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A variational problem involving the distributional determinant

Abstract. We deal with bounded $W^{1,n-1}$-maps defined in n-dimensional domains, whose graph has finite area and finite boundary mass. We show that the singular part of the distributional determinant is concentrated on a countable set of points. A related variational problem is then considered. Finally, we study the analogous problem involving the distributional minors of fixed order.

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Let $u : \Omega \to \mathbb{R}^n$ be a vector valued map defined on an n-dimensional bounded domain $\Omega \subset \mathbb{R}^n$. If u is essentially bounded and belongs to the Sobolev space $W^{1,n-1}(\Omega, \mathbb{R}^n)$, the *distributional determinant* $\operatorname{Det} \nabla u$, introduced by J.M. Ball in [**4**], is well-defined in the distributional sense by

$$\operatorname{Det} \nabla u := \frac{1}{n} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(u^j \, (\operatorname{adj} \nabla u)_i^j \right), \tag{0.1}$$

where $\nabla u = Du$ and $\operatorname{adj} \nabla u$ is the matrix of the adjoints of Du.

In [**13**] S. Müller and S. Spector studied the distributional determinant in the setting of a theory for nonlinear elasticity. They showed in particular that if u satisfies the so called "INV condition", the pointwise determinant $\det Du > 0$ a.e. in Ω and the "geometric image" $\operatorname{im}(u, \Omega)$ of u has finite perimeter, then the distributional determinant is a non-negative Radon measure, with absolute continuous part equal to the pointwise determinant, compare [**11**]; also, the singular part $(\operatorname{Det} Du)^s$ is concentrated in a countable set $\{x_l\} \subset \Omega$ and

$$\operatorname{Det} Du = \det Du \cdot dx + \sum_{l=1}^{\infty} c_l \, \delta_{x_l} \,, \tag{0.2}$$

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where $c_l \geq 0$ and δ_x is the unit Dirac mass at x. Furthermore

$$\sum_{l=1}^{\infty} c_l^{(n-1)/n} \leq C(n) \cdot \mathrm{Per}(\mathrm{im}(u,\Omega)) < \infty\,,$$

where $C(n) > 0$ is the isoperimetric constant. This representation result describes the so called *cavitations* at the points x_i.

In order to show that the singular part of the distributional determinant $(\mathrm{Det}\, Du)^s$ does not contain a diffuse part, let us say a "Cantor-type" part, in [**13**] it is made use of the following isoperimetric inequality for Caccioppoli sets. If $U := \mathrm{im}(u,\Omega)$, and $V := \mathrm{im}_T(u, B_r(x))$ denotes the topological image of the n-ball of radius r centered at x, then for a.e. $r > 0$

$$\begin{aligned} (\mathrm{Det}\, Du)^s(B_r(x)) &= \mathcal{L}^n(V \setminus U) \\ \mathcal{L}^n(V \setminus U)^{(n-1)/n} &\leq C(n)\, \mathcal{H}^{n-1}(\partial^* U \cap V) \end{aligned}$$

where $\partial^* U$ is the reduced boundary of U, see [**2**].

On the other hand, it was proved in Müller [**12**] that the singular part of the distributional determinant may in general concentrate on a set of Hausdorff measure α, for any prescribed $0<\alpha<n$. More precisely, e.g. in the case $n = 2$, there exists a bounded function $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ for every $p<2$, where $\Omega = (0,1)^2 \subset \mathbb{R}^2$, such that $\det Du = 0$ and $|Du^1|\,|Du^2| = 0$ a.e. in Ω, but

$$\mathrm{Det}\, Du = V' \otimes V'\,,$$

where V is the Cantor-Vitali function. Therefore, the derivatives of u have no masses, but the distributional determinant has a Cantor-type part and the role played by V' in the Cantor set C is played here by $\mathrm{Det}\, Du$ in $C \times C$. The "graph" of u is very similar to the graph of the Cantor-Vitali function V and, actually, has infinitely many holes; for instance, in [**8** Vol. I, Sec. 4.2.5] it is shown that the *current* G_u associated to the graph of u has in fact a boundary of infinite mass, $\boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^2) = \infty$.

The main new result of this paper is contained in Sec. 2, where we prove an isoperimetric inequality related to the singular part of the distributional determinant. According to the previously cited example by Müller [**12**], we shall assume that the graph-current G_u has a *boundary of finite mass*, i.e.,

$$\boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^n) < \infty\,, \tag{0.3}$$

and we shall obtain that for balls $B_r \subset\subset \Omega$

$$|(\mathrm{Det}\, \nabla u)^s|(\overline{B}_r) \leq c_n\, \boldsymbol{M}((\partial G_u) \llcorner \overline{B}_r \times \mathbb{R}^n)^{n/(n-1)}\,.$$

We point out that this isoperimetric inequality is false in general under the hypotheses as in [9, Prop. 3], see [**10**] and Remark 3.2 below. However, see Corollary 2.2, it holds true provided that u is a bounded function in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ and with summable pointwise determinant, $\det Du \in L^1(\Omega)$. Actually, it will be proved for a wider class of maps, see Proposition 2.1.

Our isoperimetric inequality is obtained without assuming additional hypotheses as the "INV condition", or the positivity of the pointwise determinant. In our setting, condition (0.3) plays the role of the property from [**13**] that the "geometric image" $\mathrm{im}(u,\Omega)$ of u has finite perimeter, compare [**5**].

We shall then make use of some results obtained in [**9**].

Firstly, in Sec. 3, we shall recover the representation formula (0.2), where this time the coefficients c_l are possibly negative real numbers. Notice that the (at most) countable set of points where the singular part of the distributional determinant $(\mathrm{Det}\, Du)^s$ concentrates may not be finite. More precisely, setting

$$S_0(u) := \{x_l \in \Omega \mid c_l \neq 0 \text{ in } (0.2)\},$$

in general we have $\mathcal{H}^0(S_0(u)) \leq +\infty$, whereas the total variation of $(\mathrm{Det}\, Du)^s$ satisfies

$$|(\mathrm{Det}\, Du)^s|(\Omega) = \sum_{l=1}^{\infty} |c_l| < \infty .$$

Secondly, in Sec. 4, we shall deal with some related variational problems. To this purpose, we denote by $|\overrightarrow{M}(Du)|^2$ the sum of the squares of the determinants of all the minors of Du. We will prove the existence of the minimum of functionals of the type

$$\mathcal{F}(u) := \int_\Omega \Phi(|\overrightarrow{M}(Du)|)\, dx + \int_\Omega |Du|^{n-1}\, dx + \mathcal{H}^0(S_0(u)),$$

where Φ is e.g. a non negative convex function satisfying a p-coercivity condition

$$c\,|t|^p \leq \Phi(t), \qquad p > 1, \quad c > 0 .$$

Roughly speaking, the minimum is attained on classes of functions of the type

$$\{u \in W^{1,n-1}(\Omega,\mathbb{R}^n) \cap L^\infty \mid \det Du \in L^1(\Omega),\ \|u\|_\infty + \boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^n) \leq K\},$$

for $K > 0$ fixed, under suitable Dirichlet-type boundary conditions.

In Sec. 5, using a slicing argument, we shall finally treat the analogous problem about the *distributional minors* of fixed order $k = 2, \ldots, \min(n, N)$, for vector valued Sobolev maps in $W^{1,k-1}(\Omega,\mathbb{R}^N)$.

1 - Notation and preliminary results

In this preliminary section we collect some notation and results. We refer to [**7, 14**] for general facts about geometric measure theory, and to [**8**, Vol. I] for further details.

Rectifiable sets.

Let U an open set in $\mathbb{R}^m$ and $\mathcal{H}^k$ the k-dimensional Hausdorff measure on $\mathbb{R}^m$. We say that $\mathcal{M} \subset U$ is *countably $\mathcal{H}^k$-rectifiable* if it is $\mathcal{H}^k$-measurable and $\mathcal{H}^k$-almost all of $\mathcal{M}$ is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^k$ to U, compare [**7,** 3.2.14]. Also, $\mathcal{M}$ is said to be k-rectifiable if moreover $\mathcal{H}^k(\mathcal{M}) < \infty$.

Rectifiable currents.

A current $T \in \mathcal{D}_k(U)$ is said to be of the type $(\mathcal{M}, \theta, \xi)$, $T = \tau(\mathcal{M}, \theta, \xi)$, if T acts on smooth compactly supported k-forms ω as

$$T(\omega) = \int_{\mathcal{M}} \langle \xi(x), \omega(x) \rangle \theta(x)\, d\mathcal{H}^k(x) \qquad \forall\, \omega \in \mathcal{D}^k(U)\,, \tag{1.1}$$

where $\mathcal{M} \subset U$ is countably $\mathcal{H}^k$-rectifiable, the multiplicity $\theta : \mathcal{M} \to]0, +\infty]$ is $\mathcal{H}^k$-measurable and locally $(\mathcal{H}^k \llcorner \mathcal{M})$-summable and $\xi : \mathcal{M} \to \Lambda_k \mathbb{R}^m$ is $\mathcal{H}^k$-measurable with $|\xi| = 1$ $(\mathcal{H}^k \llcorner \mathcal{M})$-a.e.. Moreover we will denote

$$\text{set}(T) := \{x \in \mathbb{R}^m \mid \theta^k(\|T\|, x) > 0\}$$

and *size* $\boldsymbol{S}$ of T the number

$$\boldsymbol{S}(T) := \mathcal{H}^k(\text{set}(T)) = \mathcal{H}^k(\mathcal{M})\,,$$

so that

$$\boldsymbol{S}_V(T) := \boldsymbol{S}(T \llcorner V) \qquad \forall\, V \quad \text{open}, \quad V \subset U\,.$$

We say that T is a *rectifiable current*, $T \in \boldsymbol{R}_k(U)$, if T has finite mass, $\boldsymbol{M}(T) < \infty$, and for $\mathcal{H}^k$-a.e. $x \in \mathcal{M}$ the unit k-vector $\xi(x) \in \Lambda_k \mathbb{R}^m$ provides an orientation to the approximate tangent space $\text{Tan}^k(\mathcal{M}, x)$. If moreover the size of T is bounded, i.e., $\boldsymbol{S}(T) < \infty$, we say that T is a *size bounded rectifiable current*, $T \in \mathcal{S}_k(U)$. If in particular the density θ takes integer values, we say that $T = \tau(\mathcal{M}, \theta, \xi)$ is an *integer multiplicity (i.m) rectifiable current*, $T \in \mathcal{R}_k(U)$.

We finally denote by $\mathcal{N}_k(U)$ the class of *normal* currents, i.e., of k-currents with finite mass and finite boundary mass, $\boldsymbol{N}(T) := \boldsymbol{M}(T) + \boldsymbol{M}(\partial T) < \infty$.

Lower semicontinuity of the size.

For our purposes, we recall a lower semicontinuity result for the size of suitable rectifiable currents with respect to a suitable flat convergence. This result was proved in [**9**, Sec. 6], using some ideas from Almgren [**1,** Prop. 2.10], see also the closure theorems in [**3**, Thm. 8.5] and [**6**]. For $T_1,\, T_2 \in \boldsymbol{R}_k(U) \cap \mathcal{N}_k(U)$, define the "flat" metric

$$d(T_1, T_2) := \inf\{\boldsymbol{M}(Q) + \boldsymbol{M}(R) \mid Q \in \mathcal{N}_k(U)\,, \quad R \in \mathcal{N}_{k+1}(U) \\ T_1 - T_2 = Q + \partial R\}\,.$$

Theorem 1.1. *Let $U \subset \mathbb{R}^n$ a bounded and open set. Let T, $\{T_h\} \subset \boldsymbol{R}_k(U) \cap \mathcal{N}_k(U)$ be such that:* (i) $\sup_h \boldsymbol{N}(T_h) < \infty$, *and* (ii) $d(T_h, T) \to 0$ *as $h \to \infty$. Then for every open set $V \subset U$ we have*

$$\boldsymbol{S}_V(T) \le \liminf_{h\to\infty} \boldsymbol{S}_V(T_h)\,.$$

Notation for multi-indices.

Let $n, N \ge 2$ integer. In the sequel, if G is an $(N \times n)$-matrix, β and α will always denote the multi-indices of row and column of G, respectively. If e.g. $\alpha = (\alpha_1, \dots, \alpha_p)$ is a multi-index of length $|\alpha| = p \le n$, with $\alpha_i \in \{1, \dots, n\}$, we say that the positive integer i belongs to α if it is one of the indices $\alpha_1, \dots, \alpha_p$. If $i \in \alpha$ we denote by $\alpha - i$ the multi-index of length $p - 1$ obtained by removing i from α. Also, $\overline{\alpha}$ is the complement of α in $(1, \dots, n)$ and $\sigma(\alpha, \overline{\alpha})$ is the sign of the permutation which reorders α and $\overline{\alpha}$. A similar notation holds for β, with n replaced by N. Then G^β_α denotes the submatrix obtained by selecting the rows and columns by β and α, respectively. For example, if $|\alpha| + |\beta| = n$, then $G^\beta_{\overline{\alpha}}$ is a square matrix and we will denote by $M^\beta_{\overline{\alpha}}(G)$ its determinant

$$M^\beta_{\overline{\alpha}}(G) := \det G^\beta_{\overline{\alpha}}\,,$$

and we set $M^0_0(G) := 1$.

Currents carried by graphs.

Let $\Omega \subset \mathbb{R}^n$ be an n-dimensional bounded domain. We shall denote by $\mathcal{A}^1(\Omega, \mathbb{R}^N)$ the class of vector-valued maps $u : \Omega \to \mathbb{R}^N$ that are a.e. approximately differentiable and such that all the minors of the Jacobian matrix ∇u are summable. We also let $\overrightarrow{M}(\nabla u)$ denote the n-vector that spans the graph of the linear maps associated to

∇u at a given point. More precisely,

$$\vec{M}(\nabla u) := \left(e_1, \sum_{j=1}^{N} \nabla_1 u^j \,\varepsilon_j\right) \wedge \cdots \wedge \left(e_n, \sum_{j=1}^{N} \nabla_n u^j \,\varepsilon_j\right),$$

$(e_1, \ldots, e_n)$, $(\varepsilon_1, \ldots, \varepsilon_N)$ being the standard basis in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, so that

$$\vec{M}(\nabla u) = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \overline{\alpha})\, M_{\overline{\alpha}}^{\beta}(\nabla u)\, e_\alpha \wedge \varepsilon_\beta \,.$$

Remark 1.2. In the case $n = N$, if $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ and $\det Du \in L^1(\Omega)$, then u belongs to the class $\mathcal{A}^1(\Omega, \mathbb{R}^n)$, and the distributional derivative Du agrees with the approximate gradient ∇u.

If $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$, the i.m. rectifiable current $G_u \in \mathcal{R}_n(\Omega \times \mathbb{R}^N)$ of the type $(\mathcal{G}_u, 1, \vec{G}_u)$ is well defined [8, Vol. I, Sec. 3.2.1]. It is given by the integration of n-forms over the *rectifiable graph*

$$\mathcal{G}_u := \{(x, \lambda_u(x)) \,:\, x \in \mathcal{L}_u \cap A_D(u) \cap \Omega\}\,,$$

where $\mathcal{L}_u$ is the set of Lebesgue points, $\lambda_u(x)$ is the Lebesgue value at x and $A_D(u)$ is the set of approximate differentiability points of u. Moreover the tangent unit n-vector is $\vec{G}_u := \dfrac{\vec{M}(\nabla u)}{|\vec{M}(\nabla u)|}$.

Also, denoting $(Id \bowtie u)(x) := (x, u(x))$, in approximate sense we have

$$G_u = (Id \bowtie u)_\# [\![\Omega]\!]\,.$$

Moreover, since $|\vec{M}(\nabla u)|$ agrees with the n-dimensional Jacobian of $Id \bowtie u$, the area formula yields that the mass of G_u is equal to the area of the graph of u, i.e.,

$$\boldsymbol{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) = \int_\Omega |\vec{M}(\nabla u)|\, dx < \infty\,.$$

Splitting of currents.

For $k = 1, \ldots, n$, every differential k-form $\omega \in \mathcal{D}^k(\Omega \times \mathbb{R}^N)$ splits as a sum $\omega = \sum_{j=0}^{\min(k,N)} \omega^{(j)}$, where the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical y-variables. Every current $T \in \mathcal{D}_k(\Omega \times \mathbb{R}^N)$ then splits as $T = \sum_{j=0}^{\min(k,N)} T_{(j)}$, where $T_{(j)}(\omega) := T(\omega^{(j)})$.

Boundary.

Graphs of smooth maps $u : \Omega \to \mathbb{R}^N$ satisfy the null-boundary condition

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^N = 0 . \tag{1.2}$$

In fact, for every $(n-1)$-form $\omega \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$, by Stokes theorem we have

$$\partial G_u(\omega) := G_u(d\omega) = \int_{\mathcal{G}_u} d\omega = \int_{\partial \mathcal{G}_u} \omega = 0 ,$$

as ω is compactly supported in $\Omega \times \mathbb{R}^N$. On the other hand, if $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$, in general the interior boundary of G_u is non zero, i.e.,

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^N \neq 0 .$$

However, if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ for some positive integer $p \in \mathbb{N}^+$, then

$$(\partial G_u)_{(j)} \llcorner \Omega \times \mathbb{R}^N = 0 \qquad \forall j \in \mathbb{N} , \quad j \le p-1 . \tag{1.3}$$

This follows from a standard density argument based on the dominated convergence theorem and on condition (1.2), that holds true for smooth maps.

Example 1.1. If e.g. $\Omega := B^n$, the unit ball, $n = N$, and $u(x) := x/|x|$, then $u \in W^{1,p}(B^n, \mathbb{R}^n)$ for any $p < n$ whereas $\det Du = 0$ a.e., so that $u \in \mathcal{A}^1(B^n, \mathbb{R}^n)$. In particular, (1.3) holds true for $j = 0, \dots, n-2$. However, (1.3) is false for $j = n-1$, and we have

$$(\partial G_u) \llcorner B^n \times \mathbb{R}^n = -\delta_0 \times [\![\mathbb{S}^{n-1}]\!] ,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in the target space, compare [8, Vol. I, Sec. 3.2.2].

Cartesian maps with fractures.

In order to discuss some variational problems, we now recall some facts from [**9**, Sec. 5], see also [8, Vol. I].

For every $p \ge 1$ consider the class

$$\mathcal{A}^p(\Omega, \mathbb{R}^N) := \{ u \in L^p(\Omega, \mathbb{R}^N) \mid u \text{ is } \mathcal{L}^n\text{-a.e. appr. diff.}, \\ M^{\beta}_{\bar\alpha}(\nabla u) \in L^p(\Omega) \quad \forall\, \alpha, \beta \text{ with } |\alpha| + |\beta| = n \} .$$

In $\mathcal{A}^p$ we introduce the "norm"

$$\|u\|_{\mathcal{A}^p} := \|u\|_{L^p(\Omega, \mathbb{R}^N)} + \| \, |\overrightarrow{M}(\nabla u)| \, \|_{L^p(\Omega)}$$

and we say that a sequence $\{u_h\}$ in $\mathcal{A}^p(\Omega, \mathbb{R}^N)$ *weakly converges in* $\mathcal{A}^p$ to $u \in \mathcal{A}^p(\Omega, \mathbb{R}^N)$, say $u_h \rightharpoonup u$ weakly in $\mathcal{A}^p$, if and only if $u_h \to u$ strongly in $L^p(\Omega)$ and for every α and β with $|\alpha| + |\beta| = n$

$$M^{\beta}_{\bar{\alpha}}(\nabla u_h) \rightharpoonup M^{\beta}_{\bar{\alpha}}(\nabla u) \qquad \text{weakly in } L^p(\Omega)\,.$$

Notice that since Ω is a bounded set, for every $p \geq 1$ we have

$$\boldsymbol{M}(G_u) \leq c(p, \Omega) \cdot (|\Omega| + \|u\|_{\mathcal{A}^p}) < \infty\,.$$

Definition 1.3. *For every exponent* $p \geq 1$ *denote*

$$\mathrm{Cf}^{\,p}(\Omega, \mathbb{R}^N) := \{u \in \mathcal{A}^p(\Omega, \mathbb{R}^N) \cap L^\infty \mid \boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) < \infty\}$$

the class of p-Cartesian maps with fractures. *For* $u \in \mathrm{Cf}^{\,p}(\Omega, \mathbb{R}^N)$ *we also define the* $\mathrm{Cf}^{\,p}$*-norm*

$$\|u\|_{\mathrm{Cf}^{\,p}} := \|u\|_{\mathcal{A}^p} + \|u\|_{L^\infty} + \boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N)\,.$$

Finally, for every $K > 0$ *denote*

$$\mathrm{Cf}^{\,p}_K(\Omega, \mathbb{R}^N) := \{u \in \mathrm{Cf}^{\,p}(\Omega, \mathbb{R}^N) \mid \|u\|_\infty + \boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) \leq K\}\,.$$

By the closure theorem for graphs, see [8, Vol. I, Sec. 3.3.2], we readily obtain that the class $\mathrm{Cf}^{\,p}_K(\Omega, \mathbb{R}^N)$ is closed under the weak convergence in the product.

Proposition 1.4. (Closure). *Let* $p \geq 1$, $K > 0$, *and* $\{u_h\}$ *be a sequence in* $\mathrm{Cf}^{\,p}_K(\Omega, \mathbb{R}^N)$. *Let* $u \in L^1(\Omega, \mathbb{R}^N)$ *be an a.e. approximately differentiable map and let* $v^{\beta}_{\bar{\alpha}} \in L^1(\Omega)$, *where* α *and* β *are multi-indices with* $|\alpha| + |\beta| = n$. *Suppose that* $u_h \to u$ *strongly in* $L^p(\Omega)$ *and* $M^{\beta}_{\bar{\alpha}}(\nabla u_h) \rightharpoonup v^{\beta}_{\bar{\alpha}}$ *weakly in* $L^p(\Omega)$ *for every* α *and* β *with* $|\alpha| + |\beta| = n$. *Then* $u \in \mathrm{Cf}^{\,p}_K(\Omega, \mathbb{R}^N)$ *and* $v^{\beta}_{\bar{\alpha}}(x) = M^{\beta}_{\bar{\alpha}}(\nabla u(x))$ *a.e. in* Ω.

The sequential weak compactness of bounded sets in L^p, $p > 1$, together with the previous result, readily yields the following

Proposition 1.5. (Compactness). *Let* $\{u_h\} \subset \mathrm{Cf}^{\,p}(\Omega, \mathbb{R}^N)$ *such that*

$$\sup_h \|u_h\|_{\mathrm{Cf}^{\,p}} < \infty \tag{1.4}$$

for some $p > 1$. *There exist a subsequence* $\{u_{h_j}\}$ *of* $\{u_h\}$ *and a map* $u \in \mathrm{Cf}^{\,p}(\Omega, \mathbb{R}^N)$ *such that* $u_{h_j} \rightharpoonup u$ *weakly in* $\mathcal{A}^p$, *with* $G_{u_{h_j}} \rightharpoonup G_u$ *weakly in* $\mathcal{D}_n(\Omega \times \mathbb{R}^N)$.

Taking into account the lower semicontinuity of the mass with respect to the weak convergence in the sense of currents, we readily obtain the following.

Proposition 1.6. (Lower semicontinuity). *Let* $\{u_h\}$ *a sequence in* $\mathrm{Cf}^p(\Omega, \mathbb{R}^N)$ *such that* (1.4) *holds for some* $p > 1$. *Let* $u \in L^1(\Omega, \mathbb{R}^N)$ *an a.e. approximately differentiable map such that* $u_h \to u$ *in* $L^1(\Omega, \mathbb{R}^N)$. *Then*

$$\|u\|_{\mathrm{Cf}^p} \le \liminf_{h\to\infty} \|u_h\|_{\mathrm{Cf}^p} .$$

More precisely, $\|u\|_{\mathcal{A}^p} \le \liminf_{h\to\infty} \|u_h\|_{\mathcal{A}^p}$, $\|u\|_\infty \le \liminf_{h\to\infty} \|u_h\|_\infty$ *and*

$$\begin{aligned} \boldsymbol{M}(G_u \llcorner \Omega \times \mathbb{R}^N) &\le \liminf_{h\to\infty} \boldsymbol{M}(G_{u_h} \llcorner \Omega \times \mathbb{R}^N) , \\ \boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) &\le \liminf_{h\to\infty} \boldsymbol{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^N) . \end{aligned}$$

Finally, if $\widetilde{\Omega}$ is a bounded open set in $\mathbb{R}^n$ such that $\Omega \subset\subset \widetilde{\Omega}$, and $\varphi : \widetilde{\Omega} \to \mathbb{R}^N$ is a given smooth function in $\mathrm{Cf}^p_K(\widetilde{\Omega}, \mathbb{R}^N)$, we shall denote

$$\mathrm{Cf}^p_{K,\varphi}(\widetilde{\Omega}, \mathbb{R}^N) := \{u \in \mathrm{Cf}^p_K(\widetilde{\Omega}, \mathbb{R}^N) \mid (G_u - G_\varphi) \llcorner (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathbb{R}^N = 0\} .$$

The distributional determinant.

Assume now $n = N \ge 2$, and denote by $\widehat{\mathbb{R}}^n$ the target space. If $u \in \mathcal{A}^1(\Omega, \widehat{\mathbb{R}}^n)$ is bounded, the distributional determinant is again well-defined by (0.1), where this time adj ∇u is the matrix of the adjoints of the approximate Jacobian ∇u, so that

$$(\mathrm{adj}\, \nabla u)^j_i := (-1)^{i+j} \det \frac{\partial(u^1, \dots, u^{j-1}, u^{j+1}, \dots, u^n)}{\partial(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)} .$$

More precisely, we set for any $\varphi \in C^\infty_c(\Omega)$

$$\langle \mathrm{Det}\, \nabla u, \varphi \rangle = -\frac{1}{n} \sum_{i,j=1}^n \int_\Omega D_i\varphi(x)\, u^j(x)\, (\mathrm{adj}\, \nabla u(x))^j_i \, dx .$$

Notice that if $u : \Omega \to \widehat{\mathbb{R}}^n$ is a smooth map, we have

$$\mathrm{Det}\, Du = \det Du \cdot dx . \tag{1.5}$$

Since in fact $\sum_{i=1}^n \frac{\partial}{\partial x_i} (\mathrm{adj}\, Du)^j_i = 0$, by the Laplace formulas we have

$$\begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} (u^j\, (\mathrm{adj}\, Du)^j_i) &= \sum_{i=1}^n \frac{\partial u^j}{\partial x_i} (\mathrm{adj}\, Du)^j_i \\ &+ u^j \sum_{i=1}^n \frac{\partial}{\partial x_i} (\mathrm{adj}\, Du)^j_i = \det Du \end{aligned}$$

for every j. Therefore, by the $W^{1,n}$-density of smooth maps, (1.5) holds true for maps $u \in W^{1,n}(\Omega, \widehat{\mathbb{R}}^n) \cap L^\infty$.

Example 1.2. If $\Omega = B^n$ and $u(x) := x/|x|$, one easily verifies that $\det \nabla u = 0$ while $\operatorname{Det} \nabla u = |B^n|\, \delta_0$, where δ_0 is the unit Dirac mass centered at the origin. Notice that the boundary of G_u is non zero, see Example 1.1. Similarly for the map

$$u(x) := (|x|^n + |a|^n)^{1/n}\, \frac{x}{|x|}\,, \qquad a > 0\,,$$

we have $\det \nabla u = 1$ a.e. and $\operatorname{Det} \nabla u = 1 \cdot dx + |B^n(0,a)|\, \delta_0$.

2 - An isoperimetric inequality

In this section we prove an isoperimetric inequality that is the main new result of the paper. We shall assume $n = N \geq 2$, and we will focus on maps satisfying

$$\boldsymbol{M}((\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^n) < \infty\,. \tag{2.1}$$

Denote by ω_n the smooth $(n-1)$-form in $\widehat{\mathbb{R}}^n$

$$\omega_n := \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} y^j \widehat{dy^j}\,, \tag{2.2}$$

where $\widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^n$. Moreover, in the sequel we will let $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projection onto the x and y coordinates, respectively.

Proposition 2.1. (Isoperimetric inequality). *Let $u \in \mathcal{A}^1(\Omega, \widehat{\mathbb{R}}^n) \cap L^\infty$ be such that* (2.1) *holds and*

$$(\partial G_u)_{(n-2)} \llcorner \Omega \times \widehat{\mathbb{R}}^n = 0\,. \tag{2.3}$$

Then for every $x_0 \in \Omega$ and for a.e. $r > 0$ such that $B_r(x_0) \subset\subset \Omega$ we have

$$|((\partial G_u) \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)(\widehat{\pi}^\# \omega_n)| \leq c_n \boldsymbol{M}((\partial G_u)_{(n-1)} \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)^{n/(n-1)}\,, \tag{2.4}$$

where $B_r = B_r(x_0)$ and $c_n > 0$ is an absolute constant.

As a consequence of Remark 1.2 and property (1.3), where $p = n - 1$, we readily obtain the following

Corollary 2.2. (Isoperimetric inequality). *Let $u \in W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n) \cap L^\infty$ be such that* $\det Du \in L^1(\Omega)$ *and* (2.1) *holds. Then* (2.4) *holds for every $x_0 \in \Omega$ and for a.e. $r > 0$ such that $B_r(x_0) \subset\subset \Omega$.*

Proof of Proposition 2.1. Let $S_u \in \mathcal{D}_n(\Omega \times \widehat{\mathbb{R}}^n)$ be the n-current

$$S_u := h_\#(\partial G_u \times [\![0,1]\!]),$$

where $h : \Omega \times \widehat{\mathbb{R}}^n \times [0,1] \to \Omega \times \widehat{\mathbb{R}}^n$ is the affine homotopy map $h(x,y,t) := (x,ty)$.

Similarly to (2.2), to every vector field $g \in C^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$ we associate the $(n-1)$-form

$$\omega_g(y) := \sum_{j=1}^n (-1)^{j-1} g^j(y)\, \widehat{dy^j}, \qquad g = (g^1, \dots, g^n),$$

so that $d\omega_g = \operatorname{div} g\, dy$, where $dy := dy^1 \wedge \cdots \wedge dy^n$. We shall make use of the following

Lemma 2.3. *For every $\eta \in \mathcal{D}^1(\Omega)$ and $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$ we have $S_u(\eta \wedge \omega_g) = 0$.*

Remark 2.4. Lemma 2.3 is false if (2.3) is not satisfied, as the function u from Remark 3.2 below shows.

Proof of Lemma 2.3. We write

$$S_u(\eta \wedge \omega_g) = (\partial G_u \times [\![0,1]\!])(\eta \wedge \widetilde{h}^\# \omega_g),$$

where $\widetilde{h}(y,t) := ty$. Moreover, we can decompose the pull-back of ω_g as

$$\widetilde{h}^\# \omega_g = \Phi(y,t) \wedge dt + \Psi(y,t),$$

where the forms $\Phi(\cdot,t) \in \mathcal{D}^{n-2}(\widehat{\mathbb{R}}^n)$ and $\Psi(\cdot,t) \in \mathcal{D}^{n-1}(\widehat{\mathbb{R}}^n)$ for every $t \in (0,1)$. By definition of cartesian product of currents we get

$$(\partial G_u \times [\![0,1]\!])(\eta \wedge \Psi(y,t)) = 0,$$

as $\eta \wedge \Psi(y,t)$ does not contain the differential dt. Moreover, we have

$$(\partial G_u \times [\![0,1]\!])(\eta \wedge \Phi(y,t) \wedge dt) = \partial G_u(\eta(x) \wedge \widetilde{\Phi}(y))$$

for some "vertical" $(n-2)$-form $\widetilde{\Phi}(y) \in \mathcal{D}^{n-2}(\widehat{\mathbb{R}}^n)$. Property (2.3) yields the assertion. □

Setting $\Pi(x,y) := (x,0)$, we have

$$\partial S_u = (-1)^{n-1} \partial G_u + (-1)^n \Pi_\#(\partial G_u) \qquad \text{on} \quad \mathcal{D}^{n-1}(\Omega \times \widehat{\mathbb{R}}^n).$$

For any $\varphi \in C_c^\infty(\Omega)$ and $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$, since $\Pi_\#(\partial G_u)(\varphi \wedge \omega_g) = 0$, we thus obtain

$$\partial S_u(\varphi \wedge \omega_g) = (-1)^{n-1} \partial G_u(\varphi \wedge \omega_g).$$

By Lemma 2.3 we then obtain

$$(-1)^{n-1} \partial G_u(\varphi \wedge \omega_g) = S_u(\varphi \wedge d\omega_g). \tag{2.5}$$

Let μ_r be the signed measure on $\widehat{\mathbb{R}}^n$

$$\langle \mu_r, g\rangle := (-1)^{n-1}((\partial G_u) \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)(\widehat{\pi}^{\#}\omega_g)\,, \qquad g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)\,.$$

Since $\|\omega_g\| = \|g\|_\infty$, by (2.1) we infer that μ_r has bounded total variation

$$|\mu_r|(\widehat{\mathbb{R}}^n) \le \boldsymbol{M}((\partial G_u)_{(n-1)} \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n) < \infty\,.$$

Moreover, taking a sequence $\{\varphi_j\} \subset C_c^\infty(\Omega)$ converging in L^1 to the characteristic function $\chi_{\overline{B}_r}$ of the closed ball $\overline{B}_r$, by (2.5) we deduce that

$$\langle \mu_r, g\rangle = (S_u \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)(\widehat{\pi}^{\#} \operatorname{div} g\, dy)\,.$$

By the boundary rectifiability theorem, the current ∂G_u is i.m. rectifiable. Therefore, it turns out that $S_u \in \mathcal{R}_n(\Omega \times \widehat{\mathbb{R}}^n)$. By using the *degree theory* from [8, Vol. I, Sec. 4.3.2], for every $x_0 \in \Omega$ and for a.e. $r > 0$ small we have

$$\langle \mu_r, g\rangle = (S_u \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)(\widehat{\pi}^{\#} \operatorname{div} g\, dy) = \int_{\widehat{\mathbb{R}}^n} \widetilde{\Delta}_r(y) \operatorname{div} g(y)\, dy$$

for some *integer valued* L^1-function $\widetilde{\Delta}_r$, namely

$$\widetilde{\Delta}_r(y) := \deg(S_u \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n, \widehat{\pi}, y) \in L^1(\widehat{\mathbb{R}}^n, \mathbb{Z})\,.$$

As a consequence, $\widetilde{\Delta}_r$ is a function of bounded variation in $\widehat{\mathbb{R}}^n$, with

$$|D\widetilde{\Delta}_r|(\widehat{\mathbb{R}}^n) = |\mu_r|(\widehat{\mathbb{R}}^n) \le \boldsymbol{M}((\partial G_u)_{(n-1)} \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n) < \infty\,.$$

By Sobolev embedding theorem, and by density of smooth maps in $BV(\widehat{\mathbb{R}}^n)$, we have

$$\|\widetilde{\Delta}_r\|_{L^{n/(n-1)}(\widehat{\mathbb{R}}^n)} \le c_n\, |D\widetilde{\Delta}_r|(\widehat{\mathbb{R}}^n)$$

whereas, taking into account that $\Delta_r(y) \in \mathbb{Z}$,

$$\int_{\widehat{\mathbb{R}}^n} |\widetilde{\Delta}_r(y)|\, dy \le \int_{\widehat{\mathbb{R}}^n} |\widetilde{\Delta}_r(y)|^{n/(n-1)}\, dy = \|\widetilde{\Delta}_r\|^{n/(n-1)}_{L^{n/(n-1)}(\widehat{\mathbb{R}}^n)}\,.$$

We thus have

$$\begin{aligned} |\langle \mu_r, g\rangle| &\le \int_{\widehat{\mathbb{R}}^n} |\widetilde{\Delta}_r(y) \operatorname{div} g(y)|\, dy \\ &\le \|\operatorname{div} g\|_\infty \int_{\widehat{\mathbb{R}}^n} |\widetilde{\Delta}_r(y)|\, dy \\ &\le \|\operatorname{div} g\|_\infty\, c_n\, (|D\widetilde{\Delta}_r|(\widehat{\mathbb{R}}^n))^{n/(n-1)}\,, \end{aligned}$$

and hence

$$|((\partial G_u) \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)(\widehat{\pi}^{\#}\omega_g)| \le c_n \, \|\mathrm{div}\, g\|_\infty \, \boldsymbol{M}((\partial G_u)_{(n-1)} \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)^{n/(n-1)}\,.$$

Finally, taking $g(y) = y/n$ on the compact set $\{y \in \widehat{\mathbb{R}}^n \,:\, |y| \le \|u\|_\infty\}$ we obtain (2.4). □

3 - A representation result

In this section, making use of the isoperimetric inequality, we prove that the singular part of the distributional determinant is concentrated on a countable set of points. We assume $n = N \ge 2$.

Proposition 3.1. *Let $u : \Omega \to \widehat{\mathbb{R}}^n$ satisfy the hypotheses of Proposition* 2.1 *or Corollary* 2.2. *Then for every $g \in C_c^\infty(\Omega)$ we have*

$$\langle \mathrm{Det}\,\nabla u, g\rangle - \langle \det \nabla u \cdot dx, g\rangle = -\pi_{\#}((\partial G_u)_{(n-1)} \llcorner \widehat{\pi}^{\#}\omega_n)(g) \tag{3.1}$$

and $\mathrm{Det}\,\nabla u$ *is a signed Radon measure with finite total variation. The density of its absolute continuous part is equal to the pointwise determinant of* ∇u

$$\mathrm{Det}\,\nabla u = \det \nabla u \cdot dx + (\mathrm{Det}\,\nabla u)^s\,, \qquad (\mathrm{Det}\,\nabla u)^s \perp \mathcal{L}^n\,. \tag{3.2}$$

Moreover the singular part is supported on an at most countable set and

$$(\mathrm{Det}\,\nabla u)^s = \sum_{l=1}^{\infty} c_l\, \delta_{x_l}\,, \qquad \sum_{l=1}^{\infty} |c_l| < \infty\,, \tag{3.3}$$

where $c_l \in \mathbb{R}$ and δ_{x_l} is the unit Dirac mass centered at the point $x_l \in \Omega$. Finally, for every open set $U \subset \Omega$ the total variation of $(\mathrm{Det}\,\nabla u)^s$ *is given by*

$$|(\mathrm{Det}\,\nabla u)^s|(U) = \sum_{x_l \in U} |c_l| = \|\pi_{\#}((\partial G_u)_{(n-1)} \llcorner \widehat{\pi}^{\#}\omega_n)\|(U)\,.$$

Remark 3.2. On account of (3.1), that holds true even if (2.3) is not satisfied, the isoperimetric inequality (2.4) reads as

$$|(\mathrm{Det}\,\nabla u)^s|(\overline{B}_r) \le c_n \boldsymbol{M}((\partial G_u)_{(n-1)} \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)^{n/(n-1)}\,. \tag{3.4}$$

We now see that (3.4) is false, if (2.3) is not satisfied. In dimension $n = 2$, this happens if e.g. $u \in BV(\Omega, \widehat{\mathbb{R}}^2)$ is not a $W^{1,1}$-function. Taking for example $\Omega = (-1,1) \times (-1,1)$ and $u(x_1, x_2) = (x_1, x_2)$ if $x_1 < 0$, whereas $u(x_1, x_2) = (x_1 + 1, x_2)$ if $x_1 > 0$, we have

$$(\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^2 = \gamma_{1\#} I - \gamma_{2\#} I\,, \qquad I := [\![-1, 1]\!]\,,$$

where $\gamma_1(\lambda) := (0, \lambda, 0, \lambda)$ and $\gamma_2(\lambda) := (0, \lambda, 1, \lambda)$, for $-1 < \lambda < 1$. Moreover, we have

$$(\mathrm{Det}\, \nabla u)^s = \frac{1}{2}\, \mathcal{H}^1 \llcorner J_u\,, \qquad J_u = \{0\} \times (-1, 1)\,.$$

As a consequence, (3.3) fails to hold, too, if (2.3) is not satisfied.

Formula (3.2) goes back to [**11**]. We recall that in general the singular part of $\mathrm{Det}\, \nabla u$ is not a sum of Dirac masses, see [**12**]. However, since $\mathrm{Det}\, \nabla u$ is a Borel measure, thanks to a classical result by L. Schwartz, to prove (3.3) it suffices to show that $(\mathrm{Det}\, \nabla u)^s$ is concentrated on a countable set.

In view of this let us first prove

Lemma 3.3. *Let λ and μ be respectively a non-negative and a signed Radon measure on Ω, with finite total variation, such that for every $x_0 \in \Omega$ and for a.e. $r > 0$ for which $B_r(x_0) \subset\subset \Omega$ we have*

$$|\mu(\overline{B_r(x_0)})| \le c\, \lambda(\overline{B_r(x_0)})^\alpha$$

for some fixed constants $c > 0$ and $\alpha > 1$. Then μ is purely atomic.

Proof. Let

$$A := \{a \in \Omega \mid \limsup_{r \to 0^+} \lambda(B_r(a)) > 0\}$$

denote the set of atoms of λ. Since λ is finite on compact sets of Ω, then A is at most countable. For every $x_0 \in \Omega$ and a.e. $r > 0$ small we have

$$\frac{\mu(\overline{B}_r)}{\lambda(\overline{B}_r)} \le c\, \lambda(\overline{B}_r)^{\alpha - 1},$$

where $B_r = B_r(x_0)$, and hence, since $\alpha > 1$, letting $r \to 0$ we infer that the density of μ with respect to λ is zero at all points x_0 which are not in A. This yields that μ is concentrated on A, as required. □

We also recall from [8, Vol. I, Sec. 3.2.3] the *integration by parts formula*

$$\partial G_u(\phi(x, y)\, \widehat{dy^j}) = (-1)^{j-1} \sum_{i=1}^{n} \int_{\Omega} \nabla_i[\phi(x, u(x))]\, (\mathrm{adj}\, \nabla u(x))_i^j\, dx\,, \tag{3.5}$$

where $j = 1, \ldots, n$, that holds true for every function $\phi \in C^1(\mathbb{R}^n \times \widehat{\mathbb{R}}^n)$ with bounded derivatives.

Proof of Proposition 3.1. Taking $\phi(x,y) := (-1)^{j-1} g(x) y^j$ in (3.5), since

$$\begin{aligned} \nabla_i[g(x)u^j(x)]\,(\mathrm{adj}\,\nabla u(x))^j_i &= D_i g(x)\, u^j(x)\,(\mathrm{adj}\,\nabla u(x))^j_i \\ &+ g(x)\,\nabla_i u^j(x)\,(\mathrm{adj}\,\nabla u(x))^j_i \end{aligned}$$

by summing over $i=1,\dots,n$ and by the Laplace formulas we have

$$\begin{aligned} \pi_\#((\partial G_u) \llcorner \widehat{\pi}^\#((-1)^{j-1}\, y^j \widehat{dy^j}))(g) &= (-1)^{j-1}\, \partial G_u(g y^j \widehat{dy^j}) \\ &= \sum_{i=1}^n \int_\Omega D_i g\, u^j\,(\mathrm{adj}\,\nabla u)^j_i\, dx + \int_\Omega g \det \nabla u\, dx \end{aligned}$$

hence, by taking the sum over j and dividing by n, we obtain (3.1).

By (2.1), and by the boundary rectifiability theorem, it turns out that the current $\pi_\#((\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^n)$ is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega)$ and hence it is supported on an $(n-1)$-rectifiable set of Ω. As a consequence (3.2) holds and $\mathrm{Det}\,\nabla u$ is a signed Radon measure in Ω. Moreover, for every point $x_0 \in \Omega$ and a.e. small radius r, so that $B_r(x_0) \subset\subset \Omega$, by (3.1) we have

$$\begin{aligned} -(\mathrm{Det}\,\nabla u)^s(\overline{B}_r) &= \pi_\#((\partial G_u) \llcorner \widehat{\pi}^\# \omega_n)(\overline{B}_r) \\ &= \langle (\partial G_u) \llcorner \widehat{\pi}^\# \omega_n, \overline{B}_r \times \widehat{\mathbb{R}}^n \rangle \\ &= ((\partial G_u) \llcorner \overline{B}_r \times \widehat{\mathbb{R}}^n)(\widehat{\pi}^\# \omega_n)\,, \end{aligned}$$

where $B_r = B_r(x_0)$, thus by the isoperimetric inequality (2.4) we obtain (3.4).

We then apply Lemma 3.3 with

$$\mu(B) := (\mathrm{Det}\,\nabla u)^s(B)\,, \qquad \lambda(B) := \boldsymbol{M}((\partial G_u)_{(n-1)} \llcorner B \times \widehat{\mathbb{R}}^n)\,,$$

taking $\alpha := n/(n-1)$, to conclude that $(\mathrm{Det}\,\nabla u)^s$ is a purely atomic measure with finite total variation. If $\mu_\pm = ((\mathrm{Det}\,\nabla u)^s)^\pm$ are the positive and negative part of $(\mathrm{Det}\,\nabla u)^s$, the Radon differentiability theorem yields that for each $a_i \in A$ the limit

$$\alpha_i^\pm = \lim_{r \to 0^+} \mu_\pm(\overline{B}(a_i, r))$$

exists and hence (3.3) holds as

$$(\mathrm{Det}\,\nabla u)^s = \sum_{a_i \in A} (\alpha_i^+ - \alpha_i^-)\, \delta_{a_i}\,.$$

Finally, the last assertion trivially holds. □

Remark 3.4. Proposition 3.1 yields that the 0-current

$$\pi_\#((\partial G_u)_{(n-1)} \llcorner \widehat{\pi}^\# \omega_n) \tag{3.6}$$

is rectifiable in $\boldsymbol{R}_0(\Omega)$. However, in general (3.6) is not size bounded. In fact, in [**9**, Sec. 7] we showed the existence of Sobolev functions u satisfying the hypotheses of Proposition 2.1, actually of Corollary 2.2, such that (3.3) holds for some countable family $\{x_l\}$ of pairwise distinct points in Ω.

4 - A variational problem

In this section we discuss a variational problems that is naturally related to the results of the previous sections. Assume again $n = N \geq 2$.

Definition 4.1. *Let* $u : \Omega \to \widehat{\mathbb{R}}^n$ *satisfy the hypotheses of Proposition* 2.1 *or Corollary* 2.2. *The* 0-dimensional singular set *of* u *is defined by the set of points of concentration of* $(\operatorname{Det} \nabla u)^s$, *i.e.,*

$$S_0(u) := \{x_l \in \Omega \mid c_l \neq 0 \;\; in \; (3.3)\}\,.$$

Remark 4.2. $S_0(u)$ is the countable set of points of positive multiplicity of the rectifiable current (3.6), and it detects the so-called points of *cavitation*. We thus infer that $\mathcal{H}^0(S_u)$ agrees with the size of the current (3.6), i.e.,

$$\mathcal{H}^0(S_u) = \boldsymbol{S}(\pi_{\#}((\partial G_u)_{(n-1)} \llcorner \widehat{\pi}^{\#}\omega_n))\,.$$

Lower semicontinuity of the size.

We fix $p \geq 1$, $K > 0$, and $\varphi \in \operatorname{Cf}^p_K(\widetilde{\Omega}, \widehat{\mathbb{R}}^n)$ smooth.

Theorem 4.3. *Let* $\{u_h\} \subset \operatorname{Cf}^1_{K,\varphi}(\widetilde{\Omega}, \widehat{\mathbb{R}}^n) \cap W^{1,n-1}$ *be such that*

$$\sup_h \boldsymbol{M}(G_{u_h}) < \infty\,, \qquad \sup_h \boldsymbol{M}((\partial G_{u_h}) \llcorner \widetilde{\Omega} \times \widehat{\mathbb{R}}^n) < \infty \tag{4.1}$$

and $G_{u_h} \rightharpoonup G_u$ *weakly in* $\mathcal{D}_n(\widetilde{\Omega} \times \widehat{\mathbb{R}}^n)$, *where* $u \in \operatorname{Cf}^1_{K,\varphi}(\widetilde{\Omega}, \widehat{\mathbb{R}}^n) \cap W^{1,n-1}$. *Then we have*

$$\mathcal{H}^0\big(S_0(u)\big) \leq \liminf_{h \to \infty} \mathcal{H}^0\big(S_0(u_h)\big)\,.$$

Proof. We wish to apply Theorem 1.1 with $U = \widetilde{\Omega}$, $m = n$, $k = 0$, and

$$T := \pi_{\#}((\partial G_u)_{(n-1)} \llcorner \widehat{\pi}^{\#}\omega_n)\,, \qquad T_h := \pi_{\#}((\partial G_{u_h})_{(n-1)} \llcorner \widehat{\pi}^{\#}\omega_n)\,.$$

By Proposition 3.1 we have T, $\{T_h\} \subset \boldsymbol{R}_0(\widetilde{\Omega}) \cap \mathcal{N}_0(\widetilde{\Omega})$. Moreover, condition (i) in Theorem 1.1 follows from (4.1), so that it remains to check condition (ii). To this purpose, notice that ∂G_{u_h}, $\partial G_u \in \mathcal{R}_{n-1}(\widetilde{\Omega} \times \widehat{\mathbb{R}}^n)$, that $\sup_h (\|u_h\|_\infty + \|u\|_\infty) < \infty$, and that for every open set W, with $\Omega \subset\subset W \subset\subset \widetilde{\Omega}$,

$$\partial(\partial G_{u_h} \llcorner W \times \widehat{\mathbb{R}}^n - \partial G_u \llcorner W \times \widehat{\mathbb{R}}^n) = 0\,.$$

Therefore, as in [**14**, 31.2], the weak convergence of ∂G_{u_h} to ∂G_u and (4.1) yield the existence of a sequence $\{\Sigma_h\} \subset \mathcal{R}_n(\widetilde{\Omega} \times \widehat{\mathbb{R}}^n)$ of integral currents, with support $\mathrm{spt}(\Sigma_h) \subset \widetilde{\Omega} \times K$ for some compact set $K \subset \widehat{\mathbb{R}}^n$, such that

$$\partial G_{u_h} - \partial G_u = \partial \Sigma_h\,.$$

Set now $Q_h := \pi_{\#}(\Sigma_h \llcorner \widehat{\pi}^{\#} dy)$ and $R_h := (-1)^{n-1} \pi_{\#}(\Sigma_h \llcorner \widehat{\pi}^{\#} \omega_n)$. Since $\{\Sigma_h\} \subset \mathcal{N}_n(\widetilde{\Omega} \times \widehat{\mathbb{R}}^n)$ satisfies

$$\sup_h \boldsymbol{M}(\partial \Sigma_h) < \infty\,, \quad \lim_{h\to\infty} \boldsymbol{M}(\Sigma_h) = 0\,, \quad \mathrm{spt}(\Sigma_h) \subset \widetilde{\Omega} \times K\,,$$

we infer that $Q_h \in \mathcal{N}_0(\widetilde{\Omega})$, $R_h \in \mathcal{N}_1(\widetilde{\Omega})$ and $\lim_h [\boldsymbol{M}(Q_h) + \boldsymbol{M}(R_h)] = 0$. Therefore, condition (ii) in Theorem 1.1 holds true if we show that

$$T_h - T = Q_h + \partial R_h\,. \tag{4.2}$$

To this aim, we notice that for every $\phi \in \mathcal{D}^0(\widetilde{\Omega} \times \widehat{\mathbb{R}}^n)$

$$(-1)^{n-1} \Sigma_h(\widehat{\pi}^{\#}\omega_n \wedge d\phi) = \Sigma_h(d(\widehat{\pi}^{\#}\omega_n \wedge \phi)) - \Sigma_h(d\widehat{\pi}^{\#}\omega_n \wedge \phi)\,,$$

whereas $d\widehat{\pi}^{\#}\omega_n = \widehat{\pi}^{\#} d\omega_n = \widehat{\pi}^{\#} dy$. We thus obtain

$$\begin{aligned}(-1)^{n-1}\partial(\Sigma_h \llcorner \widehat{\pi}^{\#}\omega_n) &= -\Sigma_h \llcorner \widehat{\pi}^{\#} dy + (\partial \Sigma_h) \llcorner \widehat{\pi}^{\#}\omega_n \\ &= -\Sigma_h \llcorner \widehat{\pi}^{\#} dy + (\partial G_{u_h}) \llcorner \widehat{\pi}^{\#}\omega_n - (\partial G_u) \llcorner \widehat{\pi}^{\#}\omega_n\,,\end{aligned}$$

which clearly yields (4.2). As a consequence, by Definition 4.1 and Theorem 1.1 we have

$$\mathcal{H}^0(S_0(u)) = \boldsymbol{S}(T) \le \liminf_{h\to\infty} \boldsymbol{S}(T_h) = \liminf_{h\to\infty} \mathcal{H}^0(S_0(u_h))\,,$$

as required. □

Existence results.

We conclude this section by proving an existence result. Let $c_1, c_2, c_3 > 0$, and consider the energy functional

$$\mathcal{F}(u) := c_1 \int_\Omega \Phi(|\overrightarrow{M}(Du)|)\, dx + c_2 \int_\Omega |Du|^{n-1}\, dx + c_3\, \mathcal{H}^0(S_0(u))\,. \tag{4.3}$$

Theorem 4.4. (Existence result I). *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non negative and convex function satisfying a p-coercivity condition*

$$c\,|t|^p \le \Phi(t)\,, \qquad c > 0\,, \quad p > 1\,.$$

Then for every $K > 0$ and $\varphi \in \mathrm{Cf}^p_K(\widetilde{\Omega}, \widehat{\mathbb{R}}^n)$ smooth, there exists a solution to the problem

$$\inf\{\mathcal{F}(u) \mid u \in \mathrm{Cf}^p_{K,\varphi}(\widetilde{\Omega}, \widehat{\mathbb{R}}^n) \cap W^{1,n-1}\}\,.$$

Proof. Let $\{u_h\} \subset \mathrm{Cf}^p_{K,\varphi}(\widetilde{\Omega}, \widehat{\mathbb{R}}^n) \cap W^{1,n-1}$ be a minimizing sequence. Since (1.4) holds true, then by the closure, compactness and lower semicontinuity properties of Propositions 1.4, 1.5 and 1.6, and possibly passing to a subsequence, we find the existence of $u \in \mathrm{Cf}^p_{K,\varphi}(\widetilde{\Omega}, \widehat{\mathbb{R}}^n) \cap W^{1,n-1}$ such that $u_h \rightharpoonup u$ weakly in $\mathcal{A}^p$ and in $W^{1,n-1}$, with $G_{u_h} \rightharpoonup G_u$ weakly in $\mathcal{D}_n(\widetilde{\Omega} \times \widehat{\mathbb{R}}^n)$. Moreover, by Theorem 4.3, by the lower semicontinuity of $u \mapsto \int_\Omega |Du|^{n-1}\,dx$, and by the standard lower semicontinuity of convex functionals with p-coercivity, we obtain

$$\mathcal{F}(u) \le \liminf_{h\to\infty} \mathcal{F}(u_h)$$

and hence the assertion. □

In a similar way one may consider functionals with more general bulk energies given by the integral of a polyconvex function of the gradient Du, satisfying a suitable growth condition. Moreover, since the zero boundary condition (2.3) is preserved by the weak convergence as currents, one may consider similar variational problems involving the wider class of functions in $\mathrm{Cf}^1_{K,\varphi}(\widetilde{\Omega}, \widehat{\mathbb{R}}^n)$ satisfying (2.3), this time taking $c_2 = 0$ in (4.3), and ∇u instead of Du.

5 - The distributional minors.

In this section we extend the previous results to the case of *distributional minors* of the Jacobian matrix ∇u of fixed order k. In the sequel u is a bounded function in $\mathcal{A}^1(\Omega, \mathbb{R}^N)$, where $n, N \ge 2$.

Let us fix the order $2 \le k \le \min(n, N)$. Also, let α and β be any multi-indices with length $|\alpha| + |\beta| = n$, where $|\beta| = k$.

Definition 5.1. *The* distributional minor *of indices $\overline{\alpha}$ and β of ∇u is defined by*

$$\mathrm{Div}^\beta_{\overline{\alpha}} u := \frac{1}{|\beta|} \sum_{j\in\beta} \sum_{i\in\overline{\alpha}} \frac{\partial}{\partial x_i} \left(u^j(x)\,((\mathrm{adj}\,\nabla u)^\beta_{\overline{\alpha}})^j_i \right)$$

i.e., for every $g \in C^1_c(\Omega)$

$$\langle \mathrm{Div}^{\beta}_{\overline{\alpha}} u, g\rangle := -\frac{1}{|\beta|}\sum_{j\in\beta}\sum_{i\in\overline{\alpha}} \langle u^j(x)\,((\mathrm{adj}\,\nabla u)^{\beta}_{\overline{\alpha}})^j_i, D_i g\rangle\,.$$

Remark 5.2. In the case $k = 1$, if $\beta = j$ and $\overline{\alpha} = i$, we have $(\mathrm{adj}\,\nabla u)^{\beta}_{\overline{\alpha}} = 1$ and $\mathrm{Div}^{\beta}_{\overline{\alpha}} u = D_i u^j$. Therefore, we have assumed $k \geq 2$, the case $k = 1$ being reduced to well-known facts from the theory of BV-functions, compare [**2**]. Moreover, if u is smooth we infer that

$$\mathrm{Div}^{\beta}_{\overline{\alpha}} u = M^{\beta}_{\overline{\alpha}}(\nabla u)\cdot dx\,, \tag{5.1}$$

where $M^{\beta}_{\overline{\alpha}}(\nabla u)$ is the pointwise determinant of the corresponding minor of ∇u. In fact, by the Laplace formulas, for every $j \in \beta$ we have

$$\begin{aligned}\sum_{i\in\overline{\alpha}}\frac{\partial}{\partial x_i}\left(u^j\,((\mathrm{adj}\,Du)^{\beta}_{\overline{\alpha}})^j_i\right) &= \sum_{i\in\overline{\alpha}}\frac{\partial u^j}{\partial x_i}\,((\mathrm{adj}\,Du)^{\beta}_{\overline{\alpha}})^j_i \\ &+ u^j\sum_{i\in\overline{\alpha}}\frac{\partial}{\partial x_i}\,((\mathrm{adj}\,Du)^{\beta}_{\overline{\alpha}})^j_i \\ &= M^{\beta}_{\overline{\alpha}}(\nabla u)\,,\end{aligned}$$

where we used that $\sum_{i\in\overline{\alpha}}\frac{\partial}{\partial x_i}\,((\mathrm{adj}\,Du)^{\beta}_{\overline{\alpha}})^j_i = 0$. Therefore, by the $W^{1,k}$-density of smooth maps, (5.1) holds true for maps $u \in W^{1,k}(\Omega, \mathbb{R}^N)\cap L^\infty$.

In Proposition 5.3, we will show that if the boundary of G_u has finite mass, see (2.1), and u satisfies the additional zero boundary condition

$$(\partial G_u)_{(k-2)} \llcorner \Omega \times \mathbb{R}^N = 0\,, \tag{5.2}$$

then $\mathrm{Div}^{\beta}_{\overline{\alpha}} u$ is a signed Radon measure with density of the absolute continuous part given by the pointwise determinant $M^{\beta}_{\overline{\alpha}}(\nabla u)$ and singular part concentrated on a countably $\mathcal{H}^{n-k}$-rectifiable set of Ω, so that $\mathrm{Div}^{\beta}_{\overline{\alpha}} u$ has no Cantor-type part. We recall that (5.2) is satisfied if $u \in W^{1,k-1}(\Omega, \mathbb{R}^N)$, see (1.3).

To this aim, we associate to every $g \in C^\infty_c(\Omega)$ the $(n-k)$-form $\omega^{\alpha}_g \in \mathcal{D}^{n-k}(\Omega)$ given by

$$\omega^{\alpha}_g(x) := (-1)^{|\alpha|}\sigma(\alpha, \overline{\alpha})\,g(x)dx^{\alpha}\,.$$

Also, we will denote by

$$\omega_\beta := \frac{1}{|\beta|}\sum_{j\in\beta}\sigma(j, \beta - j)\,y^j dy^{\beta-j}$$

the $(k-1)$-form in $\mathcal{D}^{k-1}(\mathbb{R}^N)$ associated to β, so that $d\omega_\beta = dy^\beta$. Finally we will set

$$\sigma_k^n := (-1)^{(k-1)(n-k)},$$

so that, ω_g^α and $y^j dy^{\beta-j}$ being of degree $n-k$ and $k-1$, respectively, then

$$y^j dy^{\beta-j} \wedge \omega_g^\alpha = \sigma_k^n \, \omega_g^\alpha \wedge y^j dy^{\beta-j} \, .$$

Proposition 5.3. *Let $2 \le k \le \min(n, N)$. Let $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N) \cap L^\infty$ be such that*

$$\boldsymbol{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) < \infty \, . \tag{5.3}$$

Assume in addition that $u \in W^{1,k-1}(\Omega, \mathbb{R}^N)$ or, more generally, that (5.2) *holds true. Let α and β be multi-indices with length $|\alpha| + |\beta| = n$ and $|\beta| = k$. Then for every $g \in C_c^\infty(\Omega)$ we have*

$$\langle \mathrm{Div}_{\overline{\alpha}}^{\beta} u, g \rangle - \langle M_{\overline{\alpha}}^{\beta}(\nabla u) \cdot dx, g \rangle = -\sigma_k^n \, \pi_\#((\partial G_u)_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta)(\omega_g^\alpha) \tag{5.4}$$

and $\mathrm{Div}_{\overline{\alpha}}^{\beta} u$ *is a signed Radon measure with finite total variation. The density of its absolute continuous part is equal to the pointwise determinant* $M_{\overline{\alpha}}^{\beta}(\nabla u)$

$$\mathrm{Div}_{\overline{\alpha}}^{\beta} u = M_{\overline{\alpha}}^{\beta}(\nabla u) \cdot dx + (\mathrm{Div}_{\overline{\alpha}}^{\beta} u)^s \, , \qquad (\mathrm{Div}_{\overline{\alpha}}^{\beta} u)^s \perp \mathcal{L}^n \, . \tag{5.5}$$

Also, the singular part is supported on a countably $\mathcal{H}^{n-k}$-rectifiable set. Finally, for every open set $U \subset \Omega$ the total variation of $(\mathrm{Div}_{\overline{\alpha}}^{\beta} u)^s$ *is given by*

$$|(\mathrm{Div}_{\overline{\alpha}}^{\beta} u)^s|(U) = \|(\pi_\#((\partial G_u)_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta)) \llcorner dx^\alpha\|(U) \, . \tag{5.6}$$

As a consequence, we also have

Corollary 5.4. *Let k, u and β be as in Proposition* 5.3. *Then the* $(n-k)$*-current*

$$T := \pi_\#((\partial G_u)_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta) \tag{5.7}$$

is rectifiable in $\boldsymbol{R}_{n-k}(\Omega)$.

Remark 5.5. In general the rectifiable $(n-k)$-current (5.7) is not size bounded, if $k \ge 2$, see [**9**, Sec. 7].

Proof of Proposition 5.3. The integration by parts formula

$$\begin{aligned} &\partial G_u(\phi(x,y) dx^\alpha \wedge dy^{\beta-j}) \\ &= (-1)^{|\alpha|} \sigma(\alpha, \overline{\alpha}) \, \sigma(j, \beta - j) \sum_{i \in \overline{\alpha}} \int_\Omega \nabla_i [\phi(x, u(x))] \, (\mathrm{adj}\,(\nabla u(x))_{\overline{\alpha}}^{\beta})_i^j \, dx \end{aligned} \tag{5.8}$$

holds for every $j \in \beta$ and every function $\phi \in C^1(\mathbb{R}^n \times \mathbb{R}^N)$ with bounded derivatives, compare [8, Vol. I, Sec. 3.2.3]. We apply (5.8) with $\phi(x,y) := (-1)^{|\alpha|}\sigma(\alpha,\overline{\alpha})\,\sigma(j,\beta-j)\,g(x)y^j$ and $j \in \beta$. Since

$$\sigma(j,\beta-j)\,\omega_g^\alpha \wedge y^j dy^{\beta-j} = \phi(x,y)dx^\alpha \wedge dy^{\beta-j}$$

and

$$\begin{aligned}\nabla_i[g(x)u^j(x)]\ (\mathrm{adj}\,(\nabla u(x))^\beta_{\overline{\alpha}})^j_i &= D_i g(x)\,u^j(x)\,(\mathrm{adj}\,(\nabla u(x))^\beta_{\overline{\alpha}})^j_i\\ &+ g(x)\,\nabla_i u^j(x)\ (\mathrm{adj}\,(\nabla u(x))^\beta_{\overline{\alpha}})^j_i\end{aligned}$$

by summing over $i \in \overline{\alpha}$ and by the Laplace formulas we have

$$\begin{aligned}&\sigma^n_k\,\pi_\#((\partial G_u)\llcorner \widehat{\pi}^\#\sigma(j,\beta-j)\,y^j dy^{\beta-j})(\omega^\alpha_g)\\ &\quad= \partial G_u(\sigma(j,\beta-j)\,\omega_g^\alpha \wedge y^j dy^{\beta-j})\\ &\quad= \sum_{i\in\overline{\alpha}}\int_\Omega D_i g\,u^j\,(\mathrm{adj}\,(\nabla u)^\beta_{\overline{\alpha}})^j_i\,dx + \int_\Omega g\,M^\beta_{\overline{\alpha}}(\nabla u)\,dx\,.\end{aligned}$$

Therefore, taking the sum over $j \in \beta$ and dividing by $|\beta|$ we obtain (5.4). Arguing as in Proposition 3.1, by (5.3) and by the boundary rectifiability theorem we then deduce (5.5) and that $\mathrm{Div}^\beta_{\overline{\alpha}}u$ is a signed Radon measure in Ω. It remains to show that $(\mathrm{Div}^\beta_{\overline{\alpha}}u)^s$ is concentrated on an countably $\mathcal{H}^{n-k}$-rectifiable set.

If $k = n$, setting $u^\beta := (u^{\beta_1}, \dots, u^{\beta_k})$ we have

$$\pi_\#((\partial G_u)_{(n-1)}\llcorner \widehat{\pi}^\#\omega_\beta) = \pi_\#((\partial G_{u^\beta})_{(n-1)}\llcorner \widehat{\pi}^\#\omega_\beta)\,.$$

Since $u^\beta \in W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n)$ or, more generally, by (5.2),

$$(\partial G_{u^\beta})_{(n-2)}\llcorner \Omega \times \widehat{\mathbb{R}}^n = 0\,,$$

whereas u^β satisfies (2.1), the assertion follows from Proposition 3.1 applied to $u^\beta : \Omega \to \widehat{\mathbb{R}}^n$.

If $2 \le k \le n-1$, we recall from slicing theory, see [**14**], that if $T \in \mathcal{N}_{n-k}(\Omega)$ is a normal current and $\pi_\alpha : \mathbb{R}^n \to \mathbb{R}^{n-k}$ denotes the orthogonal projection onto the α-components of x, i.e., $\pi_\alpha(x) = x_\alpha$, then

$$T\llcorner dx^\alpha = \int_{\mathbb{R}^{n-k}} \langle T, \pi_\alpha, x_\alpha\rangle\,dx_\alpha\,, \tag{5.9}$$

where $\langle T, \pi_\alpha, x_\alpha\rangle$ is for a.e. x_α the 0-current obtained by slicing T with respect to π_α. We may and will apply (5.9) to the current T in (5.7), since $\partial T = \partial\pi_\#((G_u)_{(k)}\llcorner \widehat{\pi}^\# y^\beta)$

and hence $T \in \mathcal{N}_{n-k}(\Omega)$. Moreover, without loss of generality, we assume $x_\alpha = (x_1, \dots, x_{n-k})$, so that $x = (x_\alpha, x_{\overline{\alpha}})$, and we let

$$\Omega_{x_\alpha} := \{x_{\overline{\alpha}} \in \mathbb{R}^k \mid (x_\alpha, x_{\overline{\alpha}}) \in \Omega\}$$

denote the k-dimensional section of Ω with the k-plane ${\pi_\alpha}^{-1}(x_\alpha)$. Let $u^\beta_{x_\alpha} : \Omega_{x_\alpha} \to \widehat{\mathbb{R}}^k$ given by

$$x_{\overline{\alpha}} \mapsto u^\beta_{x_\alpha}(x_{\overline{\alpha}}) := (u^{\beta_1}, \dots, u^{\beta_k})(x_\alpha, x_{\overline{\alpha}}).$$

For a.e. choice of $x_\alpha \in \mathbb{R}^{n-k}$ such that Ω_{x_α} is non-empty, we have:

1. $\langle \pi_\#((\partial G_u) \llcorner \widehat{\pi}^\# \omega_\beta), \pi_\alpha, x_\alpha\rangle(g(x)) = \pi_\#((\partial G_{u^\beta_{x_\alpha}}) \llcorner \widehat{\pi}^\# \omega_\beta)(g(x_\alpha, \cdot))$;
2. $u^\beta_{x_\alpha}$ belongs to $\mathcal{A}^1(\Omega_{x_\alpha}, \widehat{\mathbb{R}}^k) \cap L^\infty$;
3. property $\boldsymbol{M}((\partial G_{u^\beta_{x_\alpha}}) \llcorner \Omega_{x_\alpha} \times \widehat{\mathbb{R}}^k) < \infty$ holds;
4. $u^\beta_{x_\alpha}$ belongs to $W^{1,k-1}(\Omega_{x_\alpha}, \widehat{\mathbb{R}}^k)$ or, more generally, (5.2) yields that $(\partial G_{u^\beta_{x_\alpha}})_{(k-2)} \llcorner \Omega_{x_\alpha} \times \widehat{\mathbb{R}}^k = 0$.

Applying Proposition 3.1 with k and $u^\beta_{x_\alpha}$ for of n and u, we infer that $\langle \pi^\#((\partial G_u)_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta), \pi_\alpha, x_\alpha\rangle$ is concentrated on a countable set of points. Therefore, by (5.9) we conclude that $(\mathrm{Div}^\beta_{\overline{\alpha}} u)^s$ is concentrated on a countably $\mathcal{H}^{n-k}$-rectifiable set. Finally, (5.6) trivially follows. □

Proof of Corollary 5.4. Proposition 5.3 yields that the $(n-k)$-current (5.7) is of the type $(\mathcal{M}, \theta, \xi_0)$, see (1.1). Let ξ be the unit $(n-1)$-vector that provides an orientation to the i.m. rectifiable current $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N$. Define $\widetilde{\phi} := (\phi^1, \dots, \phi^{k-1}) : \mathbb{R}^{n+N} \to \mathbb{R}^{k-1}$ by

$$\widetilde{\phi}(x, y) := \sum_{j \in \beta} \sigma(j, \beta - j)\, y^{\beta - j}\,,$$

and let $\phi := \phi^1 \wedge \cdots \wedge \phi^{k-1}$. By Prop. 3 from [8, Vol. I, Sec. 5.2.2], we infer that for $\mathcal{H}^{n-1}$-a.e. $(x, y) \in \partial \mathcal{G}_u$, the $(n-k)$-vector $(\xi \llcorner \phi)(x, y)$ is simple and tangent to $\ker \widetilde{\phi}_{|\Sigma}$, where Σ is the approximate tangent $(n-1)$-space to $\partial \mathcal{G}_u$ at (x, y). As a consequence, the $(n-k)$-vector $\pi_\# \xi \llcorner \phi$, modulo a renormalization, provides an orientation to $\mathcal{M}$, as required. □

Singular set.

Similarly to Definition 4.1, we now give the following:

Definition 5.6. *Let* $2 \le k \le \min(n, N)$*. Let* $u : \Omega \to \mathbb{R}^N$ *satisfy the hypotheses of Proposition* 5.3*. The* $(n-k)$-dimensional singular set *of* u *is defined by*

$$S_{n-k}(u) := \bigcup_{|\beta|=k} \mathcal{M}_\beta(u)\,,$$

where $\mathcal{M}_\beta(u)$ *is the set of points of positive multiplicity of the rectifiable current* (5.7)*, i.e.,*

$$\mathcal{M}_\beta(u) := \operatorname{set}\big(\pi_\#((\partial G_u)_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta)\big)\,.$$

Lower semicontinuity of the size.

Fix $p \ge 1$, $K > 0$, and $\varphi \in \mathrm{Cf}^p_K(\widetilde{\Omega}, \widehat{\mathbb{R}}^n)$ smooth.

Theorem 5.7. *Let* $2 \le k \le \min(n, N)$*. Let* $\{u_h\} \subset \mathrm{Cf}^1_{K,\varphi}(\widetilde{\Omega}, \mathbb{R}^N) \cap W^{1,k-1}$ *be such that*

$$\sup_h \boldsymbol{M}(G_{u_h}) < \infty\,, \qquad \sup_h \boldsymbol{M}((\partial G_{u_h}) \llcorner \widetilde{\Omega} \times \mathbb{R}^N) < \infty$$

and $G_{u_h} \rightharpoonup G_u$ *weakly in* $\mathcal{D}_n(\widetilde{\Omega} \times \mathbb{R}^N)$*, where* $u \in \mathrm{Cf}^1_{K,\varphi}(\widetilde{\Omega}, \mathbb{R}^N) \cap W^{1,k-1}$*. Then we have*

$$\mathcal{H}^{n-k}\big(S_{n-k}(u)\big) \le \liminf_{h\to\infty} \mathcal{H}^{n-k}\big(S_{n-k}(u_h)\big)\,.$$

Proof. As in the proof of Theorem 4.3, we apply Theorem 1.1 with $n-k$ instead of k and

$$T := \pi_\#((\partial G_u)_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta)\,, \qquad T_h := \pi_\#((\partial G_{u_h})_{(k-1)} \llcorner \widehat{\pi}^\# \omega_\beta)\,,$$

for every β. Using a covering argument as in the second part of the proof of [**9**, Thm. 5.8], we obtain the assertion. □

Existence results.

We conclude this section by stating an existence result. Let $2 \le k \le \min(n, N)$, and consider the energy functional

$$\mathcal{F}_k(u) := c_1 \int_\Omega \Phi(|\overrightarrow{M}(Du)|)\, dx + c_2 \int_\Omega |Du|^{k-1}\, dx + c_3\, \mathcal{H}^{n-k}(S_{n-k}(u))\,, \tag{5.10}$$

where $c_1, c_2, c_3 > 0$. We have:

Theorem 5.8. (Existence result II). *Let Φ be as in Theorem* 4.4*, where* $p > 1$. *Then for every* $K > 0$ *and* $\varphi \in \mathrm{Cf}^p_K(\widetilde{\Omega}, \mathbb{R}^N)$ *smooth, there exists a solution to the problem*

$$\inf\{\mathcal{F}_k(u) \mid u \in \mathrm{Cf}^p_{K,\varphi}(\widetilde{\Omega}, \mathbb{R}^N) \cap W^{1,k-1}\}\,.$$

The proof of Theorem 5.8 is omitted, being similar to the one of Theorem 4.4, on account of Theorem 5.7. We notice again that one may similarly consider functionals with more general bulk energies given by the integral of a polyconvex function of the gradient Du, satisfying a suitable growth condition. Finally, since the zero boundary condition (5.2) is preserved by the weak convergence as currents, one may also consider similar variational problems involving the wider class of functions in $\mathrm{Cf}^1_{K,\varphi}(\widetilde{\Omega}, \mathbb{R}^N)$ satisfying (5.2), this time taking the constant $c_2 = 0$ in (5.10), and ∇u instead of Du.

References

[**1**] F. J. Almgren, *Deformations and multiple-valued functions*, In: "Geometric measure theory and the calculus of variations", Proc. Sympos. Pure Math. **44**, Amer. Math. Soc., Providence, RI 1986, 29-130.

[**2**] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems,* Oxford Math. Monogr., Oxford Univ. Press, New York 2000.

[**3**] L. Ambrosio and B. Kirchheim, *Currents on metric spaces*, Acta Math. **185** (2000), 1-80.

[**4**] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1976/77), 337-403.

[**5**] S. Conti and C. De Lellis, *Some remarks on the theory of elasticity for compressible Neohookean materials*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), 521-549.

[**6**] C. De Lellis, *Some fine properties of currents and applications to distributional Jacobians*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), 815-842.

[**7**] H. Federer, *Geometric measure theory,* Grundlehren der mathematischen Wissenschaften 153, Springer, New York 1969.

[**8**] M. Giaquinta, G. Modica and J. Souček, *Cartesian currents in the calculus of variations,* I, II. Ergeb. Math. Grenzgeb. (3), 37, 38, Springer, Berlin 1998.

[**9**] D. Mucci, *Fractures and vector valued maps*, Calc. Var. Partial Differential Equations **22** (2005), 391-420.

[**10**] D. Mucci, *Erratum to: Fractures and vector valued maps*, Calc. Var. Partial Differential Equations **37** (2010), 547-548.

[**11**] S. Müller, $\mathrm{Det} = \det$. *A remark on the distributional determinant*, C. R. Acad. Sci. Paris Sér I Math. **311** (1990), 13-17.

[**12**] S. MÜLLER, *On the singular support of the distributional determinant*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 657-696.

[**13**] S. MÜLLER and S. SPECTOR, *An existence theory for nonlinear elasticity that allows for cavitation*, Arch. Rational Mech. Anal. **131** (1995), 1-66.

[**14**] L. SIMON, *Lectures on geometric measure theory,* Proc. Centre Math. Anal., Vol. **3**, Australian National University, Canberra 1983.

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