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On a nonlocal $\vec{p}(.)-$Laplacian equations via genus theory

Abstract. In this work, we study a class of nonlocal anisotropic type problems involving $\vec{p}(.)-$Laplacian Dirichlet boundary condition with an additional nonlocal term, we give a result on the existence and multiplicity of solutions by using as main tool a result due to genus theory.

Keywords. Anisotropic variable exponent equation, Krasnoselskii's genus.

Mathematics Subject Classification (2010): 35J30 , 35J60, 35J92.

1 - Introduction

The study of anisotropic problems with variable exponents has become particularly interesting for many researchers so far. This is partly due to their frequent appearance in applications such as some physical phenomena which can be modelled by such kind of equations. The reader can find several models in mathematical physics where this class of problems appears in electrorheological fluids [**19, 22**], thermorheological fluids [**3**], elastic mechanics [**29**] and image restoration [**10**]. From a purely mathematical point of view these problems seem to have a great importance because many results have been obtained in this direction (see for instance [**7-9, 23, 25, 26, 28**] and the references therein). As a result of the preoccupation for the nonhomogeneous materials which behave differently on different space directions, the anisotropic spaces involving variable exponent were introduced. For more detail see [**9**] and [**23**].

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It is important to note that this kind of equations with growth conditions would make the study of such equation more complicated, since the differential operator $\Delta_{\vec{p}(x)}(u) := \sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right)$ allows a distinct behavior for partial derivatives in various directions. The anisotropic problems are more complicated than the isotropic problems.

On the other hand, nonlocal differential equations are also called Kirchhoff-type equations and introduced in [**21**],

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ_0, ρ, L and h are constants associated to the effects of the changes in the length of strings during the vibrations. It is an extension of the classical D'Alembert's wave equation.

Let consider the following anisotropic nonlinear elliptic problem

$$(1)\quad -\sum_{i=1}^{N} M_i(I_i(u))\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right) = f(x,u)\Big(F(x,u)\Big)^r \ in\ \Omega, \quad u = 0\ on\ \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded open set with smooth boundary and ν_i are the components of the outer normal unit vector and

$$I_i(u) = \int_\Omega \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\, dx,$$

$p_i,\ i = 1,...,N$ are continuous functions on $\overline{\Omega},\ r > 0$ is a real parameter and for each $i = 1,...,N,\ \ M_i:\ [0,\infty) \to [0,\infty),\ \ f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions with the potential

$$F(x,t) = \int_0^t f(x,s)\, ds.$$

Problem like (1) is a generalization of the $p(x)-$Kirchhoff equation. This type of Kirchhoff problems with stationary process has received considerable attention by several researchers. Among the numerous literature on the subject we would only mention the papers [**1, 2, 4-6, 10-14, 17, 18, 24, 28**].

The underlying ideas of the present paper are essentially the same as those of [**6, 12, 18**]. Obviously, our problem (1) is more general. That means that the theory of the anisotropic variable exponent Sobolev spaces $W^{1,\vec{p}(.)}(\Omega)$ and $W_0^{1,\vec{p}(.)}(\Omega)$ is needed.

The planning is organized as follows. In the next section, we present a brief review on variable exponent Sobolev and anisotropic variable exponent Sobolev spaces. Afterwards, we give the main result about the existence of weak solutions. The last part of this paper is aimed at giving the proof of the main result.

2 - Preliminaries and main result

In order to deal with the problem (1), we recall some auxiliary results. For convenience, we only recall some basic facts which will be used later, we refer to [**15, 16, 20**].

For $q \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent defined by

$$L^{q(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{q(x)}\,dx < \infty\},$$

where

$$C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}, \mathbb{R}) : \inf_{x\in\Omega} q(x) > 1\}.$$

This space, endowed with the Luxemburg norm,

$$|u|_{q(.)} = \|u\|_{L^{q(\cdot)}(\Omega)} = \inf\{\mu > 0 : \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{q(x)}\,dx \le 1\},$$

is a separable and reflexive Banach space. We also have an embedding result.

Proposition 2.1. *Assume that Ω is bounded and q_1, $q_2 \in C_+(\overline{\Omega})$ such that $q_1 \le q_2$ in Ω. Then the embedding $L^{q_2(\cdot)}(\Omega) \hookrightarrow L^{q_1(\cdot)}(\Omega)$ is continuous.*

Furthermore, the Hölder-type inequality

$$(2) \qquad \Big|\int_\Omega u(x)v(x)\,dx\Big| \le 2\|u\|_{L^{q(\cdot)}(\Omega)}\|v\|_{L^{q'(\cdot)}(\Omega)}$$

holds for all $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, where $L^{q'(\cdot)}(\Omega)$ is the conjugate space of $L^{q(\cdot)}(\Omega)$, with $1/q(x) + 1/q'(x) = 1$.

Moreover, we denote

$$q^+ = \sup_{x\in\Omega} q(x), \quad q^- = \inf_{x\in\Omega} q(x)$$

and for $u \in L^{q(\cdot)}(\Omega)$, we have the following properties:

$$\|u\|_{L^{q(\cdot)}(\Omega)} < 1\ (=1;\ >1) \ \Leftrightarrow\ \int_\Omega |u(x)|^{q(x)}\,dx < 1\ (=1;\ >1); \tag{3}$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} > 1 \ \Rightarrow\ \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{q(x)}\,dx \le \|u\|^{q^+}_{L^{q(\cdot)}(\Omega)}; \tag{4}$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} < 1 \ \Rightarrow\ \|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{q(x)}\,dx \le \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}; \tag{5}$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} \to 0\ (\to\infty) \ \Leftrightarrow\ \int_\Omega |u(x)|^{q(x)}\,dx \to 0\ (\to \infty). \tag{6}$$

To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,q(\cdot)}(\Omega)$, we set

$$W^{1,q(\cdot)}(\Omega) = \{u \in L^{q(\cdot)}(\Omega) : \partial_{x_i} u \in L^{q(\cdot)}(\Omega) \text{ for all } i \in \{1,\dots,N\}\},$$

endowed with the norm

$$\|u\|_{W^{1,q(\cdot)}(\Omega)} = \|u\|_{L^{q(\cdot)}(\Omega)} + \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{q(\cdot)}(\Omega)}.$$

The space $\big(W^{1,r(\cdot)}(\Omega),\ \|\cdot\|_{W^{1,r(\cdot)}(\Omega)}\big)$ is a separable and reflexive Banach space.

Now, we consider $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ to be the vectorial function

$$\vec{p}(x) = (p_1(x), \dots, p_N(x))$$

with $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1,\dots,N\}$ and we put

$$p_M(x) = \max\{p_1(x),\dots,p_N(x)\}, \quad p_m(x) = \min\{p_1(x),\dots,p_N(x)\}.$$

The anisotropic space with variable exponent is

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_M(\cdot)}(\Omega) : \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1,\dots,N\}\}$$

and it is endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_M(\cdot)}(\Omega)} + \sum_{i=1}^N \big\|\partial_{x_i} u\big\|_{L^{p_i(\cdot)}(\Omega)}.$$

The space $\big(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\big)$ is a reflexive Banach space. Furthermore, an embedding theorem takes place for all the exponents that are strictly less than a variable critical exponent, which is introduced with the help of the notations

$$\bar{p}(x) = \frac{N}{\sum_{i=1}^N 1/p_i(x)}, \quad q^\star(x) = \begin{cases} Nq(x)/[N-q(x)] & \text{if } q(x) < N, \\ \infty & \text{if } q(x) \ge N. \end{cases}$$

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and for all $i = 1, 2, ..., N$ and $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \ldots, N\}$. If $q \in C(\overline{\Omega}, \mathbb{R})$, $1 \leq q(x) < \max\{\overline{p}^*(x), p_M(x)\}$ for all $x \in \overline{\Omega}$, then we have the compact and continuous embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.*

Remark 2.1. We make the following notations,

$$\mathcal{F}_1 = \{i \in \{1, \ldots, N\} : \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}(\Omega)} \leq 1\},$$
$$\mathcal{F}_2 = \{i \in \{1, \ldots, N\} : \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}(\Omega)} > 1\}.$$

Then, by (3), (4) and (5),

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)} dx &= \sum_{i \in \mathcal{F}_1} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)} dx + \sum_{i \in \mathcal{F}_2} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)}\, dx \\ &\geq \sum_{i \in \mathcal{F}_1} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_M^+} + \sum_{i \in \mathcal{F}_2} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-} \\ &\geq \sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-} - \sum_{i \in \mathcal{F}_1} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-}.\end{aligned}$$

Thus,

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)} dx \geq \sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-} - N. \tag{7}$$

We denote by $X = W_0^{1,\vec{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. According to [**16**], the space $\big(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}\big)$ is a reflexive Banach space, where

$$\|u\| = \|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} = \sum_{i=1}^{N} \big\|\partial_{x_i} u\big\|_{L^{p_i(\cdot)}(\Omega)}.$$

3 - The main result

The problem (1) is considered in the case when $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfying:
There exist $\alpha(x), \beta(x), \sigma(x)$ and $\delta(x) \in C_+(\overline{\Omega})$ such that
(H_1) For each $i = 1, ..., N$

$$A_0 + A_i t^{\alpha(x)} \leq M_i(t) \leq B_0 + B_i t^{\beta(x)}$$

and

$$Q_1 t^{\delta(x)-1} \leq f(x,t) \leq Q_2 t^{\sigma(x)-1}$$

for all $t \geq 0$ and $x \in \overline{\Omega}$, where Q_1, Q_2, A_i and $B_i,\ i = 0, ..., N$ are positive constants.

(H_2) $p_m^- > \sigma^+(r+1)$, and $f(x,-t) = -f(x,t)$ for all $t \in \mathbb{R}$ and $x \in \overline{\Omega}$.

Definition 3.1. We define the weak solution of problem (1) as a function $u \in X$ satisfying:

$$\sum_{i=1}^{N} M_i \left(\int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} \, dx \right) \int_{\Omega} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \, \partial_{x_i} v \, dx$$
$$- \Big[\int_{\Omega} F(x,u) \, dx \Big]^r \int_{\Omega} f(x,u) v \, dx = 0$$

for all $v \in X$.

Our main result in this section is the following.

Theorem 3.1. *Assume* (H_1) *and* (H_2), *then the problem* (1) *has a sequence of solutions* $(\pm u_n)_n$ *such that* $J(u_n) < 0$ *for* $n = 1, 2, \ldots$.

4 - proofs

Since X is a separable and reflexive Banach space [18], there exist $\{e_j\}_{j=1}^{\infty} \subset X$ and $\{e_j^*\}_{j=1}^{\infty} \subset X^*$ such that

$$e_i^*(e_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

$$X = \overline{\text{span}}\{e_j : j = 1, 2, \ldots\}, \quad X^* = \overline{\text{span}}^{W^*}\{e_j^* : j = 1, 2, \ldots\}.$$

Define for $i = 1, ..., N$ the functionals

$$\widehat{M}_i(t) = \int_0^t M_i(s) \, ds, \quad \forall t \geq 0,$$

$$I_i(u) = \int_{\Omega} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} \, dx.$$

Consider the functional energy

$$J(u)=\sum_{i=1}^{N}\widehat{M}_i(I_i(u))-\frac{1}{r+1}\Big[\int_\Omega F(x,u)\,dx\Big]^{r+1},\quad \forall\, u\in X.$$

Obviously, $J\in C^1(X,\mathbb{R})$ and

$$J'(u).v=\sum_{i=1}^{N}M_i\left(\int_\Omega\frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}\,dx\right)\int_\Omega|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\,\partial_{x_i}v\,dx$$
$$-\Big[\int_\Omega F(x,u)\,dx\Big]^{r}\int_\Omega f(x,u)v\,dx,$$

for all $u,v\in X$.

Claim 1. J is bounded from below and even:
By (H_1) and (H_2), we have

$$J(u)=\sum_{i=1}^{N}\widehat{M}_i(I_i(u))-\Big[\int_\Omega F(x,u)\,dx\Big]^{r}$$
$$\geq\sum_{i=1}^{N}\int_0^{\int_\Omega\frac{|\partial_{x_i}u|^{p_i(x)}\,dx}{p_i(x)}}[A_0+A_i|s|^{\alpha(x)}]\,ds]-\frac{1}{r+1}\left[\Big(Q_2\int_0^u|s|^{\sigma(x)-1}ds\Big)\right]^{r+1}$$
$$\geq A_0\sum_{i=1}^{N}\int_\Omega\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\,dx+\frac{\min_{1\leq i\leq N}A_i}{\alpha^++1}\sum_{i=1}^{N}\left(\int_\Omega\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\,dx\right)^{\alpha(x)+1}$$
$$-\frac{1}{r+1}\Big(\frac{Q_2}{\sigma^-}\Big)^{r+1}\Big(\int_\Omega|u|^{\sigma(x)}\Big)^{r+1}.$$

For $\|u\|>1$ we define

$$\xi_i=\begin{cases}p_M^+\ \textit{if}\ |\partial_{x_i}u|_{p_i(.)}<1,\\ p_m^-\ \textit{if}\ \ |\partial_{x_i}u|_{p_i(.)}>1.\end{cases}$$

We recall the following inequality

$$\sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(.)}^{p_m^-}\geq N\left(\frac{\sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(.)}}{N}\right)^{p_m^-}\tag{8}$$
$$\geq\frac{\|u\|^{p_m^-}}{N^{p_m^--1}}$$

for all $u \in X$. By this inequality, for $u \in X$ with $\|u\| > 1$ we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |u|^{p_i(x)}\,dx &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}^{\xi_i} \\ &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}^{p_m^-} - \sum_{i\in\mathcal{F}_1} |\partial_{x_i} u|_{p_i(.)}^{p_M^+} \\ &\geq \frac{\|u\|^{p_m^-}}{N^{p_m^- -1}} - N,\end{aligned} \tag{9}$$

then,

$$J(u) \geq \frac{\min_{1\leq i\leq N} A_i}{(p_M^+)^{\alpha^+ +1}(\alpha^+ + 1)} \frac{\|u\|^{p_m^-(\alpha^- +1)}}{N^{p_m^- -1}} + \frac{A_0}{p_M^+}\left(\frac{\|u\|^{p_m^-}}{N^{p_m^- -1}} - N\right) - C_1\|u\|^{(r+1)\sigma^{\pm}} - C_2.$$

Since

$$p_m^-(\alpha^+ + 1 > p_m^+ > (r+1)\sigma^+,$$

it follows that J is bounded from below.

Claim 2. J satisfies the condition of Palais-Smale (P.S):
Let $(u_n)_n \subset X$ be a (P.S) sequence of the functional J, that is,

$$J(u_n) \to c, \quad \text{and} \quad J'(u_n) \to 0.$$

Thus, there exists $c_1 > 0$ such that

$$\begin{aligned}c_1 \geq& J(u_n) \\ \geq& \frac{\min_{1\leq i\leq N} A_i}{(p_M^+)^{\alpha^+ +1}(\alpha^+ + 1)} \left[\frac{\|u_n\|^{p_m^-}}{N^{p_m^- -1}}\right]^{\alpha^- +1} - C_1\|u_n\|^{(r+1)\sigma^{\pm}} - C_2.\end{aligned} \tag{10}$$

Because $p_m^-(\alpha^+ + 1) > (r+1)\sigma^+$, we see that $(u_n)_n$ is bounded in X. Up to a subsequence, still denoted by (u_n), $u_n \rightharpoonup u$ (i.e. weakly) in X.

On the one hand we have

$$J'(u_n) \to 0,$$

then

$$\langle J'(u_n), u_n - u\rangle \to 0,$$

so it leads to

$$
\begin{aligned}
&\sum_{i=1}^{N} M_i(I_i(u_n)) \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n \partial_{x_i}(u_n - u)\, dx \\
&- \left[\int_{\Omega} F(x, u_n)\, dx \right]^r \times \int_{\Omega} f(x, u_n)(u_n - u)\, dx \to 0.
\end{aligned} \tag{11}
$$

Using the Hölder inequality, we get

$$
\Big| \int_{\Omega} f(x, u_n)(u_n - u)\, dx \Big| \leq Q_2 \Big| \int_{\Omega} |u_n|^{\sigma(x)-1}(u_n - u)\, dx \Big|,
$$

$$
\Big| \int_{\Omega} |u_n|^{\sigma(x)-1}(u_n - u)\, dx \Big| \leq 2|u|_{\sigma'(x)} |u_n - u|_{\sigma(x)}.
$$

In view of the condition $1 < \sigma(x) < p^*(x)$ for all $x \in \overline{\Omega}$, it yields $X \hookrightarrow L^{\sigma(x)}(\Omega)$ with compact embedding. Hence $u_n \to u$ in $L^{\sigma(x)}(\Omega)$ and thus

$$
\Big| \int_{\Omega} f(x, u_n)(u_n - u)\, dx \Big| \to 0. \tag{12}
$$

Since the Banach space X is reflexive, from the boundedness of $(u_n)_n$ in X, we may assume that

$$
\int_{\Omega} \sum_{i=1}^{N} \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)}\, dx \to c_* \geq 0.
$$

If $c_* = 0$, so $u_n \to 0$ in X and it is will be done.

In the case when $c_* > 0$, there exist C_4 and $C_5 > 0$ such that

$$
C_4 < M_i(I_i(u)) \leq C_5, \quad i = 1, ..., N.
$$

From (11) and (12), we obtain

$$
\mathcal{K}_{p(x)} u_n.(u_n - u) = \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n \quad \partial_{x_i}(u_n - u)\, dx \to 0
$$

and

$$
\mathcal{K}_{p(x)} u.(u_n - u) = \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \quad \partial_{x_i}(u_n - u)\, dx \to 0,
$$

what implies that

$$
\langle \mathcal{K}_{p(x)} u_n - \mathcal{K}_{p(x)} u, u_n - u \rangle \to 0, \quad as \quad n \to \infty,
$$

that is

$$\sum_{i=1}^{N}\int_{\Omega}\Big(|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n-|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big)\Big(\partial_{x_i}u_n-\partial_{x_i}u\Big)\,dx\to 0.$$

Applying the following inequality

$$(|x|^{q-2}x-|y|^{q-2}y)\geq 2^{-q}|x-y|^{-q},\quad x,y\in\mathbb{R}^N,\ q\geq 2,$$

we deduce that $u_n\to u$ in X.

Proof of Theorem 1.1. From (H_1) and (H_2) we obtain

$$J(u)\leq B_0\sum_{i=1}^{N}\int_{\Omega}\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\,dx+\frac{B}{\beta(x)+1}\sum_{i=1}^{N}\left(\int_{\Omega}\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\,dx\right)^{\beta(x)+1}$$
$$-\frac{1}{r+1}\Big(\frac{Q_1}{\delta^+}\Big)^{r+1}\Big(\int_{\Omega}|u|^{\delta(x)}\,dx\Big)^{r+1}.$$

Taking $\|u\|$ small enough we get $|\partial_{x_i}u|_{p(.)}<1$ for $i=1,...,N$, then we have

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}u|^{p_i(x)}\,dx\leq\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}u|^{p_i^-}$$
$$\leq N\left(\frac{\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}u|_{p_i(.)}}{N}\right)^{p_m^-}$$
$$=\frac{\|u\|^{p_m^-}}{N^{p_m^--1}}.$$

Since X_k is finite dimensional, let $C_0>0$ such that

$$\int_{\Omega}|u|^{\delta(x)}\,dx\geq C_0\|u\|^{\delta^+}.$$

Hence,

$$J(u)\leq\frac{B_0}{p_m^-}\frac{\|u\|^{p_m^-}}{N^{p_m^--1}}+\frac{B}{(p_m^-)^{\alpha^-+1}(\alpha^-+1)}\frac{\|u\|^{p_m^-(\alpha^-+1)}}{N^{p_m^--1}}-C_0\|u\|^{(r+1)\delta^+}.$$

That is,

$$J(u)\leq\|u\|^{(r+1)\delta^+}\Big[\Big(\frac{B_0}{p_m^-}\frac{1}{N^{p_m^--1}}+\frac{B}{(p_m^-)^{\alpha^-+1}(\alpha^-+1)N^{p_m^--1}}\Big)\|u\|^{p_m^--(r+1)\delta^+}-C_0\Big].$$

Let R be a positive constant such that

$$\Big[\frac{B_0}{p_m^-}\frac{1}{N^{p_m^- -1}}+\frac{B}{(p_m^-)^{\alpha^-+1}(\alpha^-+1)N^{p_m^- -1}}\Big]R^{p_m^- -(r+1)\delta^+}<C_0\,.$$

Thus, for $0<\|u\|<R$ we get

$$J(u)\le\Big[\frac{B_0}{p_m^-}\frac{1}{N^{p_m^- -1}}+\frac{B}{(p_m^-)^{\alpha^-+1}(\alpha^-+1)}\frac{1}{N^{p_m^- -1}}\Big]R^{p_m^- -(r+1)\delta^+}-C_0<J(0)=0.$$

Now, let n be any integer given. It is well known that $C_0^\infty(\Omega)$ is infinite dimensional subspace of X. Taking $Y_n\subset C_0^\infty(\Omega)$ of dimension is n. Set $S_{n-1}=\{u\in Y_n:\ \|u\|=1\}$. Therefore, for every $v\in S_{n-1}$, we can find a nonnegative constant τ such that $J(\tau v)<0$. As S_{n-1} is compact, there exists $\tau_*>0$ with $J(\tau_* v)<0$ for every $v\in S_{n-1}$. Taking $K_n=\tau_* S_{n-1}$. Hence, $\gamma(K_n)=n$ and $\sup_{u\in K_n}J(u)<0$, where $\gamma(.)$ is the Krasnoselskii genus. By the Ljusternik-Schnirelman category theorem (cf. [**27**]), J has at least n pair of different critical points.

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