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On the functional solutions of a system of Partial Differential Equations relevant in mathematical physics

Abstract. We study the system of P.D.E.

$$\nabla \cdot \left[\sum_{j=1}^{n} a_{ij}(\boldsymbol{u}, w) \nabla u_j + b_i(\boldsymbol{u}, w) \nabla w \right] = 0, \; i = 1, \ldots, n$$

$$\nabla \cdot (K(\boldsymbol{u}, w) \nabla w) = 0 \text{ in } \Omega$$

and the class of its solutions $(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x})) = (u_1(\boldsymbol{x}), \ldots, u_n(\boldsymbol{x}), w(\boldsymbol{x}))$ which occurs when a functional relation between $\boldsymbol{u}(\boldsymbol{x})$ and $w(\boldsymbol{x})$ of the form $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{U}(w(\boldsymbol{x}))$ exists. If the solution satisfies constant boundary conditions $\boldsymbol{U}(w)$ is shown to exist and to satisfy a Volterra-Fredholm integral equation. An application to the thermistor problem and to a bifurcation problem of filtration in porous media is also given.

Keywords. Functional solutions, system of PDE, Thermistor problem, Volterra-Fredholm integral equation.

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1 - Introduction

The autonomous boundary value problem (P)

$$\nabla \cdot \left[\sum_{j=1}^{n} a_{ij}(\boldsymbol{u}, w) \nabla u_j + b_i(\boldsymbol{u}, w) \nabla w \right] = 0 \text{ in } \Omega, \; \boldsymbol{u} = (u_1, \ldots, u_n), \; i = 1, \ldots, n \tag{1.1}$$

$$\nabla \cdot (K(\boldsymbol{u}, w) \nabla w) = 0 \text{ in } \Omega \tag{1.2}$$

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$$\boldsymbol{u} = 0 \text{ on } \Gamma_0,\ \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \Gamma_1, \boldsymbol{q}_i \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_2 \tag{1.3}$$

$$w = 0 \text{ on } \Gamma_0,\ w = \bar{w} \text{ on } \Gamma_1,\ K(\boldsymbol{u}, w)\frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_2, \tag{1.4}$$

where $\nabla w = (w_{x_1}, \dots, w_{x_N})$, Ω is an open and bounded subset of $\boldsymbol{R}^N$ with a regular boundary Γ which consists of three disjoint hypersurfaces Γ_0, Γ_1 and Γ_2, $\bar{\boldsymbol{u}} = (\bar{u}_1, \dots, \bar{u}_n)$ is a constant vector of $\boldsymbol{R}^n$, the $\boldsymbol{q}_i$'s are given by

$$\boldsymbol{q}_i = \sum_{j=1}^{n} a_{ij}(\boldsymbol{u}, w)\nabla u_j + b_i(\boldsymbol{u}, w)\nabla w,\ i = 1, \dots, n$$

and $\boldsymbol{n}$ is the unit vector of the normal to Γ_2, has many physical motivations. We recall, in particular, the equations of the electrical heating of a conductor, the so-called "thermistor problem" [**9**], [**7**]

$$\nabla \cdot (\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega \tag{1.5}$$

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega \tag{1.6}$$

$$u = 0 \text{ on } \Gamma_0,\ u = \bar{u} \text{ on } \Gamma_1,\ \kappa(u)\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \tag{1.7}$$

$$\varphi = 0 \text{ on } \Gamma_0,\ \varphi = \bar{\varphi} \text{ on } \Gamma_1,\ \sigma(u)\frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_2, \tag{1.8}$$

where $u(\boldsymbol{x})$ and $\varphi(\boldsymbol{x})$, $\boldsymbol{x} = (x, y, z)$ are the temperature and the electric potential. The electric conductivity $\sigma(u)$ and the thermal conductivity $\kappa(u)$ are given functions of the temperature. Also the flow of an incompressible fluid in a porous medium, taking into account the Soret and Dufour effects, leads [**1**], [**3**] to a problem of the form (P). More precisely we have, with $\Gamma_2 = \emptyset$,

$$\nabla \cdot [\nabla u + S\nabla v + f(u, v)\nabla w] = 0 \text{ in } \Omega \tag{1.9}$$

$$\nabla \cdot [\nabla v + D\nabla u + g(u, v)\nabla w] = 0 \text{ in } \Omega \tag{1.10}$$

$$\nabla \cdot (K\nabla w) = 0 \text{ in } \Omega \tag{1.11}$$

$$u = 0 \text{ on } \Gamma_0,\ u = \bar{u} \text{ on } \Gamma_1 \tag{1.12}$$

$$v = 0 \text{ on } \Gamma_0,\ v = \bar{v} \text{ on } \Gamma_1 \tag{1.13}$$

$$w = 0 \text{ on } \Gamma_0,\ w = \bar{w} \text{ on } \Gamma_1. \tag{1.14}$$

Here u is the concentration, v the temperature and w the pressure. Equation (1.11) follows from the incompressibility assumption and the Darcy's law, (1.9) and (1.10) are balance equations, in which S and D are the Soret and Dufour coefficients.

If Ω is a subset of $\boldsymbol{R}^2$, the equations (1.1) and (1.2) are invariant under all conformal mappings $\phi(z)$ such that $|\phi'(z)| \neq 0$, as noticed in the case of problem (1.5)-(1.7) in [**6**]. Suppose in problem (P)

$$\Gamma_2 = \emptyset,\ K = 1,\ n = 1,\ a_{11} = 1,\ b_1 = b(u),\ \bar{w} = 1$$

and consider the conformal transformation which maps Ω onto the annulus $\tilde{\Omega}$ of radii 1 and $R > 1$. In $\tilde{\Omega}$ problem (1.2), (1.4) is immediately solved in polar coordinates ρ, θ by

$$w = \frac{\log \rho}{\log R}. \tag{1.15}$$

Moreover, if we restrict our attention to radial solutions, equation (1.1) gives

$$\frac{d}{d\rho}\Big(\rho\frac{du}{d\rho}\Big) + b_u(u)\frac{du}{d\rho}\Big(\frac{1}{log\, R}\Big) = 0. \tag{1.16}$$

From (1.15) we have $\rho = R^w$, thus (1.16) gives

$$\frac{d}{dw}\Big(\frac{du}{dw}\Big) + b_u(u)\frac{du}{dw} = 0, \tag{1.17}$$

where every reference to the domain Ω has disappeared and u is considered as a function of w. This elementary remark suggests the following

Definition 1. *If* $(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x}))$ *is a solution to problem* (P) *and there exist* n *functions* $(\mathcal{U}_1(w), \dots, \mathcal{U}_n(w))$ *of class* $C^1([0, \bar{w}])$ *such that*

$$u_1(\boldsymbol{x}) = \mathcal{U}_1(w(\boldsymbol{x})), \dots, u_n(\boldsymbol{x}) = \mathcal{U}_n(w(\boldsymbol{x}))$$

we say that $(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x}))$ *is a* functional solution *of problem* (P).

In a previous work [3] the link between problem (P) and a two-point problem for a system of ordinary differential equations (TPP) is established. We prove in Section 2 that this correspondence between the functional solutions of (P) and the solutions of (TPP) is biunivocal. Moreover, problem (TPP) can be restated as a Volterra-Fredholm integral equation, for which a theorem of existence is proved in [3], when $a_{ij} = \delta_{ij}$. In Section 3 we extend that result to cover the case when $a_{ij}(\boldsymbol{u}, w)$ is an uniformly positive-definite matrix. Section 4 deals with an application to the thermistor problem and presents a slight generalization of the usual condition of existence and nonexistence [4]. Section 5 applies the method to a problem of bifurcation relevant in the flow of fluid in a porous medium using the Liapunoff-Schmidt technique for the corresponding two-point problem.

2 - The two-point problem associated with problem $(\boldsymbol{P})$

All functional solutions of problem (P) can be computed explicitly if we can find the solutions $(\boldsymbol{U}(w), \boldsymbol{k})$, $\boldsymbol{k} = (k_1, \dots, k_n)$ of the following two-point problem (TPP)

$$\sum_{j=1}^{n} a_{ij}(\boldsymbol{U}, w)\frac{d\mathcal{U}_i}{dw} + b_i(\boldsymbol{U}, w) = k_i K(\boldsymbol{U}, w),\ i = 1, \dots, n,\ \boldsymbol{U} = (\mathcal{U}_1, \dots, \mathcal{U}_n) \tag{2.1}$$

$$\boldsymbol{U}(0) = 0,\ \boldsymbol{U}(\bar{w}) = \bar{\boldsymbol{u}} \tag{2.2}$$

and the solution of

$$\varDelta z = 0 \text{ in } \varOmega \tag{2.3}$$

$$z = 0 \text{ on } \varGamma_0,\ z = 1 \text{ on } \varGamma_1, \frac{\partial z}{\partial n} = 0 \text{ on } \varGamma_2 \tag{2.4}$$

is known. First order ordinary differential equations depending on a parameter whose solutions must satisfy two boundary conditions have been considered, among others, in [**14**] and [**13**]. We assume

$$K_1 \geq K(\boldsymbol{U}, w) \geq K_0 > 0,\ \text{for all } \boldsymbol{U} \in \boldsymbol{R}^n,\ \text{and for all } w \in\ [0, \bar{w}] \tag{2.5}$$

and denote with $\{TPP\}$ the set of all solutions of problem (2.1), (2.2) and with $\{P\}$ the set of all functional solutions of problem (1.1)-(1.4). We claim that

$$card\{TPP\} = card\{P\}. \tag{2.6}$$

Let the map $\mathcal{I} : \{TPP\} \to \{P\}$ be defined as follows. Suppose $(\boldsymbol{U}(w), \boldsymbol{k}) \in \{TPP\}$, and consider the nonlinear boundary value problem in the unknown $w(\boldsymbol{x})$

$$\nabla \cdot (K(\boldsymbol{U}(w), w)\nabla w) = 0 \text{ in } \varOmega \tag{2.7}$$

$$w = 0 \text{ on } \varGamma_0,\ w = \bar{w} \text{ in } \varGamma_1,\ \frac{\partial w}{\partial n} = 0 \text{ in } \varGamma_2. \tag{2.8}$$

(2.7), (2.8) has one and only one solution. For, let

$$v = F(w) = \int_0^w K(\boldsymbol{U}(t), t)dt. \tag{2.9}$$

By (2.5) F maps one-to-one $[0, \bar{w}]$ onto $[0, F(\bar{w})]$. Moreover, under (2.9) problem (2.7), (2.8) transforms into

$$\varDelta v = 0 \text{ in } \varOmega$$

$$v = 0 \text{ on } \varGamma_0,\ v = F(\bar{w}) \text{ on } \varGamma_1,\ \frac{\partial v}{\partial n} = 0 \text{ on } \varGamma_2$$

and

$$v(\boldsymbol{x}) = F(\bar{w})z(\boldsymbol{x}),$$

with $z(\boldsymbol{x})$ given by (2.3), (2.4). Therefore, the only solution of problem (2.7)-(2.8) is

$$w(\boldsymbol{x}) = F^{-1}(F(\bar{w})z(\boldsymbol{x})). \tag{2.10}$$

We claim that

$$(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x})) = (\boldsymbol{U}(w(\boldsymbol{x})), w(\boldsymbol{x})) \in \{P\}. \tag{2.11}$$

For, by (2.1) and (2.7) we have

$$\nabla \cdot \left[\left(\sum_{j=1}^{n} a_{ij}(\boldsymbol{U}(w), w)\frac{d\mathcal{U}_i}{dw} + b_i(\boldsymbol{U}(w), w)\right)\nabla w\right] = k_i \nabla \cdot (K(\boldsymbol{U}(w), w)) = 0.$$

Moreover, (2.11) satisfies all the boundary conditions in problem (P). Hence $\mathcal{I}$ is well defined. On the other hand $\mathcal{I}$ is one-to-one. For, let $(\boldsymbol{U}(w), \boldsymbol{k}) \in \{TPP\}$, $(\tilde{\boldsymbol{U}}(w), \tilde{\boldsymbol{k}}) \in \{TPP\}$, and $(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x})) = \mathcal{I}(\boldsymbol{U}(w), \boldsymbol{k})$, $(\tilde{\boldsymbol{u}}(\boldsymbol{x}), \tilde{w}(\boldsymbol{x})) = \mathcal{I}(\tilde{\boldsymbol{U}}(w), \tilde{\boldsymbol{k}})$. If

$$\mathcal{I}(\boldsymbol{U}(w), \boldsymbol{k}) = \mathcal{I}(\tilde{\boldsymbol{U}}(w), \tilde{\boldsymbol{k}})$$

we have $\boldsymbol{u}(\boldsymbol{x}) = \tilde{\boldsymbol{u}}(\boldsymbol{x})$ and $w(\boldsymbol{x}) = \tilde{w}(\boldsymbol{x})$ and also $\boldsymbol{U}(w) = \tilde{\boldsymbol{U}}(w)$ since $w(\boldsymbol{x})$ takes all values between 0 and $\bar{w}$. In addition we have: $\boldsymbol{k} = \tilde{\boldsymbol{k}}$. We prove that $\mathcal{I}$ is onto. Let $(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x})) = (\boldsymbol{U}(w(\boldsymbol{x})), w(\boldsymbol{x})) \in \{P\}$ and define

$$\theta_i = \int_0^w \sum_{j=1}^{n} a_{ij}(\boldsymbol{U}(t), t)\frac{d\mathcal{U}_i}{dw}(t) + b_i(\boldsymbol{U}(t), t)dt \tag{2.12}$$

$$\zeta = \int_0^w K(\boldsymbol{U}(t), t)dt \tag{2.13}$$

$$\Theta_i(\boldsymbol{x}) = \theta_i(w(\boldsymbol{x})),\ Z(\boldsymbol{x}) = \zeta(w(\boldsymbol{x})).$$

By (1.1) and (1.2) we have

$$\Delta\Theta_i = 0 \text{ in } \Omega,\ \Delta Z = 0 \text{ in } \Omega$$

$$\Theta_i = 0,\ Z = 0 \text{ on } \Gamma_0,\ \Theta_i = C_i,\ Z = C \text{ on } \Gamma_1,\ \frac{\partial\Theta_i}{\partial n} = 0,\ \frac{\partial Z}{\partial n} = 0 \text{ on } \Gamma_2,$$

where the C_i and C are constants and $C \neq 0$ by (2.5). If $z(\boldsymbol{x})$ is given by (2.3), (2.4), we have $\Theta_i(\boldsymbol{x}) = C_i z(\boldsymbol{x}),\ Z(\boldsymbol{x}) = Cz(\boldsymbol{x})$ and setting

$$k_i = \frac{C_i}{C} \tag{2.14}$$

we have

$$\Theta_i = k_i Z. \tag{2.15}$$

By (2.12), (2.13) and (2.15) we obtain

$$\int_0^w \left[\sum_{j=1}^n a_{ij}(\boldsymbol{U}(t),t) \frac{d\mathcal{U}_i}{dw} + b_i(\boldsymbol{U}(t),t) \right] dt = k_i \int_0^w K(\boldsymbol{U}(t),t) dt. \tag{2.16}$$

Taking the derivative of (2.16) with respect to w we obtain (2.1). Hence, with $\boldsymbol{k}$ given by (2.14), we have

$$\mathcal{I}(\boldsymbol{U}(w), \boldsymbol{k}) = (\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x})).$$

Remark 1. *As an immediate consequence of* (2.6) *we can say that problem* (P) *has one and only one solution if and only if the same is true for problem* (TPP). *Moreover, the search of the functional solutions of problem* (P) *is broken into two steps:* (i) *to find the solutions* $(\boldsymbol{U}(w), \boldsymbol{k})$ *of problem* (TPP) *and* (ii) *to solve the problem*

$$\Delta z = 0 \text{ on } \Omega, \; z = 0 \text{ on } \Gamma_0, \; z = 1 \text{ on } \Gamma_1, \; \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_2.$$

The solutions of problem (P) *are then given by* $(\boldsymbol{u}(\boldsymbol{x}), w(\boldsymbol{x})) = (\boldsymbol{U}(w(\boldsymbol{x})), w(\boldsymbol{x}))$, *where* $w(\boldsymbol{x})$ *is given by* (2.10). *Step* (ii) *contains, so-to-speak, the geometrical part of* (P) *and step* (i) *the nonlinear part.*

3 - Existence of solutions for the two-point problem (*TPP*)

In this section we prove the existence of at least one solution for the two-point problem

$$A(\boldsymbol{U}, w)\frac{d\boldsymbol{U}}{dw} = K(\boldsymbol{U}, w)\boldsymbol{k} - \boldsymbol{b}(\boldsymbol{U}, w), \; \boldsymbol{k} = (k_1, \dots, k_n)^T, \; \boldsymbol{b} = (b_1, \dots, b_n)^T \tag{3.1}$$

$$\boldsymbol{U}(0) = \boldsymbol{0} \tag{3.2}$$

$$\boldsymbol{U}(\bar{w}) = \tilde{\boldsymbol{u}}, \; \bar{w} > 0, \tag{3.3}$$

where $A(\boldsymbol{U}, w) = (a_{ij}(\boldsymbol{U}, w))$ is a symmetric $n \times n$ matrix continuous function of $\boldsymbol{U} = (\mathcal{U}_1, \dots, \mathcal{U}_n)$ and of $w \in [0, \bar{w}]$ which satisfies

$$a_M |\xi|^2 \geq \sum_{ij=1}^n a_{ij}(\boldsymbol{U}, w)\xi_i \xi_j \geq a_m |\xi|^2, \; a_m > 0, \; \forall \; \xi \in \boldsymbol{R}^n \tag{3.4}$$

for all $\boldsymbol{U} \in \boldsymbol{R}^n$ and for all $w \in [0, \bar{w}]$. Moreover, we assume $K(\boldsymbol{U}, w) \in C^0(\boldsymbol{R}^n \times [0, \bar{w}])$, $\boldsymbol{b}(\boldsymbol{U}, w) \in C^0(\boldsymbol{R}^n \times [0, \bar{w}]; \boldsymbol{R}^n)$ and

$$K_1 \geq K(\boldsymbol{U}, w) \geq K_0 > 0 \tag{3.5}$$

$$|\boldsymbol{b}(\boldsymbol{U}, w)| \leq \beta_0 + \beta_1 |\boldsymbol{U}|^a, \ 0 \leq a < 1, \ \beta_1 > 0. \tag{3.6}$$

We transform (3.1)-(3.3) in a Volterra-Fredholm integral equation. By (3.4) $A(\boldsymbol{U}, w)$ is invertible and the inverse $C(\boldsymbol{u}, w) = A^{-1}(\boldsymbol{U}, w)$ satisfies

$$c_M |\xi|^2 \geq \sum_{ij=1}^{n} c_{ij}(\boldsymbol{U}, w)\xi_i \xi_j \geq c_m |\xi|^2, \ \forall \xi \in \boldsymbol{R}^n. \tag{3.7}$$

Thus we can rewrite (3.1) as follows

$$\frac{d\boldsymbol{U}}{dw} = E(\boldsymbol{U}, w)\boldsymbol{k} - C(\boldsymbol{U}, w)\boldsymbol{b}(\boldsymbol{U}, w), \tag{3.8}$$

where $E(\boldsymbol{U}, w) = K(\boldsymbol{U}, w)C(\boldsymbol{U}, w)$. Integrating (3.8) we obtain

$$\boldsymbol{U}(w) = \int_0^w E(\boldsymbol{U}(t), t)dt\boldsymbol{k} - \int_0^w C(\boldsymbol{U}(t), t)\boldsymbol{b}(\boldsymbol{U}(t), t)dt. \tag{3.9}$$

Recalling (3.3) we have

$$\int_0^{\bar{w}} E(\boldsymbol{U}(t), t)dt\boldsymbol{k} = \bar{\boldsymbol{u}} + \int_0^{\bar{w}} C(\boldsymbol{U}(t), t)\boldsymbol{b}(\boldsymbol{U}(t), t)dt. \tag{3.10}$$

Moreover, by (3.5) and (3.7)

$$\left(\int_0^{\bar{w}} E(\boldsymbol{U}(t), t)dt \right)^{-1}$$

exists. Solving (3.10) with respect to $\boldsymbol{k}$ and substituting in (3.9) we arrive at

$$\boldsymbol{U}(w) = \int_0^w E(\boldsymbol{U}(t), t)dt \left(\int_0^{\bar{w}} E(\boldsymbol{U}(t), t)dt \right)^{-1} \left[\bar{\boldsymbol{u}} + \int_0^{\bar{w}} C(\boldsymbol{U}(t), t)\boldsymbol{b}(\boldsymbol{U}(t), t)dt \right] - \int_0^{\bar{w}} C(\boldsymbol{U}(t), t)\boldsymbol{b}(\boldsymbol{U}(t), t)dt. \tag{3.11}$$

To find "a priori" estimates for the solutions of (3.11) we need the following

Lemma 1. *Let* $G(t) = \big(g_{ij}(t)\big)$ *be a symmetric and continuous matrix satisfying*

$$g_M|\xi|^2 \geq \sum_{ij=1}^{n} g_{ij}(\boldsymbol{U}, w)\xi_i\xi_j \geq g_m|\xi|^2, \text{ for all } \xi \in \boldsymbol{R}^n,\ g_m > 0. \tag{3.12}$$

Then

$$\Big\| \Big(\int_0^w G(t)dt \Big) \Big(\int_0^{\bar{w}} G(t)dt \Big)^{-1} \Big\| \leq \frac{g_M}{g_m}.$$

Proof. Let $\lambda_1(t) \leq \lambda_2(t) \leq \ldots \leq \lambda_n(t)$ be the eigenvalues of $G(t)$. By (3.12) and the mini-max property of the eigenvalues we have $g_m \leq \lambda_1(t),\ \lambda_n(t) \leq g_M$ and

$$\lambda_n(t)|\xi|^2 \geq \sum_{ij=1}^{n} g_{ij}(t)\xi_i\xi_j \geq \lambda_1(t)|\xi|^2, \text{ for all } \xi = \in \boldsymbol{R}^n. \tag{3.13}$$

Integrating (3.13) we have

$$\int_0^w \lambda_n(t)dt|\xi|^2 \geq \sum_{ij=1}^{n} \int_0^w g_{ij}(t)dt\xi_i\xi_j \geq \int_0^w \lambda_1(t)dt|\xi|^2, \text{ for all } \xi \in \boldsymbol{R}^n.$$

If $\Lambda_1(w) \leq \Lambda_2(w) \leq \ldots \leq \Lambda_n(w)$ are the eigenvalues of the matrix $\int_0^w G(t)dt$ we have

$$\int_0^w \lambda_1(t)dt \leq \Lambda_1(t),\ \Lambda_n(t) \leq \int_0^w \lambda_n(t)dt. \tag{3.14}$$

Hence, by (3.14)

$$\Big\| \int_0^w G(t)dt \Big(\int_0^{\bar{w}} G(t)dt \Big)^{-1} \Big\| \leq \Big\| \int_0^w G(t)dt \Big\| \, \Big\| \Big(\int_0^{\bar{w}} G(t)dt \Big)^{-1} \Big\|$$

$$\leq \frac{\Lambda_n(w)}{\Lambda_1(\bar{w})} \leq \frac{\int_0^w \lambda_n(t)dt}{\int_0^{\bar{w}} \lambda_1(t)dt} \leq \frac{g_M w}{g_m \bar{w}} \leq \frac{g_M}{g_m}.$$

□

Theorem 1. *If* (3.4), (3.5) *and* (3.6) *hold then the problem* (3.1)-(3.3) *has at least one solution.*

Proof. Let $\mathcal{B}=C^0([0,\bar{w}];\boldsymbol{R}^n)$. Define the operator $\mathcal{A}:\mathcal{B}\to\mathcal{B}$, $\mathcal{A}=(\mathcal{A}_1,\dots,\mathcal{A}_n)$

$$\mathcal{A}(\boldsymbol{U})(w)=\int_0^w E(\boldsymbol{U}(t),t)dt\left(\int_0^{\bar{w}} E(\boldsymbol{U}(t),t)dt\right)^{-1}\left[\bar{\boldsymbol{u}}+\int_0^{\bar{w}} C(\boldsymbol{U}(t),t)\boldsymbol{b}(\boldsymbol{U}(t),t)dt\right] - \int_0^w C(\boldsymbol{U}(t),t)\boldsymbol{b}(\boldsymbol{U}(t),t)dt. \tag{3.15}$$

Let $0\le\mu\le1$. We claim that all solutions of the equation

$$\boldsymbol{U}=\mu\mathcal{A}(\boldsymbol{U}),\ \ \boldsymbol{U}\in\mathcal{B} \tag{3.16}$$

are bounded in $\mathcal{B}$. For, let $\boldsymbol{U}(w)$ be any solution of (3.16). By Lemma 1 and (3.7) we have

$$|\boldsymbol{U}(w)|\le\frac{K_1}{K_0}\frac{c_M}{c_m}|\bar{\boldsymbol{u}}|+\left(1+\frac{K_1}{K_0}\frac{c_M}{c_m}\right)\int_0^{\bar{w}} c_M\left(\beta_0+\beta_1|\boldsymbol{U}(t)|^a\right)dt.$$

Hence

$$|\boldsymbol{U}(w)|\le C_1\left[1+\int_0^{\bar{w}}|\boldsymbol{U}(t)|^a dt\right]. \tag{3.17}$$

Let $m\ge1$. Then by Hölder inequality we obtain from (3.17)

$$|\boldsymbol{U}(w)|\le C_2\left[1+\bar{w}^{\frac{m-a}{m}}\left(\int_0^{\bar{w}}|\boldsymbol{U}(t)|^m dt\right)^{\frac{a}{m}}\right]$$

and integrating from 0 to $\bar{w}$

$$\left(\int_0^{\bar{w}}|\boldsymbol{U}(t)|^m dt\right)^{\frac{1}{m}}\le\left[1+\left(\int_0^{\bar{w}}|\boldsymbol{U}(t)|^m dt\right)^{\frac{a}{m}}\right].$$

By (3.6) we have

$$\left(\int_0^{\bar{w}}|\boldsymbol{U}(t)|^m dt\right)^{\frac{1}{m}}\le C_4,$$

where C_4 does not depend on m. Hence

$$\|\boldsymbol{U}\|_{\mathcal{B}}\le C_4. \tag{3.18}$$

The operator $\mathcal{A}$ is also compact and continuous. To see that, let $\{\boldsymbol{U}^{(k)}(w)\}$ be a sequence bounded in $\mathcal{B}$ and $\boldsymbol{V}^{(k)}=\mathcal{A}(\boldsymbol{U}^{(k)})$. From (3.5), (3.6) and Lemma 1 we obtain that $\{\boldsymbol{V}^{(k)}\}$ is also bounded in $\mathcal{B}$

$$\|\boldsymbol{V}^{(k)}\|_{\mathcal{B}} \leq C_6. \tag{3.19}$$

On the other hand, $V_i^{(k)}$ is absolutely continuous and we have from (3.15)

$$\boldsymbol{V}^{(k)'}(w) = E(\boldsymbol{U}(w), w)\Big(\int_0^{\bar{w}} E(\boldsymbol{U}(t), t)dt\Big)^{-1}\Big[\bar{\boldsymbol{u}} + \int_0^{\bar{w}} C(\boldsymbol{U}(t), t)\boldsymbol{b}(\boldsymbol{U}(t), t)dt\Big]$$
$$- C(\boldsymbol{U}(w), w)\boldsymbol{b}(\boldsymbol{U}(w), w), \text{ a.e.} \tag{3.20}$$

By (3.18) the right hand side of (3.20) is bounded in $\mathcal{B}$. Hence by the Arzela's theorem the operator $\mathcal{A}$ is compact. Therefore by the Schauder fixed point theorem in Schaefer's form [**10**] $\mathcal{A}$ has a fixed point which gives a solution to problem (TPP). □

4 - Example 1

We apply the present method to the case, quoted in the introduction, of the electrical heating of a conductor. We have

$$\nabla \cdot (\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega \tag{4.1}$$

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega \tag{4.2}$$

$$u = 0 \text{ on } \Gamma_0,\ u = \bar{u} \text{ on } \Gamma_1, \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \tag{4.3}$$

$$\varphi = 0 \text{ on } \Gamma_0,\ \varphi = \bar{\varphi} \text{ on } \Gamma_1, \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_2. \tag{4.4}$$

The associated two-point problem is the following

$$\kappa(\mathcal{U})\frac{d\mathcal{U}}{d\varphi} + \varphi\sigma(\mathcal{U}) = k\sigma(\mathcal{U}) \tag{4.5}$$

$$\mathcal{U}(0) = 0 \tag{4.6}$$

$$\mathcal{U}(\bar{\varphi}) = \bar{u}. \tag{4.7}$$

Let us assume $\bar{\varphi} > 0$, $\bar{u} > 0$, $\kappa(u) \in C^0(\boldsymbol{R}^1)$, $\sigma(u) \in C^0(\boldsymbol{R}^1)$ and

$$\kappa(u) > 0,\ \sigma(u) > 0,\ u \in [0, \infty).$$

Define for $u > 0$

$$F(u) = \int_0^u \frac{\kappa(t)}{\sigma(t)}dt,\ L = F(\bar{u}),\ M = \int_0^\infty \frac{\kappa(t)}{\sigma(t)}dt.$$

By separation of variables, from (4.5) we have

$$F(\mathcal{U}) = -\frac{\varphi^2}{2} + k\varphi. \tag{4.8}$$

Using (4.7), we obtain from (4.8)

$$F(\mathcal{U}) = -\frac{\varphi^2}{2} + \left(\frac{\bar{\varphi}}{2} + \frac{L}{\bar{\varphi}}\right)\varphi.$$

Thus, if a solution of (4.5)-(4.7) exists, it can only be given by

$$\mathcal{U} = F^{-1}\left(-\frac{\varphi^2}{2} + \left(\frac{\bar{\varphi}}{2} + \frac{L}{\bar{\varphi}}\right)\varphi\right).$$

Examining the graphs of $F(\mathcal{U})$ and of the parabola $f(\varphi) = -\frac{\varphi^2}{2} + \left(\frac{\bar{\varphi}}{2} + \frac{L}{\bar{\varphi}}\right)\varphi$, recalling that $\varphi \in [0, \bar{\varphi}]$, we have

Lemma 2. *If* $M = \infty$ *or* $M < \infty$ *and* $\bar{\varphi}^2 < 2L$, *then problem* (4.5)-(4.7) *has one and only one solution. Let* $M < \infty$ *and* $\bar{\varphi}^2 \geq 2L$. *If*

$$\frac{1}{8}\bar{\varphi}^2 + \frac{L}{2} + \frac{L^2}{4\bar{\varphi}^2} < M$$

then problem (4.5)-(4.7) *has one and only one solution. If*

$$\frac{1}{8}\bar{\varphi}^2 + \frac{L}{2} + \frac{L^2}{4\bar{\varphi}^2} \geq M$$

problem (4.5)-(4.7) *has no solution.*

When problem (4.5)-(4.7) has a solution the corresponding functional solution of (4.1)-(4.4) is $(\varphi(\boldsymbol{x}), u(\boldsymbol{x}))$, where $\varphi(\boldsymbol{x})$ is computed from

$$\nabla \cdot \left[\sigma\left(F^{-1}\left(-\frac{\varphi^2}{2} + \left(\frac{\bar{\varphi}}{2} + \frac{L}{\bar{\varphi}}\right)\varphi\right)\right)\nabla\varphi\right] = 0 \text{ in } \Omega \tag{4.9}$$

$$\varphi = 0 \text{ on } \Gamma_0, \ \varphi = \bar{\varphi} \text{ on } \Gamma_1, \ \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_2 \tag{4.10}$$

and

$$u(\boldsymbol{x}) = F^{-1}\left(-\frac{\varphi^2(\boldsymbol{x})}{2} + \left(\frac{\bar{\varphi}}{2} + \frac{L}{\bar{\varphi}}\right)\varphi(\boldsymbol{x})\right).$$

In addition, the solution of (4.9), (4.10) can be expressed in terms of $z(\boldsymbol{x})$, the solution of (2.3), (2.4). A question naturally arises: is the functional solution of problem (4.1)-(4.4) (or more generally of problem (P)) the only solution? The answer is affirmative [3], at least for problem (4.1)-(4.4).

5 - Example 2

Setting $S=0, D=0, \bar{w}=\lambda, \bar{u}=0, \bar{v}=0$ in (1.9)-(1.13) we arrive at the following nonlinear eigenvalue problem (P_1)

$$\nabla\cdot[\nabla u+\lambda f(u,v)\nabla z]=0 \text{ in } \Omega,\ u=0 \text{ on } \Gamma$$

$$\nabla\cdot[\nabla v+\lambda g(u,v)\nabla z]=0 \text{ in } \Omega,\ v=0 \text{ on } \Gamma$$

$$\Delta z=0 \text{ in } \Omega \tag{5.1}$$

$$z=0 \text{ on } \Gamma_0,\ z=1 \text{ on } \Gamma_1,\ \frac{\partial z}{\partial n}=0 \text{ on } \Gamma_2. \tag{5.2}$$

By (5.1) there is no loss in generality in assuming

$$f(0,0)=0,\ g(0,0)=0.$$

(P_1) has for every λ the solution

$$(u(\boldsymbol{x}),v(\boldsymbol{x}),z(\boldsymbol{x}))=(0,0,z(\boldsymbol{x})), \tag{5.3}$$

where $z(\boldsymbol{x})$ is given by (5.1), (5.2). Do exist branches of non-trivial solutions bifurcating from (5.3)?

The two-point problem (TPP_1) associated with (P_1) is

$$\frac{d\mathcal{U}}{dz}+\lambda f(\mathcal{U},\mathcal{V})=\gamma,\ \mathcal{U}(0)=0,\ \mathcal{U}(1)=0 \tag{5.4}$$

$$\frac{d\mathcal{V}}{dz}+\lambda g(\mathcal{U},\mathcal{V})=\mu,\ \mathcal{V}(0)=0,\ \mathcal{V}(1)=0 \tag{5.5}$$

or, equivalently,

$$\frac{d^2\mathcal{U}}{dz^2}+\lambda\Big[f_u(\mathcal{U},\mathcal{V})\frac{d\mathcal{U}}{dz}+f_v(\mathcal{U},\mathcal{V})\frac{d\mathcal{V}}{dz}\Big]=0,\ \mathcal{U}(0)=0,\ \mathcal{U}(1)=0$$

$$\frac{d^2\mathcal{V}}{dz^2}+\lambda\Big[g_u(\mathcal{U},\mathcal{V})\frac{d\mathcal{U}}{dz}+g_v(\mathcal{U},\mathcal{V})\frac{d\mathcal{V}}{dz}\Big]=0,\ \mathcal{V}(0)=0,\ \mathcal{V}(1)=0.$$

Let

$$\boldsymbol{F}: D(\boldsymbol{F})\subseteq L^2((0,1);\boldsymbol{R}^2)\to L^2((0,1);\boldsymbol{R}^2),$$

$$D(\boldsymbol{F})=\boldsymbol{R}^1\times H_0^1(0,1)\times H^2(0,1),\ \boldsymbol{U}=(\mathcal{U},\mathcal{V})$$

and

$$\boldsymbol{F}(\lambda,\boldsymbol{U})=\big(\mathcal{U}''+\lambda[f_u(\mathcal{U},\mathcal{V})\mathcal{U}'+f_v(\mathcal{U},\mathcal{V})\mathcal{V}'],\mathcal{V}''+\lambda[g_u(\mathcal{U},\mathcal{V})\mathcal{U}'+g_v(\mathcal{U},\mathcal{V})\mathcal{V}']\big).$$

Lemma 3. *The differential of* $\boldsymbol{F}(\lambda, \boldsymbol{U})$

$$\boldsymbol{F}'_{\boldsymbol{U}}(\lambda, \mathbf{0})[\boldsymbol{H}] = \boldsymbol{H}'' + \lambda A\boldsymbol{H}',\ \boldsymbol{H} = (\mathcal{H}, \mathcal{K}),$$

where

$$A = \begin{bmatrix} f_u(0,0) & f_v(0,0) \\ g_u(0,0) & g_v(0,0) \end{bmatrix}$$

is invertible if and only if A *has purely imaginary proper values* $\pm ib$, $b \neq 0$ *and*

$$\lambda = \lambda_n,\ \lambda_n = \frac{2\pi n}{b},\ n \in \boldsymbol{N}. \tag{5.6}$$

The eigenvalues λ_n *are all double with the eigenfunctions*

$$\begin{aligned} \boldsymbol{y}_1^{(n)}(z) &= \Big(\sin\frac{2\pi}{b}nz, 1 - \cos\frac{2\pi}{b}nz\Big), \\ \boldsymbol{y}_2^{(n)}(z) &= \Big(1 - \cos\frac{2\pi}{b}nz, -\sin\frac{2\pi}{b}nz\Big). \end{aligned} \tag{5.7}$$

Proof. Let us consider the two-point problem

$$\frac{d\boldsymbol{H}}{dz} + \lambda A\boldsymbol{H} = \boldsymbol{k},\ \boldsymbol{k} = (k_1, k_2)^T \tag{5.8}$$

$$\boldsymbol{H}(0) = 0 \tag{5.9}$$

$$\boldsymbol{H}(1) = 0. \tag{5.10}$$

The solution of the Cauchy problem (5.8), (5.9) is given by

$$\boldsymbol{H}(z) = \Big(e^{-\lambda Az}\int_0^z e^{\lambda At}dt\Big)\boldsymbol{k}.$$

Thus, in view of condition (5.10), nontrivial solutions of problem (5.8)-(5.10) occur if and only if

$$det\Big(\int_0^1 e^{\lambda Jt}dt\Big) = 0,$$

where J is the Jordan form of A. Of the four possible, the only case which needs to be considered is when

$$J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

We have

$$det\Big(\int_0^1 e^{\lambda J t}dt\Big) = \frac{1}{\lambda^2(a^2+b^2)}\Big[e^{2\lambda a} - 2e^{\lambda a}\cos(\lambda b) + 1\Big].$$

This is different from zero if $a \neq 0$. When $a = 0$, we have the countable sequence of double eigenvalues (5.6) with the eigensolutions (5.7). □

By Lemma 3 bifurcation for problem (5.4)-(5.5) can be present only when the matrix A has purely imaginary eigenvalues and $\lambda = \lambda_n$. Let $\boldsymbol{H} = (\mathcal{H}, \mathcal{K})$ and define the operator

$$\boldsymbol{B} : D(\boldsymbol{B}) \subseteq L^2((0,1);\boldsymbol{R}^2) \to L^2((0,1);\boldsymbol{R}^2),\ \boldsymbol{B}(\boldsymbol{H}) = \Big(\mathcal{H}'' - \frac{2\pi}{b}n\mathcal{K}', \mathcal{K}'' + \frac{2\pi}{b}n\mathcal{H}'\Big).$$

Lemma 4. *$\boldsymbol{B}$ is symmetric.*

Proof. Let $\tilde{\boldsymbol{H}} = (\tilde{\mathcal{H}}, \tilde{\mathcal{K}}) \in D(\boldsymbol{B})$. We have, integrating by parts,

$$\int_0^1 \boldsymbol{B}(\boldsymbol{H})\cdot\tilde{\boldsymbol{H}}dz = \int_0^1 \mathcal{H}\Big(\tilde{\mathcal{H}}'' - \frac{2\pi}{b}n\tilde{\mathcal{K}}'\Big) + \mathcal{K}\Big(\tilde{\mathcal{K}}'' + \frac{2\pi}{b}n\tilde{\mathcal{H}}'\Big)dz = \int_0^1 \boldsymbol{H}\cdot\boldsymbol{B}(\tilde{\boldsymbol{H}})dz$$

(here the dot denotes the scalar product in $\boldsymbol{R}^2$). □

Hereafter we examine the specific case of a quadratic nonlinear terms i.e. we assume

$$f(u,v) = -bv + A_1u^2 + A_2uv + A_3v^2,\ g(u,v) = bu + B_1u^2 + B_2uv + B_3v^2.$$

The relevant two-point problem is

$$\mathcal{U}'' + \frac{2\pi}{b}n\Big[-b\mathcal{V}' + A_1\mathcal{U}\mathcal{U}' + A_2(\mathcal{U}\mathcal{V}' + \mathcal{V}\mathcal{U}') + A_3\mathcal{V}\mathcal{V}'\Big] = 0$$

$$\mathcal{V}'' + \frac{2\pi}{b}n\Big[b\mathcal{U}' + B_1\mathcal{U}\mathcal{U}' + B_2(\mathcal{U}\mathcal{V}' + \mathcal{V}\mathcal{U}') + B_3\mathcal{V}\ \mathcal{V}'\Big] = 0$$

$$\mathcal{U}(0) = \mathcal{V}(0) = 0,\ \mathcal{U}(1) = \mathcal{V}(1) = 0.$$

We apply the Liapunoff-Schmidt procedure and the Newton method [**11**], [**12**], [8], [**5**] to assess if bifurcation occurs or not. In view of the symmetry of $\boldsymbol{B}$ the nullspace of the adjoint $\boldsymbol{B}^*$ is spanned by $\boldsymbol{y}_1^{(n)}$ and $\boldsymbol{y}_2^{(n)}$. Let

$$\boldsymbol{G}(\boldsymbol{U},\lambda) = \Big(\mathcal{U}'' + \lambda[-b\mathcal{V}' + A_1\mathcal{U}\mathcal{U}' + A_2(\mathcal{U}\mathcal{V}' + \mathcal{V}\mathcal{U}') + A_3\mathcal{V}\mathcal{V}'],$$

$$\mathcal{V}'' + \lambda[b\mathcal{U}' + B_1\mathcal{U}\mathcal{U}' + B_2(\mathcal{U}\mathcal{V}' + \mathcal{V}\mathcal{U}') + B_3\mathcal{V}\mathcal{V}']\Big),\ \boldsymbol{U} = (\mathcal{U},\mathcal{V})$$

and

$$\boldsymbol{G}_{10} = \boldsymbol{G}'_{\boldsymbol{u}}, \ \boldsymbol{G}_{11} = \frac{1}{2!}\boldsymbol{G}''_{\boldsymbol{u}\lambda}, \ \boldsymbol{G}_{20} = \frac{1}{2!}\boldsymbol{G}''_{\boldsymbol{uu}}, \ \boldsymbol{G}_{21} = \frac{1}{3!}\boldsymbol{G}'_{\boldsymbol{uu}\lambda}.$$

Let $\mu = \lambda - \lambda_n$. The Liapunoff-Schmidt equations corresponding to problem (5.4) are

$$\int_0^1 \Big(\boldsymbol{G}_{20}(\boldsymbol{0}, \lambda_n)[\boldsymbol{U}, \boldsymbol{U}] \cdot \boldsymbol{y}_1^{(n)} + \boldsymbol{G}_{11}(\boldsymbol{0}, \lambda_n)[\boldsymbol{U}, \mu] \cdot \boldsymbol{y}_1^{(n)} + \boldsymbol{G}_{21}(\boldsymbol{0})[\boldsymbol{U}, \boldsymbol{U}, \mu] \cdot \boldsymbol{y}_1^{(n)}\Big) dz = 0 \tag{5.11}$$

$$\int_0^1 \Big(\boldsymbol{G}_{20}(\boldsymbol{0}, \lambda_n)[\boldsymbol{U}, \boldsymbol{U}] \cdot \boldsymbol{y}_2^{(n)} + \boldsymbol{G}_{11}(\boldsymbol{0}, \lambda_n)[\boldsymbol{U}, \mu] \cdot \boldsymbol{y}_2^{(n)} + \boldsymbol{G}_{21}(\boldsymbol{0})[\boldsymbol{U}, \boldsymbol{U}, \mu] \cdot \boldsymbol{y}_2^{(n)}\Big) dz = 0. \tag{5.12}$$

Set

$$\boldsymbol{U} = \xi_1 \varphi_1^{(n)} + \xi_2 \varphi_2^{(n)} + \text{higher order terms.} \tag{5.13}$$

Substituting (5.13) in (5.12) and in (5.11) we find

$$\Phi_{20}(\xi_1, \xi_2) + \Phi_{11}(\xi_1, \xi_2)\mu + o(|\xi|^i |\mu|^j) = 0, \ \ i + j > 2 \tag{5.14}$$

$$\Psi_{20}(\xi_1, \xi_2) + \Psi_{11}(\xi_1, \xi_2)\mu + o(|\xi|^i |\mu|^j) = 0, \ \ i + j > 2, \tag{5.15}$$

where

$$\Phi_{20}(\xi_1, \xi_2) = 2n\pi^2 \Big[\Big(A_3 - B_2\Big)\xi_1^2 + \Big(A_1 + B_2\Big)\xi_2^2 + \Big(B_3 - B_1 + 2A_2\Big)\xi_1\xi_2\Big]$$

$$\Phi_{11}(\xi_1, \xi_2) = 2n\pi\xi_1$$

$$\Psi_{20}(\xi_1, \xi_2) = 2n\pi^2 \Big[\Big(-A_2 - B_2\Big)\xi_1^2 + \Big(A_2 - B_1\Big)\xi_2^2 + \Big(A_3 - 2B_2 - A_1\Big)\xi_1\xi_2\Big]$$

$$\Psi_{11}(\xi_1, \xi_2) = -2n\pi\xi_2.$$

We are interested in those solutions (ξ_1, ξ_2) of (5.14), (5.15) which tend to zero as $\mu \to 0$ i.e. in small solutions. Therefore we need only to consider the decreasing part of the Newton diagram, which in this case reduces to the segment joining the points $(1, 1)$ and $(2, 0)$. Hence the number of small solutions is given by the points of intersections different from $(0, 0)$ in the plane (ξ_1, ξ_2) of the two conics

$$\Phi_{20}^{(n)}(\xi_1, \xi_2) + \Phi_{11}^{(n)}(\xi_1, \xi_2)\mu = 0 \tag{5.16}$$

$$\Psi_{20}^{(n)}(\xi_1, \xi_2) + \Psi_{11}^{(n)}(\xi_1, \xi_2)\mu = 0. \tag{5.17}$$

From a generic point of view, we have either one or three branches of small solutions starting from each eigenvalue. Consider, as an example, the problem

$$\Delta u + 2\pi\nabla\cdot\big((v+u^2-uv+v^2)\nabla z\big)=0 \text{ in } \Omega,\ u=0 \text{ on } \Gamma$$
$$\Delta v + 2\pi\nabla\cdot\big((-u+u^2-v^2)\nabla z\big)=0 \text{ in } \Omega,\ v=0 \text{ on } \Gamma$$
$$\Delta z=0 \text{ in } \Omega,\ z=0 \text{ on } \Gamma_0,\ z=1 \text{ on } \Gamma_1,\ \frac{\partial z}{\partial n}=0 \text{ on } \Gamma_2.$$

The conics (5.16), (5.17) are, in this case, the hyperbolas

$$\xi_2^2+\xi_1^2-4\xi_1\xi_2-\frac{\mu}{\pi}\xi_1=0$$
$$-2\xi_2^2+2\xi_1^2-\frac{\mu}{\pi}\xi_2=0.$$

The only point of intersection different from $(0,0)$, is $\Big(-\frac{\mu}{3\pi},-\frac{2\mu}{3\pi}\Big)$, to which it corresponds the branch of small solutions

$$u(\boldsymbol{x})=-\frac{\mu}{3\pi}\sin 2\pi z(\boldsymbol{x})-\frac{2\mu}{3\pi}(1-\cos 2\pi z(\boldsymbol{x}))+o(|\mu|),$$
$$v(\boldsymbol{x})=-\frac{\mu}{3\pi}(1-2\cos z(\boldsymbol{x}))+\frac{2\mu}{3\pi}\sin 2\pi z(\boldsymbol{x})+o(|\mu|).$$

However, exceptional cases in which bifurcation does not occur are possible as in the following problem

$$\Delta u + 2\pi\nabla\cdot\big((v+u^2)\nabla z\big)=0 \text{ in } \Omega,\ u=0 \text{ on } \Gamma \tag{5.18}$$
$$\Delta v + 2\pi\nabla\cdot\big((-u+v^2)\nabla z\big)=0 \text{ in } \Omega,\ v=0 \text{ on } \Gamma \tag{5.19}$$
$$\Delta z=0 \text{ in } \Omega,\ z=0 \text{ on } \Gamma_0,\ z=1 \text{ on } \Gamma_1,\ \frac{\partial z}{\partial n}=0 \text{ on } \Gamma_2. \tag{5.20}$$

In this case the conics (5.16), (5.17) are the hyperbolae

$$\xi_2^2+\xi_1\xi_2-\frac{\mu}{\pi}\xi_1=0$$
$$\xi_1^2+\xi_1\xi_2\frac{\mu}{\pi}\xi_2=0$$

with parallel asymptotes. Their only point of intersection is $(0,0)$. Thus the trivial solution of problem (5.18)-(5.20) is isolated.

References

[1] A. BEJAN and D. A. NIELD, *Convection in porous media*, Springer-Verlag, New York 1999.

[2] G. CIMATTI, *An application of Levy-Caccioppoli's global inversion theorem to a*

boundary value problem for the Soret effect, Ann. Mat. Pura Appl., Online First, DOI: 10.1007/s10231-010-0144-3.

[3] G. CIMATTI, *Remark on existence, uniqueness and semi-explicit solvability of system of autonomous partial differential equations in divergence form with constant boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

[4] G. CIMATTI, *The mathematics of the Thomson effect*, Quart. Appl. Math. **67** (2009), 617-626.

[5] L. M. GRAVES, *Remarks on singular points of functional equations*, Trans. Amer. Math. Soc. **79** (1955), 150-157.

[6] S. HOWISON, *A note on the thermistor problem in two space dimensions*, Quart. Appl. Math. **47** (1989), 509-512.

[7] S. D. HOWISON, J. F. RODRIGUES and M. SHILLOR, *Stationary solutions to the thermistor problem*, J. Math. Anal. Appl. **174** (1993), 573-588.

[8] T. MA and S. WANG, *Bifurcation theory and applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ 2005.

[9] J. OCKENDON, S. HOWISON, A. LACEY and A. MOVCHAN, *Applied partial differential equations*, Oxford Unversity Press, Oxford 2003.

[10] H. SCHAEFER, *Über die methode der a priori-Schranken*, Math. Ann. **129** (1955), 415-416.

[11] M. M. VAINBERG and V. A. TRENOGIN, *The Ljapunov and Schmidt methods in the theory of non-linear equations and their subsequent development*, Uspehi Mat. Nauk **17** (1962), no. 2, 13-75.

[12] M. M. VAINBERG and V. A. TRENOGIN, *Theory of branching of solutions of non-linear equations*, Noordhoff International Publishing, Leyden 1974.

[13] K. ZAWISCHA, *Über die Differentialgleichung* $\frac{dx}{dy} = kf(x, y)$, *deren Lösungskurve durch zwei gegebene Punkte hindurchgehen soll.*, Monatsh. Math. Phys. **37** (1930), 103-124.

[14] G. ZWIRNER, *Un criterio d'esistenza e unicità per gli integrali dell'equazione* $y' = \lambda f(x, y)$ *passanti per due punti assegnati* (Italian), Boll. Un. Mat. Ital. **3** (1948), 15-18.

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