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Ekman equations for wind-driven currents: a theoretical analysis of some related numerical integration schemes

Abstract. The Ekman equations for wind driven ocean currents are summarized, and the associated boundary conditions are discussed. The discrete space equations of the numerical approximation are formulated in both real and complex form, because both these forms are used in the analysis of their integration in time. Time integration procedures for the discrete space equations to simulate different responses of the system, either undamped or damped oscillations, are illustrated. Upper bound conditions on the size of the time step are given for the stability and convergence of the numerical schemes.

Keywords. Ekman equations, wind-driven currents, time integration schemes, stability.

Mathematics Subject Classification (2000): 65M06, 65M12.

1 - Introduction: basic equations and associated boundary conditions

We refer the reader back to the book by Pedlosky, 1979 (p. 174), and to the introduction of the paper by Buffoni et al., 2010 a, for an analysis of the basic assumptions supporting the following depth dependent equations for ocean currents. Here we confine ourselves to summarize the basic equations of the pioneer model by Ekman (Ekman,1905), and discuss the boundary conditions associated to it.

Let the horizontal wind driven velocity field be denoted by $[u(t,z),\ v(t,z)]$, depending only on time t and on the the vertical upward coordinate z, $z \in (-H,\ 0)$, with $H = \textit{water layer depth}$. The initial boundary value problem for the velocity

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field is written as

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial z} K \frac{\partial u}{\partial z} - fv \;=\; 0, \tag{1}$$

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial z} K \frac{\partial v}{\partial z} + fu \;=\; 0, \tag{2}$$

$$-K\frac{\partial u}{\partial z}\bigg|_{z=0} = -\frac{\tau_x}{\rho_w}, \quad -K\frac{\partial v}{\partial z}\bigg|_{z=0} = -\frac{\tau_y}{\rho_w}, \tag{3}$$

$$-K\frac{\partial u}{\partial z}\bigg|_{z=-H} = -\alpha\,[u(t,-H) - \hat{u}], \quad -K\frac{\partial v}{\partial z}\bigg|_{z=-H} = -\alpha\,[v(t,-H) - \hat{v}], \tag{4}$$

$$u(0,z) = u^0(z), \quad v(0,z) = v^0(z), \tag{5}$$

where K is the vertical eddy viscosity, and f is the Coriolis parameter. The boundary conditions are given specifying the diffusive flux $-K\partial[u,v]/\partial z$ at $z = -H, 0$. The wind stress $[\tau_x,\ \tau_y]$ at the sea surface in the boundary conditions (3), where ρ_w is the water density, is the driving force of the motion.

This force is obtained from the experimental wind data (speed and direction) by means of empirical formulas of varying intricacy (Kochanski et al., 2006). The symplest expression (Pond and Pickard, 1983) is

$$[\tau_x,\ \tau_y] = \rho_a C_D \sqrt{w_x^2 + w_y^2}\,[w_x,\ w_y], \tag{6}$$

where $\rho_a = 1.3\,kg\,m^{-3}$ is the air density, $C_D = 1.4\,10^{-3}$ is an adimensional drag coefficient, and $[w_x,\ w_y]$ is the horizontal wind velocity in ms^{-1} taken at a reference height of 10 m.

The boundary conditions (4) describe the interaction between the water layer $(-H, 0)$ considered and the region below $z = -H$, which can be either the physical sea bottom or a deeper water layer. In these boundary conditions the parameter α is nonnegative; when $\alpha > 0$, it represents a friction factor for the motion (its dimensions are those of a velocity). The reference velocities $\hat{u}$ and $\hat{v}$ may assume different meanings, depending on the physical situations. The conditions (4) imply that both $|u(t,z) - \hat{u}|$ and $|v(t,z) - \hat{v}|$ are decreasing as $z \to -H$; moreover, when $\alpha \to +\infty$, these conditions become the Dirichlet conditions $u(t,-H) = \hat{u}$, $v(t,-H) = \hat{v}$.

Typical range of K (Nihoul, 1975, p. 75; Pedlosky, 1979, p. 174; Sheng, 1983, p. 43) is $K \in [0.001,\ 0.1]\,m^2s^{-1}$. The depth H is of the order of some units of the Ekman layer given by $\sqrt{2K/f}$, so that with $f = 10^{-4}\,s^{-1}$ we have that $H \leq 100\,m$.

Under the action of a constant wind tress, or when $[\tau_x(t), \tau_y(t)] \to [\tau_{x\infty}, \tau_{y\infty}]$, there exists an equilibrium state $[u_\infty(z),\ v_\infty(z)]$ for the velocity field. It can be shown that $[u_\infty(z),\ v_\infty(z)]$ is asymptotically stable when $\alpha > 0$, and unstable when $\alpha = 0$. Thus, the boundary conditions assigned at the bottom of the water layer considered, influence the asymptotic behaviour of the motions: when $\alpha > 0$ we have damped oscillations around $[u_\infty(z),\ v_\infty(z)]$, with damping time scales depending on $\alpha,\ K,\ H$ (Buffoni et al., 2010 b), while when $\alpha = 0$ undamped oscillations.

The case of no-slip boundary condition at the bottom of the water layer (i.e. when $\alpha = +\infty$ and $\hat{u} = 0,\ \hat{v} = 0$), has been assumed by Lewis and Belcher (2004); they investigated both the cases of a layer of finite and infinite depth. The other limit case $\alpha = 0$, i.e. the condition of zero stress fixed at the bottom of the water layer, has been assumed by Price and Sundermeyer (1999); this model shows that an impulsively started wind stress induces a flow, which continues to oscillate undamped. Lewis and Belcher (2004) say that "phisically, this seems somewhat implausible", so they use the no-slip boundary condition. Here we use the formulation (4), which admits the models by Price and Sundermeyer (1999) and Lewis and Belcher (2004) as limit cases for $\alpha = 0$ and $\alpha \to +\infty$, respectively.

Software, both in FORTRAN and in MATLAB, for solving physical oceanography problems, in particular for educational purposes, is proposed in the web (Pawlowicz, 2010; Price, 2010). Often, any justification about stability properties, convergence to a steady state, or supporting undamped oscillations, of the numerical procedures are not given. The main objective of this work is a theoretical analysis of some numerical integration schemes of equations (1)-(5), to illustrate their behaviours in different situations mainly characterized by the value of the parameter α.

The paper is organized as follows. In Section 2 the discrete space equations are derived. In Section 3 time integration schemes are illustrated, and their stability properties and ability to reproduce oscillations when friction is absent are proved. In Section 4 some concluding remarks can be found.

2 - Numerical approximations: discrete space equations

Here, for one dimensional spatial problems, we prefer to derive the numerical approximation by the finite difference method, obtaining equations with the same basic properties of the finite element equations. To simplify the notation, and also to obtain analytical expressions for some basic parameters, we consider a constant eddy viscosity K; however, matrix equations with the same properties are obtained when K is considered depth dependent. The wind stress, which is the driving force of the motion, only acts in the boundary conditions (3); so that in the discrete space equations it will appear as source terms. The discrete space equations associated to (1)-(5) are first derived, and then time integration is performed.

2.1 - *Equations in real form*

Let us define in the interval $(-H,0)$ a uniform grid $z_\ell = -H + (\ell-1)\delta$, $\ell = 1,\dots,n$, where $\delta = h/(n-1)$ is the mesh spacing and n is the order of approximation (e.g.: $H = 100\ m$, $\delta = 2.5\ m$, $n = 41$). Let

$$[u_\ell(t),\ v_\ell(t)] \ = \ [u(t,z_\ell),\ v(t,z_\ell)], \quad [u_\ell^0,\ v_\ell^0] \ = \ [u^0(z_\ell),\ v^0(z_\ell)],$$

and

$$\chi \ = \ \frac{K}{\delta^2}.$$

The following set of ode are obtained from the continuous problem (1)-(4) by the finite volume method (or integration method, Varga, 1962, p. 167):

$$\frac{1}{2}\frac{du_1}{dt} + \Big(\chi + \frac{\alpha}{\delta}\Big)u_1 - \chi u_2 - \frac{1}{2}\, f v_1 = \frac{\alpha \hat{u}}{\delta}, \tag{7}$$

$$\frac{du_\ell}{dt} - \chi u_{\ell-1} + 2\chi u_\ell - \chi u_{\ell+1} - f v_\ell = 0, \quad \ell = 2,3,\dots,n-1, \tag{8}$$

$$\frac{1}{2}\frac{du_n}{dt} - \chi u_{n-1} + \chi u_n - \frac{1}{2}\, f v_n = \frac{\tau_x}{\rho_w\,\delta}, \tag{9}$$

$$\frac{1}{2}\frac{dv_1}{dt} + \Big(\chi + \frac{\alpha}{\delta}\Big)v_1 - \chi v_2 + \frac{1}{2}\, f u_1 = \frac{\alpha \hat{v}}{\delta}, \tag{10}$$

$$\frac{dv_\ell}{dt} - \chi v_{\ell-1} + 2\chi v_\ell - \chi v_{\ell+1} + f u_\ell = 0, \quad \ell = 2,3,\dots,n-1, \tag{11}$$

$$\frac{1}{2}\frac{dv_n}{dt} - \chi v_{n-1} + \chi v_n + \frac{1}{2}\, f u_n = \frac{\tau_y}{\rho_w\,\delta}, \tag{12}$$

$$[u_\ell(0),\ v_\ell(0)] \ = \ [u_\ell^0,\ v_\ell^0], \quad \ell = 1,2,\dots,n. \tag{13}$$

Let us define the vectors

$$\boldsymbol{u}(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T, \quad \boldsymbol{v}(t) = [v_1(t), v_2(t), \dots, v_n(t)]^T,$$

$$\boldsymbol{u}^0 = [u_1^0, u_2^0, \dots, u_n^0]^T, \quad \boldsymbol{v}^0 = [v_1^0, v_2^0, \dots, v_n^0]^T,$$

$$\boldsymbol{q}_x = \left[\frac{\alpha\hat{u}}{\delta}, 0, \dots, 0, \frac{\tau_x}{\rho_w\delta}\right]^T, \quad \boldsymbol{q}_y = \left[\frac{\alpha\hat{v}}{\delta}, 0, \dots, 0, \frac{\tau_y}{\rho_w\delta}\right]^T,$$

the symmetric matrix

$$A = \begin{bmatrix} \chi + \alpha/\delta & -\chi & 0 & 0 & \dots & 0 \\ -\chi & 2\chi & -\chi & 0 & \dots & 0 \\ & & \dots & & & \\ 0 & 0 & \dots & -\chi & 2\chi & -\chi \\ 0 & 0 & \dots & 0 & -\chi & \chi \end{bmatrix},$$

and the diagonal matrix D with $diag\, D = (1/2, 1, \dots, 1, 1/2)$. Then, the discrete equations (7)-(13) are written in vectorial form as

$$D\,\frac{d\boldsymbol{u}}{dt} + A\boldsymbol{u} - f\,D\boldsymbol{v} \;=\; \boldsymbol{q}_x, \tag{14}$$

$$D\,\frac{d\boldsymbol{v}}{dt} + A\boldsymbol{v} + f\,D\boldsymbol{u} \;=\; \boldsymbol{q}_y, \tag{15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \boldsymbol{v}(0) = \boldsymbol{v}^0. \tag{16}$$

2.2 - *Equations in complex form*

Equations (14)-(16) can be written in an equivalent complex form. Let

$$\boldsymbol{p}(t) = \boldsymbol{u}(t) + i\boldsymbol{v}(t), \quad \boldsymbol{p}^0 = \boldsymbol{u}^0 + i\boldsymbol{v}^0, \quad \boldsymbol{q} = \boldsymbol{q}_x + i\boldsymbol{q}_y.$$

Then, we have

$$D\,\frac{d\boldsymbol{p}}{dt} + (A + if\,D)\boldsymbol{p} \;=\; \boldsymbol{q}, \quad \boldsymbol{p}(0) \;=\; \boldsymbol{p}^0. \tag{17}$$

By letting

$$C \;=\; D^{-1}A + if\,I, \tag{18}$$

equation (17) is written as

$$\frac{d\boldsymbol{p}}{dt} + C\boldsymbol{p} \;=\; D^{-1}\boldsymbol{q}, \quad \boldsymbol{p}(0) = \boldsymbol{p}^0. \tag{19}$$

When $\boldsymbol{q}$ is time independent, the unique steady state $\boldsymbol{p}^*$ is solution to the stationary equation

$$(A + if\,D)\boldsymbol{p}^* \;=\; \boldsymbol{q}.$$

It can be easily obtained by means of the algorithm for solving a linear system with a tridiagonal matrix (Varga, 1962, p. 195).

3 - Time integration

Now we will review some basic methods for solving the problem (14)-(16) or the equivalent (17), under the assumption of a constant wind stress. For some time integration methods, the analysis can be equivalently performed starting from the equations either in real or complex form, while for others only by using the real equations. For the various time iterative processes considered, we will adopt, without ingenerating misunderstanding, the same notation for the iteration matrix, its eigenvalues, and for some other quantities introduced in the proofs.

As stated in the introduction, when the wind stress is time independent, the system will produce well defined asymptotic behaviours, depending on the value of the parameter α introduced in the boundary conditions at the bottom depth. These behaviours are determined by the spectrum of the matrix equations (14)-(16) or the equivalent (17). Thus, a preliminary analisys of the eigenvalue problem associated to the discrete space equations (14)-(16), or the equivalent (17), is here performed.

3.1 - *The eigenvalue problem associated to the discrete space equations*

The eigenvalue problem associated to (14)-(15) is written as

$$\lambda \boldsymbol{U} + D^{-1} A \boldsymbol{U} - f\,\boldsymbol{V} = 0, \tag{20}$$

$$\lambda \boldsymbol{V} + D^{-1} A \boldsymbol{V} + f\,\boldsymbol{U} = 0. \tag{21}$$

The eigenpairs solutions to (20), (21) are given by

$$\lambda_j^{\pm} = -\mu_j \pm if, \quad [\boldsymbol{U}_j^{\pm},\ \boldsymbol{V}_j^{\pm}] = [\boldsymbol{e}_j,\ \pm i\boldsymbol{e}_j], \tag{22}$$

for $j = 1, 2, \ldots, n$, where $[\mu_j, \boldsymbol{e}_j]$ are the eigenpairs associated to $D^{-1}A$. The eigenvalues μ_j are real and nonnegative (Gantmacher, 1960, p. 310). It can be shown that

$$\mu_j = \frac{2K}{\delta^2}(1 - \cos(\hat{v}_j \delta)),$$

where $\hat{v}_j$ are the nonnegative solutions of the equation

$$\frac{\alpha}{K} = \frac{\sin(v\delta)}{\delta} \tan(vH),$$

and

$$e_{j\ell} = e_j^0 \cos(\hat{v}_j\, \delta(n - \ell)), \quad \ell = 1, 2, \ldots, n,$$

where e_j^0 are constants. As $\delta \to 0$, $\hat{v}_j \to v_j$ and $\mu_j \to K v_j^2$. When $\alpha > 0$ the zero eigenvalue is missing, while when $\alpha = 0$ the eigenvalue $\mu_1 = 0$ exists. The corresponding eigenvector is $\boldsymbol{e}_1 = constant\ [1, 1, \ldots, 1]$; its components are all equal to a constant, i.e. it is independent of depth.

The eigenvalues $\lambda_j^{\pm}$ in (22) are all different from zero for any $\alpha \geq 0$; thus, when the wind stress is time independent, there always exists a unique steady state solution to (14)-(15).

3.2 - *Numerical integration schemes*

(**I**) Let

$$C = D^{-1}A + ifI, \tag{23}$$

so that (17) is now written as

$$\frac{d\boldsymbol{p}}{dt} + C\boldsymbol{p} = D^{-1}\boldsymbol{q}, \quad \boldsymbol{p}(0) = \boldsymbol{p}^0. \tag{24}$$

The classical two levels explicit-implcit schemes are summarized in the equation

$$[I + \omega \Delta t C]\boldsymbol{p}(t + \Delta t) = [I - (1-\omega)\Delta t C]\boldsymbol{p}(t) + \Delta t D^{-1}\boldsymbol{q}, \tag{25}$$

where $0 \leq \omega \leq 1$, and Δt is the time step. In particular, for $\omega = 0$ we have the explicit method, for $\omega = 1$ the fully implicit method, while for $\omega = 1/2$ the Crank-Nicolson method (Varga, 1962, p. 264).

The asymptotic properties of $\boldsymbol{p}(t)$ are determined by the behaviour of

$$\boldsymbol{r}(t) = \boldsymbol{p}(t) - \boldsymbol{p}^*, \tag{26}$$

which satisfies the iterative process

$$\boldsymbol{r}(t + \Delta t) = M(\omega, \Delta t)\, \boldsymbol{r}(t), \quad \boldsymbol{r}(0) = \boldsymbol{p}^0 - \boldsymbol{p}^*, \tag{27}$$

where

$$M(\omega, \Delta t) = [I + \omega \Delta t C]^{-1}\, [I - (1-\omega)\Delta t C] \tag{28}$$

is the iteration matrix. In appendix B it is shown that a process of type (27) may produce asymptotic undamped oscillations only when the eigenvalue of $M(\omega, \Delta t)$ of maximum modulus is complex, and its modulus, i.e. the spectral radius $\rho(M)$, is equal to 1: $\rho(M) = 1$. In the following theorem conditions on Δt and ω to have the moduli of the eigenvalues of $M(\omega, \Delta t)$ less or equal to one are given.

Theorem 1. *Let μ_j, λ_j be the eigenvalues of $D^{-1}A$ and $M(\omega, \Delta t)$, respectively. Then, for $\alpha \geq 0$, $0 \leq \omega \leq 1$ and $\mu_j > 0$ we have*

$$(1 - 2\omega)\, \Delta t < \frac{2\mu_j}{\mu_j^2 + f^2} \iff |\lambda_j| < 1. \tag{29}$$

Moreover, when $\alpha = 0$ *we have* $\rho(M) = |\lambda_1| = 1$ *only for* $\omega = 1/2$*, obtaining*

$$\lambda_1 = e^{i\theta} \quad \textit{with} \quad \tan\theta = \frac{-f\varDelta t}{1 - f^2\varDelta t^2/4} \quad \textit{for } \varDelta t \neq \frac{2}{f}, \tag{30}$$

or $\theta = -\pi/2$ *for* $\varDelta t = 2/f$.

Proof. From the expression of $M(\omega, \varDelta t)$ it follows that its eigenvectors are $a_j\boldsymbol{e}_j$, where a_j are constants and $\boldsymbol{e}_j$ are the eigenvectors of $D^{-1}A$, and its eigenvalues λ_j are given by

$$\lambda_j = \frac{1 - (1-\omega)\varDelta t\,(\mu_j + if)}{1 + \omega\varDelta t\,(\mu_j + if)}. \tag{31}$$

Thus, the squared moduli of $|\lambda_j|$ are

$$|\lambda_j|^2 = \frac{[1 - (1-\omega)\mu_j\varDelta t]^2 + [(1-\omega)f\varDelta t]^2}{[1 + \omega\mu_j\varDelta t]^2 + [\omega f\varDelta t]^2}. \tag{32}$$

From (32), by direct calculation we obtain (29).

Assume now $\alpha = 0$, so that there exists the eigenvalue $\mu_1 = 0$, and $\mu_j > 0,\ j > 1$. As the right member of (32) is decreasing with μ_j, we have

$$\rho(M) = |\lambda_1| = \frac{1 + [(1-\omega)f\varDelta t]^2}{1 + [\omega f\varDelta t]^2}. \tag{33}$$

Thus, we have $|\lambda_1| = 1$ only when $\omega = 1/2$; the eigenvalue λ_1 is then given by

$$\lambda_1 = \frac{1 - if\varDelta t/2}{1 + if\varDelta t/2} = \frac{1 - f^2\varDelta t^2/4 - if\varDelta t}{1 + f^2\varDelta t^2/4} = e^{i\theta}, \tag{34}$$

with θ given by (30). □

(**II**) A suitable variant of the explicit method (defined by $\boldsymbol{\omega} = 0$ in (25)) leads to undamped oscillations in the case $\alpha = 0$ (Price, 2010). The analysis can be performed only by using the equations (14), (15) in real form. Let us consider the following explicit discretization scheme of equations (14), (15)

$$\begin{aligned} \boldsymbol{u}(t + \varDelta t) &= [I - \varDelta t D^{-1}A]\boldsymbol{u}(t) + f\varDelta t\boldsymbol{v}(t) + \varDelta t D^{-1}\boldsymbol{q}_x, \\ \boldsymbol{v}(t + \varDelta t) &= [I - \varDelta t D^{-1}A]\boldsymbol{v}(t) - f\varDelta t\boldsymbol{u}(t + \varDelta t) + \varDelta t D^{-1}\boldsymbol{q}_y. \end{aligned}$$

The asymptotic properties of $[\boldsymbol{u}(t),\ \boldsymbol{v}(t)]$ depend on the behaviour of

$$[\boldsymbol{r}_x(t),\ \boldsymbol{r}_y(t)] = [\boldsymbol{u}(t) - \boldsymbol{u}^*,\ \boldsymbol{v}(t) - \boldsymbol{v}^*],$$

which satisfies the iterative process

$$[\boldsymbol{r}_x(t+\Delta t),\ \boldsymbol{r}_y(t+\Delta t)]^T \ = \ M(\Delta t)\,[\boldsymbol{r}_x(t),\ \boldsymbol{r}_y(t)]^T,$$

where the iteration matrix is given by

$$M(\Delta t) = \begin{vmatrix} I - \Delta t D^{-1}A & f\Delta t\, I \\ -f\Delta t\,(I - \Delta t D^{-1}A) & (1 - f^2\Delta t^2)\, I - \Delta t D^{-1}A \end{vmatrix}.$$

Theorem 2. *Let μ_j, $\lambda_j^{\pm}$, be the eigenvalues of $D^{-1}A$ and $M(\Delta t)$, respectively. Then, for $\alpha \geq 0$ and $\mu_j > 0$*

$$\Delta t < \frac{2}{\mu_j + f} \iff |\lambda_j^{\pm}| < 1. \tag{35}$$

Moreover, when $\alpha = 0$ and Δt satisfies the inequality in (35) *we have $\rho(M) = |\lambda_1^{\pm}| = 1$ and*

$$\lambda_1^{\pm} \ = \ e^{\pm i\theta} \quad with \quad \tan\theta \ = \ \frac{f\Delta t\sqrt{4 - f^2\Delta t^2}}{2 - f^2\Delta t^2}, \quad for\ \Delta t \neq \frac{\sqrt{2}}{f}, \tag{36}$$

or $\theta = \pm\pi/2$ for $\Delta t = \sqrt{2}/f$.

Proof. From the expression of $M(\Delta t)$ it follows that its eigenvectors are given by $[a_j\boldsymbol{e}_j,\ b_j\boldsymbol{e}_j]$, where a_j and b_j are constants and $\boldsymbol{e}_j$ are the eigenvectors of $D^{-1}A$, and the eigenvalue equation is

$$(\lambda - 1 + \mu_j\Delta t)^2 + \lambda f^2\Delta t^2 = 0.$$

Thus,

$$\lambda_j^{\pm} \ = \ \varphi_j(\Delta t) - \frac{1}{4} f^2\Delta t^2 \ \pm\ f\Delta t\sqrt{-\varphi_j(\Delta t)}$$

where

$$\varphi_j(\Delta t) \ = \ 1 - \mu_j\Delta t - \frac{f^2\Delta t^2}{4}.$$

For

$$\Delta t \ < \ T_j \ = \ \frac{2}{f^2}\left(-\mu_j + \sqrt{\mu_j^2 + f^2}\right)$$

we have that $\varphi_j(\Delta t) > 0$ and consequently the eigenvalues $\lambda_j^{\pm}$ are complex. By direct calculation we obtain

$$|\lambda_j^{\pm}|^2 \ = \ (1 - \mu_j\Delta t)^2.$$

It follows that $|\lambda_j^{\pm}| < 1$ when $\mu_j > 0$ and $\Delta t < 2/\mu_j$; the last inequality is satisfied when $\Delta t < T_j$, because $T_j < 2/\mu_j$.

For $\Delta t \geq T_j$, $\varphi_j(\Delta t) \leq 0$, the eigenvalues $\lambda_j^{\pm}$ are real and

$$|\lambda_j^+|<|\lambda_j^-| = \left[\frac{f\Delta t}{2} + \sqrt{-\varphi_j(\Delta t)}\right]^2.$$

It follows that $|\lambda_j^-|<1$ when $\Delta t<2/(\mu_j+f)$. It can be verified that $T_j<2/(\mu_j+f)$; so that $|\lambda_j^-|<1$ for

$$T_j < \Delta t < \frac{2}{\mu_j+f},$$

and then the thesis (35).

Assume now $\alpha = 0$, so that there exists the eigenvalue $\mu_1 = 0$, and $\mu_j > 0,\ j > 1$. If Δt satisfies the inequality in (35), then we have $\rho(M) = |\lambda_1^{\pm}| = 1$, so that $\lambda_1^{\pm} = e^{i\theta}$, with θ given by (36). □

(**III**) Here three levels explicit-implicit schemes are written in complex form directly for the vector $\boldsymbol{r}(t)$ as defined in (26)

$$[I + 2\omega\Delta t\, D^{-1}A]\boldsymbol{r}(t+\Delta t) = [I - 2(1-\omega)\Delta t\, D^{-1}A]\boldsymbol{r}(t-\Delta t) - 2if\Delta t\boldsymbol{r}(t), \tag{37}$$

assuming $\boldsymbol{r}(0)$ and $\boldsymbol{r}(\Delta t)$ given. $\boldsymbol{r}(\Delta t)$ can be obtained from $\boldsymbol{r}(0)$ by using a fully explicit method. The process (37) may be written in the form

$$[\boldsymbol{r}(t),\ \boldsymbol{r}(t+\Delta t)]^T = M(\omega,\Delta t)\,[\boldsymbol{r}(t-2\Delta t),\ \boldsymbol{r}(t-\Delta t)]^T, \tag{38}$$

where $M(\omega,\Delta t)$ is the iteration matrix

$$M(\omega,\Delta t) = \begin{vmatrix} I+2\omega\Delta t D^{-1}A & 0 \\ 2if\Delta t\, I & I+2\omega\Delta t D^{-1}A \end{vmatrix}^{-1}$$

$$\times \begin{vmatrix} I-2(1-\omega)\Delta t D^{-1}A & -2if\Delta t\, I \\ 0 & I-2(1-\omega)\Delta t D^{-1}A \end{vmatrix}.$$

Theorem 3. *Let μ_j, $\lambda_j^{\pm}$, be the eigenvalues of $D^{-1}A$ and $M(\omega,\Delta t)$, respectively. Then, for $\alpha \geq 0$, $0 \leq \omega \leq 1$, $\mu_j > 0$ we have that*

$$\Delta t < T_j(\omega) = \frac{(2\omega-1)\mu_j + \sqrt{\mu_j^2+f^2}}{4\omega(1-\omega)\mu_j^2+f^2} \implies |\lambda_j^{\pm}|<1. \tag{39}$$

Moreover, when $\alpha = 0$ and Δt satisfies the inequality in (39) *we have $\rho(M) = |\lambda_1^{\pm}| = 1$ and*

$$\lambda_1^{\pm} = e^{\pm i\theta} \quad \textit{with} \quad \tan\theta = \frac{2f\Delta t\sqrt{1-f^2\Delta t^2}}{1-2f^2\Delta t^2}, \quad \textit{for } \Delta t \neq \frac{1}{2f}, \tag{40}$$

or $\theta = \pm\pi/2$ for $\Delta t = 1/(2f)$.

Proof. From the expression of $M(\omega, \Delta t)$ it follows that its eigenvectors are given by $[a_j \boldsymbol{e}_j,\ b_j \boldsymbol{e}_j]$, where a_j and b_j are constants and $\boldsymbol{e}_j$ are the eigenvectors of $D^{-1}A$, and the eigenvalue equation is

$$[\lambda(1+2\omega\mu_j\Delta t) - (1-2(1-\omega)\mu_j\Delta t)]^2 + 4\lambda f^2\Delta t^2 = 0.$$

Thus,

$$\lambda_j^{\pm} = \frac{\varphi_j(\omega,\Delta t) - f^2\Delta t^2 \pm 2f\Delta t\sqrt{-\varphi_j(\omega,\Delta t)}}{(1+2\omega\mu_j\Delta t)^2},$$

where

$$\varphi_j(\omega,\Delta t) = 1 - 2(1-2\omega)\mu_j\Delta t - [4\omega(1-\omega)\mu_j^2 + f^2]\Delta t^2.$$

For $\Delta t < T_j(\omega)$, defined in (39), we have that $\varphi_j(\omega,\Delta t) > 0$ and consequently the eigenvalues $\lambda_j^{\pm}$ are complex. By direct calculation we obtain

$$|\lambda_j^{\pm}|^2 = \left[\frac{1-2(1-\omega)\mu_j\Delta t}{1+2\omega\mu_j\Delta t}\right]^2.$$

It follows that $|\lambda_j^{\pm}| < 1$ when $\mu_j > 0$ and $(1-2\omega)\mu_j\Delta t < 1$. Thus, $|\lambda_j^{\pm}| < 1$ either for $1/2 \le \omega \le 1$ or for $0 \le \omega < 1/2$ and $\Delta t < 1/[(1-2\omega)\mu_j]$; the last inequality is satisfied when $\Delta t < T_j(\omega)$ because $T_j(\omega) < 1/[(1-2\omega)\mu_j]$.

Assume now $\alpha = 0$, so that there exists the eigenvalue $\mu_1 = 0$, and $\mu_j > 0,\ j > 1$. If Δt satisfies the inequality in (39), then we have $\rho(M) = |\lambda_1^{\pm}| = 1$, so that $\lambda_1^{\pm} = e^{i\theta}$, with θ given by (40). □

(**IV**) Predictor-corrector type solvers for ODE can be applied for solving (14)-(16). The subroutine DGEAR (Hindmarsh, 1974) of the IMSL library has been used successfully, for both $\alpha = 0$ and $\alpha > 0$. In this subroutine the implicit linear multistep Adams method is implemented; the time step size is adjusted automatically.

4 - Concluding remarks

Some specific remarks can be drawn about the results obtained from the previous analysis of the time integration schemes.

(**i**) Assume $\alpha > 0$, so that $\mu_j > 0,\ j \ge 1$. Thus, from (29) it follows that $\rho(M) < 1$, and then $\boldsymbol{r}(k\Delta t) \to 0$ as $k \to +\infty$, either when $\omega \ge 1/2$ for arbitrary Δt or when $\omega < 1/2$ and Δt satisfies the inequality (29). The second option $\omega < 1/2$ is not recommended.

(**ii**) Since $\max_j \ \mu_j \leq 4K/\delta^2$, the bound (35) for Δt becomes

$$\Delta t \ < \ \frac{2}{4K/\delta^2 + f}.$$

When K/δ^2 is of the same order of f (for example with $K \simeq 10^{-2} m^2 s^{-1}$, $\delta \simeq 5\,m$, $f \simeq 10^{-4} s^{-1}$) it is not a strict limitation for Δt.

(**iii**) For $\Delta t \geq T_j(\omega)$ defined in (39), $\varphi_j(\omega, \Delta t) \leq 0$, the eigenvalues $\lambda_j^{\pm}$ are real and

$$|\lambda_j^+| < |\lambda_j^-| \ = \ \left[\frac{f\Delta t + \sqrt{-\varphi_j(\omega, \Delta t)}}{1 + 2\omega\mu_j \Delta t} \right]^2.$$

However, starting from this expression of $|\lambda_j^-|$, no improvement of the bound for Δt in (39) has been obtained.

(**iv**) Let us consider the expression of $T_j(\omega)$ defined in (39) for $\omega = 0,\ 1/2,\ 1$:

$$T_j(0) = \frac{-\mu_j + \sqrt{\mu_j^2 + f^2}}{f^2}, \quad T_j(1/2) = \frac{1}{\sqrt{\mu_j^2 + f^2}}, \quad T_j(1) = \frac{\mu_j + \sqrt{\mu_j^2 + f^2}}{f^2}.$$

It can be verified that $T_j(0) \leq T_j(1/2) \leq T_j(1)$, where the equalities hold only when $\alpha = 0$ and $j = 1$; in this case we have $T_1(0) = T_1(1/2) = T_1(1) = 1/f$.

We have that $T_j(0)$ and $T_j(1/2)$ are decreasing with μ_j, so that the inequality in (39) is satisfied for all μ_j when

$$\Delta t \ < \ \max_{\mu_j > 0} \ [T_j(0),\ T_j(1/2)].$$

On the contrary, $T_j(1)$ is increasing with μ_j, so that the inequality in (39) is satisfied for all μ_j when

$$\Delta t \ < \ \min_{\mu_j > 0} \ T_j(1).$$

A comparison with the upper bound $2/(\mu_j + f)$ for Δt in (35) gives $T_j(1/2) < 2/(\mu_j + f)$ for all μ_j, while $T_j(1) < 2/(\mu_j + f)$ only for small μ_j.

(**v**) Assume $\alpha = 0$. From (30), (36), (40) we have that if $f\Delta t << 1$ then $\theta \simeq f\Delta t$. Thus, from

$$\lambda_1^k = e^{ik\theta} = e^{ikf\Delta t}$$

we obtain the period $k_{period}\Delta t = 2\pi/f$ (see appendix). The eigenvector corresponding to λ_1 is *constant* $\boldsymbol{e}_1$, independent of depth.

Few general remarks about the application of the numerical procedures are now given.

Steady state solutions, when they exist, can be easily obtained by writing the stationary equations in complex form, and then use the algorithm for solving a linear system with a tridiagonal matrix (Varga, 1962, p. 195).

When the wind stress is time independent, all the numerical integration schemes analyzed in Subsection 3.2 produce nearly the same outcomes for the same problem. Obviously, when they are suitable to solve the problem, and under the assumptions on the bounds of the time step requested for their correct working.

When the wind stress is time dependent, we obtained reliable results by using both the fully implicit method for parabolic equations (Varga, 1962, p. 264), and routines for ODE, such the IMSL routine DGEAR.

Results of numerical simulations, in particular for time dependent winds, either deterministic or stochastic processes, can be found in Buffoni et al., 2010 a, b.

Appendix

Let M be an $n \times n$ (real or complex) matrix, and let us consider the following iterative process

$$\boldsymbol{a}^k = M\,\boldsymbol{a}^{k-1} = M^k\,\boldsymbol{a}^0, \quad k = 1, 2, \ldots; \quad \boldsymbol{a}^0 \text{ given.} \tag{41}$$

Let $(\lambda_j, \boldsymbol{b}_j)$ be the eigenpairs associated to M

$$M\,\boldsymbol{b}_j = \lambda_j\,\boldsymbol{b}_j, \quad j = 1, 2, \ldots, n. \tag{42}$$

Assume

$$|\lambda_1| \; > (=) \; |\lambda_2| \; \geq (>) \; |\lambda_3| \; \geq \; \ldots$$

The relations in parentheses should hold when λ_2 is the complex conjugate of λ_1, as it happens when the matrix M is real. The asymptotic properties of $\boldsymbol{a}^k$ depend on the spectral radius $\rho(M) = |\lambda_1|$ of M. We have that

$$\rho(M) \; > 1, \; < 1 \Longrightarrow \|\boldsymbol{a}^k\| \rightarrow \; +\infty, \; 0.$$

Let λ_2 be not the complex conjugate of λ_1. The process (41) may produce undamped oscillations in the sequence $\boldsymbol{a}^k$ when

$$\lambda_1 \text{ is complex with } |\lambda_1| = 1, \; i.e. \; \lambda_1 = e^{i\theta} \text{ with } \theta \neq 0.$$

In fact, assume that the initial vector $\boldsymbol{a}^0$ could be written as

$$\boldsymbol{a}^0 = \sum_{j=1}^{n} \alpha_j\,\boldsymbol{b}_j.$$

Then, from (41) we have

$$\boldsymbol{a}^k = \sum_{j=1}^{n} \alpha_j \lambda_j^k \boldsymbol{b}_j = \alpha_1 e^{ik\theta} \boldsymbol{b}_1 + \boldsymbol{c}^k, \tag{43}$$

where $\boldsymbol{c}^k$ is the contribution due to the eigenvectors $\boldsymbol{b}_j$ with $j > 1$. Thus, $\boldsymbol{c}^k \to 0$ as $k \to +\infty$, and the sequence $\boldsymbol{a}^k$ present undamped oscillations. When the ratio $2\pi/\theta$ is an integer, the period is given by $k_{period} = 2\pi/\theta$.

When λ_2 is the complex conjugate of λ_1, the expression (43) becomes

$$\boldsymbol{a}^k = \alpha_1^+ e^{ik\theta} \boldsymbol{b}_1^+ + \alpha_1^- e^{-ik\theta} \boldsymbol{b}_1^- + \boldsymbol{c}^k. \tag{44}$$

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