begin{cases} \int_\Omega \alpha(u)(dx) + \int_\Omega |u|^p \zeta(dx) - \langle f, u\rangle & \text{if } u \in D_o(\Omega) \\ +\infty & \text{elsewhere in } L^p(\Omega, m). \end{cases}$$

It is obvious that m_n and m_o in (4.3), (4.4) realize the minima of the functionals $G_{\zeta_n}(\cdot,f)$ and $G_\zeta(\cdot,f)$. Moreover G_{ζ_n} Γ-converges to G_ζ if and only if $G_{\zeta_n}(\cdot,f)$ Γ-converges to $G_\zeta(\cdot,f)$.

5 - Preliminary general results

The results of this section refer to a stronger definition with respect to the Γ-convergence. They are due to [**11**], [**13**], [**14**] and they can be applied in our context with only few changes. We will not prove them but we limit ourselves to indicate the few modifications which we need to adapt them to our setting.

Definition 5.1. *Let Y be an arbitrary metric space. Let (F_n) be a sequence of functions from Y to $\bar{\mathbb{R}}$. Let u be an element of Y and $a^-(u) \in \bar{\mathbb{R}}$. We set*

$$a^-(u) = \Gamma(N^-, Y^-) \lim_{\substack{n\to\infty \\ v\to u}} F_n(v)$$

if and only if

(i) *for every sequence* (u_n) *converging to* u *in* Y

$$a^-(u) \leq \liminf_{n\to+\infty} F_n(u_n);$$

(ii) *there exists a sequence* (u_n) *converging to* u *in* Y *such that*

$$a^-(u) = \lim_{n\to+\infty} F_n(u_n).$$

If $a^+(u) \in \bar{\mathbb{R}}$ *we set*

$$a^+(u) = \Gamma(N^+, Y^-) \lim_{\substack{n\to\infty \\ v\to u}} F_n(v)$$

if the previous conditions (i) *and* (ii) *are satisfied with* lim sup *instead of* lim inf. *Moreover if* $a(u) \in \bar{\mathbb{R}}$ *we set*

$$a(u) = \Gamma(Y^-) \lim_{\substack{n\to\infty \\ v\to u}} F_n(v)$$

if and only if $a(u) = a^-(u) = a^+(u)$.

Let us observe that a sequence of functions (G_n), from the metric space Y to $\bar{\mathbb{R}}$ Γ-converges to the function G in Y (according to Definition 4.1) if and only if for every $u \in Y$

$$G(u) = \Gamma(N^-, Y^-) \lim_{\substack{n\to\infty \\ v\to u}} G_n(v).$$

It is evident from the above Definitions that

$$G(u) = \Gamma(Y^-) \lim_{\substack{n\to\infty \\ v\to u}} G_n(v)$$

for every $u \in Y$, is a stronger condition with respect to

$$G = \Gamma - \lim_{n\to\infty} G_n$$

in Y, according to Definition 4.1.

We shall denote by $\mathcal{A}$ the class of open subsets of Ω and by $\mathcal{B}$ the class of Borel subsets of Ω.

Definition 5.2. *Let* π *be a function from* $\mathcal{B}$ *into* $[0, +\infty[$. *We shall denote by* $\mathcal{F}_\pi$ *the class of functionals* $F : L^p(\Omega, m) \times \mathcal{B} \to [0, +\infty]$ *with the following properties:*

(a) *for each* $u \in L^p(\Omega, m)$ *the function* $B \to F(u, B)$ *is increasing on* $\mathcal{B}$ *and* $F(u, \emptyset) = 0$*;*

(b) *if* $u, v \in L^p(\Omega, m)$*,* $A \in \mathcal{A}$*,* $u|_A = v|_A$ *a.e. on* A*, then* $F(u, B) = F(v, B)$ *for every* $B \in \mathcal{B}$ *with* $B \subseteq A$*;*

(c) *for each* $B \in \mathcal{B}$ *and each* $A \in \mathcal{A}$*, with* $B \subseteq A$*, the function* $u \to F(u, B)$ *is convex and lower semicontinuous on* $D(A)$*;*

(d) *for every* $A \in \mathcal{A}$

$$\inf_{u \in D_o(\Omega)} \left[\|u\|^p_{D(\Omega)} + F(u, A) \right] \leq \pi(A);$$

(e) *there exists a functional* $\tilde{F} : L^p(\Omega, m) \times \mathcal{B} \to [0, +\infty]$ *such that for every* $u \in L^P(\Omega, m)$ *the function* $B \to \tilde{F}(u, B)$ *is countably additive on* $\mathcal{B}$*, and for every* $u \in L^p(\Omega, m)$*,* $A \in \mathcal{A}$ *we have* $F(u, A) = \tilde{F}(u, A)$*;*

(f) *for every* $A \in \mathcal{A}$ *and every* $u, v, w \in D(A)$*, with* $w \geq 0$ *a.e. on* A

$$F((u \wedge v) + w, A) + F(u \vee v, A) \leq F(u + w, A) + F(v, A).$$

EXAMPLE 5.3. Let ζ, ν be positive Radon measures, with $\zeta \in D^{-1}(\Omega)$ and let $g : \Omega \times \mathbb{R} \to [0, +\infty]$ be a Borel function, with $g(x, t)$ convex and lower semicontinuous in t for each $x \in \Omega$. For every $u \in L^p(\Omega, m)$ and for every $B \in \mathcal{B}$ we set

$$F(u, B) = \int_B g(x, \tilde{u}(x)) \zeta(dx) + \nu(B)$$

where $\tilde{u}$ is the quasi-continuous representative of u. If there exists $u \in D(\Omega)$ such that $F(u, A) < +\infty$ for any $A \in \mathcal{A}$, then F belongs to the class $\mathcal{F}_\pi$ for a suitable function $\pi : \mathcal{A} \to [0, +\infty[$ [**11**, Remark 2.5].

Definition 5.4. *A subset* $\tilde{\mathcal{B}}$ *of* $\mathcal{B}$ *is said to be* rich in $\mathcal{B}$ *if, for every family* (B_t)*,* $t \in \mathbb{R}$*, of elements of* $\mathcal{B}$*, with* $\mathcal{B}_s \subset\subset \dot{\mathcal{B}}_t$ *for* $s < t$*, the set* $\{t \in \mathbb{R} : B_t \notin \tilde{\mathcal{B}}\}$ *is at most countable.*

Theorem 5.5 (Compactness of $\mathcal{F}_\pi$). *For every sequence* (F_n) *of functionals of the class* $\mathcal{F}_\pi$ *there exists a subsequence* (F_{n_k})*, a functional* F *of the class* $\mathcal{F}_\pi$ *and a subset* $\tilde{\mathcal{B}}$ *of* $\mathcal{B}$*, rich in* $\mathcal{B}$*, such that*

$$\int_A \alpha(u)(dx) + F(u, B) = \Gamma(L^p(A, m)^-) \lim_{\substack{k \to \infty \\ v \to u}} \left[\int_A \alpha(v)(dx) + F_{n_k}(v, B) \right] \tag{5.1}$$

for every $A \in \mathcal{A}$*,* $u \in D(A)$ *and for every* $B \in \tilde{\mathcal{B}}$ *with* $B \subset\subset A$ *or with* $B = A$*.*

The proof of Theorem 5.5 is a slight modification of the proof of Theorem 2.10 of [**11**].

Theorem 5.6 (Integral representation). *For each functional F of the class $\mathcal{F}_\pi$ there exist*

(i) *two positive Radon measures ζ and ν, with $\zeta \in D^{-1}(\Omega)$;*

(ii) *a Borel function $g : \Omega \times \mathbb{R} \to [0, +\infty]$, with $t \to g(x,t)$ convex and lower semicontinuous on $\mathbb{R}$ for each $x \in \Omega$;*

(iii) *a subset $\tilde{\mathcal{B}}$ of $\mathcal{B}$, rich in $\mathcal{B}$ and containing $\mathcal{A}$ such that for every $A \in \mathcal{A}$, $u \in D(A)$ we have*

$$F(u,B) = \int_B g(x, \tilde{u}(x))\, \zeta(dx) + \nu(B) \tag{5.2}$$

for every $B \in \tilde{\mathcal{B}}$ with $B \subset\subset A$ or with $B = A$.

Theorem 5.6 can be obtained as in [**11**, Theorem 3.1], that is from Lemmas 4.5, 4.6 of [**13**] and from Theorem 3.3 of [**11**].

Theorem 5.7 (Convergence of the minima). *Let (F_n) be a sequence of functionals of the class $\mathcal{F}_\pi$ and let F be a functional of the class $\mathcal{F}_\pi$ such that*

$$\int_A \alpha(u)\,(dx) + F(u,B)$$
$$= \Gamma(L^p(\Omega,m)^-) \lim_{\substack{n\to\infty \\ v\to u}} \left[\int_A \alpha(v)(dx) + F_n(v,B) \right]$$

for all $A \in \mathcal{A}$, $u \in D(A)$, $B \in \tilde{\mathcal{B}}$, with $B \subset\subset A$ or with $B = A$.

Let the integral representation of F described in Theorem 5.6 be in force. Let $f \in D^{-1}(\Omega)$.

Let (v_n) be a sequence in $D(\Omega)$ converging in $L^p(\Omega, m)$ to an element $v_o \in D(\Omega)$.

Suppose that there exists a compact subset K of Ω such that

$$\limsup_{n\to\infty} \left[\int_{\Omega\setminus K} \alpha(v_n)\,(dx) + F_n(v_n, \Omega \setminus K) \right] < +\infty.$$

For every $n \in \mathbb{N}$, let u_n be the minimum point of the functional

$$\int_\Omega \alpha(u)(dx) + F_n(u,\Omega) - \langle f, u \rangle$$

on the set $\{u \in D(\Omega) : u - v_n \in D_o(\Omega)\}$.

Then u_n converges in $L^p(\Omega, m)$ (and weakly in $D(\Omega)$) to the minimum point u_o of the functional

$$\int_\Omega \alpha(u)(dx) + F(u, \Omega) - \langle f, u \rangle$$

on the set $\{u \in D(\Omega) : u - v_o \in D_o(\Omega)\}$. Moreover if in addition

$$\limsup_{n\to\infty} \left[\int_{\Omega\setminus K} \alpha(v_n)\,(dx) + F_n(v_n, \Omega \setminus K) \right] \leq \int_{\Omega\setminus K} \alpha(v_o)\,(dx) + F(v_o, \Omega \setminus K)$$

then

$$\int_\Omega \alpha(u_o)(dx) + F(u_o, \Omega) - \langle f, u_o \rangle = \lim_{n\to\infty} \left[\int_\Omega \alpha(u_n)(dx) + F_n(u_n, \Omega) - \langle f, u_n \rangle \right].$$

Theorem 5.7 is proved in [**11**, Corollary 4.4]. For sake of completeness we stated it in the above general form. But we are interested to apply it in the particular case of $v_n = v_o = 0$ and functionals

$$F_n(v, E) = \int_E |v|^p \zeta_n(dx)$$

which belong to the class $\mathcal{F}_\pi$ as we observed in the Example 5.3.

6 - The results

Let us return to the Γ-convergence described in Definition 4.1. Our aim in this Section is to prove the following result about the compactness of the set of measures $\mathfrak{M}_0^p(\Omega)$ with respect to the γ-convergence, Definition 4.3. As a byproduct we will obtain the convergence of the sequence of minima (4.3) to the minimum (4.4).

Theorem 6.1 (Compactness of $\mathfrak{M}_0^p(\Omega)$ and convergence of the minima). *For every sequence (ζ_n) in $\mathfrak{M}_0^p(\Omega)$ there exists a subsequence still denoted by (ζ_n) and a* *measure* $\zeta \in \mathfrak{M}_0^p(\Omega)$ *such that*

$$\text{(6.1)} \qquad \zeta_n \ \gamma - \textit{converges} \quad \textit{to} \quad \zeta \ \textit{in} \ \mathfrak{M}_0^p(\Omega).$$

This is equivalent to

(6.2) *the sequence of minima* (4.3) *converges to the minimum* (4.4).

Moreover the minimum point u_n *in* $D_o(\Omega)$ *of the functional on the right hand side of* (4.3) *converges in* $L^p(\Omega, m)$, *weakly in* $D_o(\Omega)$, *to the minimum point* u_o *in* $D_o(\Omega)$ *of the functional on the right hand side of* (4.4).

The proof is similar to that of [**14**, Theorem 4.14] but we describe it for convenience of the reader.

Lemma 6.2. *Let* Y *be an arbitrary metric space. Let* (G_n), $n \in \mathbb{N}$ *be a sequence of functions from* Y *to* $\bar{\mathbb{R}}$ *which* Γ*-converges in* Y *to a function* G *and let* H *from* Y *to* $\mathbb{R}$ *be a continuous function. Suppose that for every* $t \in \mathbb{R}$ *there exists a compact set* $K_t \subseteq Y$ *such that*

$$\{v \in Y \ : \ G_n(v) + H(v) \le t\} \subseteq K_t$$

for every $n = 1, 2, \ldots$ *Then* $G + H$ *attains its minimum in* Y *and*

$$\lim_{n \to +\infty} \inf_Y [G_n(v) + H(v)] = \min_Y [G(v) + H(v)].$$

Suppose in addition that each function $G_n + H$ *attains a minimum point* u_n *in* Y *and* $G + H$ *attains a unique minimum point* u *in* Y. *Then* (u_n) *converges to* u *in* Y.

Lemma 6.2 is proved in [**15**, Corollary 2.4] (see also [**14**, Proposition 4.2]). The following Lemma is proved in [**1**, Corollary 3.13] (see also [**14**, Proposition 4.3]).

Lemma 6.3. *Let* Y, Z *be Banach spaces, with* $Z \subset Y$ *and* Z *dense in* Y. *Suppose that* Z *is reflexive and separable, and that the imbedding of* Z *into* Y *is compact. For every* $n \in \mathbb{N}$ *let* G_n *be a function from* Y *to* $\bar{\mathbb{R}}$ *which is convex and such that* $G_n(u) \ge \|u\|_Z^p$ *for every* $u \in Z$ *and* $G_n(u) = +\infty$ *for every* $u \notin Z$. *Suppose that there exists a lower semicontinuous convex function* G *from* Y *to* $\bar{\mathbb{R}}$ *such that*

$$\lim_{n \to +\infty} \inf_Y [G_n(v) + H(v)] = \min_Y [G(v) + H(v)]$$

for every continuous linear function H *from* Y *to* $\mathbb{R}$. *Then* G_n Γ*-converges to* G *in* Y.

Remark 6.4. *In Lemma* 6.3 *the condition* $G_n(u) = +\infty$ *for every* $u \notin Z$ *reduces* $\inf_Y [G_n(v) + H(v)]$ *to* $\inf_Z [G_n(v) + H(v)]$ *and* $\min_Y [G(v) + H(v)]$ *to* $\min_Z [G(v) + H(v)]$. *Moreover* H *could be a continuous linear function from* Z *to* $\mathbb{R}$.

The same considerations hold about Lemma 6.2 *if the condition* $G_n(u) = +\infty$ *for every* $u \notin Z$ *is required in addition.*

Lemma 6.5. *A sequence* (ζ_n) *in* $\mathfrak{M}_0^p(\Omega)$ γ*-converges to* $\zeta \in \mathfrak{M}_0^p(\Omega)$ *if and only if for every* $f \in D^{-1}(\Omega)$ *the sequence of minima* (4.3) *converges to the minimum* (4.4).

Proof. It is enough to apply Lemma 6.2, Lemma 6.3 and Remark 6.4 with $Y = L^p(\Omega, m)$ and $Z = D_o(\Omega)$. Moreover $G = G_\zeta$, $G_n = G_{\zeta_n}$ of Definition 4.2 and $H(v) = -\langle f, v \rangle$.

Proof of Theorem 6.1. Given an arbitrary sequence (ζ_n) in $\mathfrak{M}_0^p(\Omega)$, let (F_n) be the sequence of functions from $L^p(\Omega, m)$ to $\bar{\mathbb{R}}$ of the class $\mathcal{F}_\pi$ associated to (ζ_n) defined by

$$F_n(u) = \int_\Omega |u|^p \zeta_n(dx).$$

In virtue of Theorem 5.5 there exists a subsequence of (ζ_n), still denoted by (ζ_n) and a function F of the class $\mathcal{F}_\pi$ such that (5.1) is satisfied.

In virtue of Theorem 5.6 there exists a positive Radon measure $\tilde{\zeta} \in D^{-1}(\Omega)$, a Borel function $g : \Omega \times \mathbb{R} \to [0, +\infty]$, with $t \to g(x, t)$ convex and lower semicontinuous on $\mathbb{R}$ for each $x \in \Omega$, such that (5.2) is satisfied.

In virtue of Theorem 5.7, where $v_n = v_o = 0$, for every $f \in D^{-1}(\Omega)$ the sequence of minima (4.3) converges to the minimum value in $D_o(\Omega)$ of the functional

$$\int_\Omega \alpha(u) m(dx) + \int_\Omega g(x, u) \tilde{\zeta}(dx) - \langle f, v \rangle \tag{6.3}$$

and the minimum point u_n in $D_o(\Omega)$ of the functional on the right hand side of (4.3) converges in $L^p(\Omega, m)$, weakly in $D_o(\Omega)$, to the minimum point u_o in $D_o(\Omega)$ of the functional (6.3).

The proof is concluded if we prove that $g(x, u)\tilde{\zeta}(dx) = |u|^p \zeta(dx)$ for a suitable measure $\zeta \in \mathfrak{M}_0^p(\Omega)$.

Applying Lemma 6.3, Remark 6.4 with $Y = L^p(\Omega, m)$ and $Z = D_o(\Omega)$, it follows that the sequence of functionals

$$G_{\zeta_n}(u) = \begin{cases} \displaystyle\int_\Omega \alpha(u)(dx) + \int_\Omega |u|^p \tilde{\zeta}_n(dx) & \text{if } u \in D_o(\Omega) \\ +\infty & \text{elsewhere in } L^p(\Omega, m) \end{cases}$$

Γ-converges to the functional $\tilde{G}_{\tilde{\zeta}}$ defined by

$$\tilde{G}_{\tilde{\zeta}}(u) = \begin{cases} \int_{\Omega} \alpha(u)(dx) + \int_{\Omega} g(x,u)\tilde{\zeta}(dx) & \text{if } u \in D_o(\Omega) \\ +\infty & \text{elsewhere in } L^p(\Omega, m). \end{cases}$$

Let us observe that the functionals G_{ζ_n} are homogeneous of degree p, and this property is preserved by Γ-convergence.

Therefore the functional $\tilde{G}_{\tilde{\zeta}}$ is homogeneous of degree p. It follows that the functional

$$u \to \int_{\Omega} g(x,u)\tilde{\zeta}(dx)$$

is homogeneous of degree p on $D_o(\Omega)$. Therefore, using an approximation argument and Eulero's Theorem, it results of the form

$$\int_{\Omega} g(x,u)\zeta(dx) = \int_{\Omega} a(x)|u|^p\tilde{\zeta}(dx)$$

for a suitable Borel function $a : \Omega \to [0, +\infty]$.

Now it is enough to define $\zeta = a\tilde{\zeta}$. It is obvious that $\zeta \in \mathfrak{M}_0^p(\Omega)$.

The equivalence between (6.1) and (6.2) follows from Lemma 6.5.

References

[1] H. ATTOUCH, *Variational convergence for functions and operators*, Pitman, Boston, MA 1984.

[2] M. BIROLI, C. PICARD and N. TCHOU, *Asymptotic behavior of some nonlinear subelliptic relaxed Dirichlet problems*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **26** (2002), 55-113.

[3] M. BIROLI and S. MARCHI, *Asymptotic behavior of relaxed Dirichlet problems related to p-homogeneous strongly locals forms*, preprint.

[4] M. BIROLI and U. MOSCO, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. (4) **169** (1995), 125-181.

[5] M. BIROLI and U. MOSCO, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **6** (1995), no. 1, 37-44.

[6] M. BIROLI and U. MOSCO, *Sobolev inequalities on homogeneous spaces*, Potential Anal. **4** (1995), 311-324.

[7] M. BIROLI and P. VERNOLE, *Strongly local nonlinear Dirichlet functionals and forms*, Adv. Math. Sci. Appl. **15** (2005), 655-682.

[8] M. BIROLI and P. VERNOLE, *Harnack inequality for harmonic functions relative to a nonlinear p-homogeneous Riemannian Dirichlet form*, Nonlinear Analysis **64** (2006), 51-68.

[9] A. BRAIDES, *Γ-convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, 22, Oxford University Press, Oxford 2002.

[10] R. R. COIFMAN and G. WEISS, *Analyse harmonique non-commutative sur certaines espaces homogènes*, Lectures Notes in Math., 242, Springer-Verlag, Berlin-New York 1971.

[11] G. DAL MASO, *Asymptotic behaviour of minimum problems with bilateral obstacles*, Ann. Mat. Pura Appl. (4) **129** (1981), 327-366.

[12] G. DAL MASO, *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications, 8, Birkhäuser Boston, Inc., Boston, MA 1993.

[13] G. DAL MASO and P. LONGO, *Γ-limits of obstacles*, Ann. Mat. Pura Appl. (4) **128** (1981), 1-50.

[14] G. DAL MASO and U. MOSCO, *Wiener's criterion and Γ-convergence*, Appl. Math. Optim. **15** (1987), 15-63.

[15] E. DE GIORGI and T. FRANZONI, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), 842-850, and, Rend. Sem. Mat. Brescia **3** (1979), 63-101.

[16] M. FUKUSHIMA, Y. ŌSHIMA and M. TAKEDA, *Dirichlet forms and symmetric Markov processes*, de Gruyter Studies in Mathematics, 19, Walter de Gruyter & Co., Berlin 1994.

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