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On preperiodic points of rational functions defined over $\mathbb{F}_p(t)$

Abstract. Let $P \in \mathbb{P}_1(\mathbb{Q})$ be a periodic point for a monic polynomial with coefficients in $\mathbb{Z}$. With elementary techniques one sees that the minimal periodicity of P is at most 2. Recently we proved a generalization of this fact to the set of all rational functions defined over $\mathbb{Q}$ with good reduction everywhere (i.e. at any finite place of $\mathbb{Q}$). The set of monic polynomials with coefficients in $\mathbb{Z}$ can be characterized, up to conjugation by elements in $\mathrm{PGL}_2(\mathbb{Z})$, as the set of all rational functions defined over $\mathbb{Q}$ with a totally ramified fixed point in $\mathbb{Q}$ and with good reduction everywhere. Let p be a prime number and let $\mathbb{F}_p$ be the field with p elements. In the present paper we consider rational functions defined over the rational global function field $\mathbb{F}_p(t)$ with good reduction at every finite place. We prove some bounds for the cardinality of orbits in $\mathbb{F}_p(t) \cup \{\infty\}$ for periodic and preperiodic points.

Keywords. preperiodic points, function fields.

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1 - Introduction

In arithmetic dynamic there is a great interest about periodic and preperiodic points of a rational function $\phi: \mathbb{P}_1 \to \mathbb{P}_1$. Let $\mathbb{N}$ be the set of non-negative integer numbers. For every $n \in \mathbb{N}$, we denote by ϕ^n the n-iterate of ϕ (in particular ϕ^0 is the identity on $\mathbb{P}_1$). A point P is said to be *periodic* for ϕ if there exists an integer $n > 0$ such that $\phi^n(P) = P$. The minimal number n with the above properties is called *minimal* or *primitive period*. We say that P is a *preperiodic point* for ϕ if its

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(forward) orbit $O_\phi(P) = \{\phi^n(P) \mid n \in \mathbb{N}\}$ contains a periodic point. In other words P is preperiodic if its orbit $O_\phi(P)$ is finite. In this context an orbit is also called a cycle and its size is called the length of the cycle.

Let p be a prime and, as usual, let $\mathbb{F}_p$ be the field with p elements. We denote by K a global field, i. e. a finite extension of the field of rational numbers $\mathbb{Q}$ or a finite extension of the field $\mathbb{F}_p(t)$. Let D be the degree of K over the base field (respectively $\mathbb{Q}$ in characteristic 0 and $\mathbb{F}_p(t)$ in positive characteristic). We denote by $\mathrm{PrePer}(\phi, K)$ the set of K−rational preperiodic points for ϕ. By considering the notion of height, one sees that the set $\mathrm{PrePer}(\phi, K)$ is finite for any rational map $\phi\colon \mathbb{P}_1 \to \mathbb{P}_1$ defined over K (see for example [**13**] or [**5**]). The finiteness of the set $\mathrm{PrePer}(f, K)$ follows from [**5**, Theorem B.2.5, p. 179] and [**5**, Theorem B.2.3, p. 177] (even if these last theorems are formulated in the case of number fields, they have a similar statement in the function field case). Anyway, the bound deduced by those results depends strictly on the coefficients of the map ϕ (see also [**13**, Exercise 3.26 p. 99]). So, during the last twenty years, many dynamists have searched for bounds that do not depend on the coefficients of ϕ. In 1994 Morton and Silverman stated a conjecture known with the name "Uniform Boundedness Conjecture for Dynamical Systems": for any number field K, the number of K-preperiodic points of a morphism $\phi\colon \mathbb{P}_N \to \mathbb{P}_N$ of degree $d \geq 2$, defined over K, is bounded by a number depending only on the integers d, N and $D = [K : \mathbb{Q}]$. This conjecture is really interesting even for possible application on torsion points of abelian varieties. In fact, by considering the Lattès map associated to the multiplication by two map [2] over an elliptic curve E, one sees that the Uniform Boundedness Conjeture for $N = 1$ and $d = 4$ implies Merel's Theorem on torsion points of elliptic curves (see [**6**]). The Lattès map has degree 4 and its preperiodic points are in one-to-one correspondence with the torsion points of $E/\{\pm 1\}$ (see [**11**]). So a proof of the conjecture for every N, could provide an analogous of Merel's Theorem for all abelian varieties. Anyway, it seems very hard to solve this conjecture, even for $N = 1$.

Let R be the ring of algebraic integers of K. Roughly speaking: we say that an endomorphism ϕ of $\mathbb{P}_1$ has (simple) good reduction at a *place* $\mathfrak{p}$ if ϕ can be written in the form $\phi([x : y]) = [F(x, y), G(x, y)]$, where $F(x, y)$ and $G(x, y)$ are homogeneous polynomial of the same degree with coefficients in the local ring $R_\mathfrak{p}$ at $\mathfrak{p}$ and such that their resultant $\mathrm{Res}(F, G)$ is a $\mathfrak{p}$−unit. In Section 3 we present more carefully the notion of good reduction.

The first author studied some problems linked to Uniform Boundedness Conjecture. In particular, when $N = 1$, K is a number field and $\phi : \mathbb{P}_1 \to \mathbb{P}_1$ is an endomorphism defined over K, he proved in [**3**, Theorem 1] that the length of a cycle of a preperiodic point of ϕ is bounded by a number depending only on the cardinality of the set of places of bad reduction of ϕ.

A similar result in the function field case was recently proved in [**4**]. Furthermore in the same paper there is a bound proved for number fields, that is slightly better than the one in [**3**].

Theorem 1.1 (Theorem 1, [4]). *Let K be a global field. Let S be a finite set of places of K, containing all the archimedean ones, with cardinality $|S| \geq 1$. Let p be the characteristic of K. Let $D = [K : \mathbb{F}_p(t)]$ when $p > 0$, or $D = [K : \mathbb{Q}]$ when $p = 0$. Then there exists a number $\eta(p, D, |S|)$, depending only on p, D and $|S|$, such that if $P \in \mathbb{P}_1(K)$ is a preperiodic point for an endomorphism ϕ of $\mathbb{P}_1$ defined over K with good reduction outside S, then $|O_\phi(P)| \leq \eta(p, D, |S|)$. We can choose*

$$\eta(0, D, |S|) = \max\Big\{(2^{16|S|-8} + 3)[12|S| \log(5|S|)]^D, [12(|S| + 2)\log(5|S| + 5)]^{4D}\Big\}$$

in zero characteristic and

$$\eta(p, D, |S|) = (p|S|)^{4D} \max\Big\{(p|S|)^{2D}, p^{4|S|-2}\Big\} \tag{1}$$

in positive characteristic.

Observe that the bounds in Theorem 1.1 do not depend on the degree d of ϕ. As a consequence of that result, we could give the following bound for the cardinality of the set of K-rational preperiodic points for an endomorphism ϕ of $\mathbb{P}_1$ defined over K.

Corollary 1.1 (Corollary 1.1, [4]). *Let K be a global field. Let S be a finite set of places of K of cardinality $|S|$ containing all the archimedean ones. Let p be the characteristic of K. Let D be the degree of K over the rational function field $\mathbb{F}_p(t)$, in the positive characteristic, and over $\mathbb{Q}$, in the zero characteristic. Let $d \geq 2$ be an integer. Then there exists a number $C = C(p, D, d, |S|)$, depending only on p, D, d and $|S|$, such that for any endomorphism ϕ of $\mathbb{P}_1$ of degree d, defined over K and with good reduction outside S, we have*

$$\#\mathrm{PrePer}(\phi, \mathbb{P}_1(K)) \leq C(p, D, d, |S|).$$

Theorem 1.1 extends to global fields and to preperiodic points the result proved by Morton and Silverman in [**7**, Corollary B]. The condition $|S| \geq 1$ in its statement is only a technical one. In the case of number fields, we require that S contains the archimedean places (i.e. the ones at infinity), then it is clear that the cardinality of S is not zero. In the function field case any place is non archimedean. Recall that the place at infinity in the case $K = \mathbb{F}_p(t)$ is the one associated to the valuation given by the prime element $1/t$. When K is a finite extension of $\mathbb{F}_p(t)$, the places at infinity of K are the ones that extend the place of $\mathbb{F}_p(t)$ associated to $1/t$. The arguments used in the proof of Theorem 1.1 and Corollary 1.1 work also when S does not contain all the places at infinity. Anyway, the most important situation is when all the ones at infinity are in S. For example, in order to have that any polynomial in $\mathbb{F}_p(t)$ is an S–integer, we have to put in S all those places. Note that in the number field case the quantity $|S|$ depends also on the degree D of the extension K of $\mathbb{Q}$, because S contains all archimedean places (whose amount depends on D).

Even when the cardinality of S is small, the bound in Theorem 1.1 is quite big. This is a consequence of our searching for some uniform bounds (depending only on $p, D, |S|$). The bound $C(p, D, d, |S|)$ in Corollary 1.1 can be effectively given, but in this case too the bound is big, even for small values of the parameters $p, D, d, |S|$. For a much smaller bound see for instance the one proved by Benedetto in [**1**] for the case where ϕ is a polynomial. In the more general case when ϕ is a rational function with good reduction outside a finite S, the bound in Theorem 1.1 can be significantly improved for some particular sets S. For example if $K = \mathbb{Q}$ and S contains only the place at infinity, then we have the following bounds (see [**4**]):

- If $P \in \mathbb{P}_1(\mathbb{Q})$ is a periodic point for ϕ with minimal period n, then $n \leq 3$.
- If $P \in \mathbb{P}_1(\mathbb{Q})$ is a preperiodic point for ϕ, then $|O_\phi(P)| \leq 12$.

Here we prove some analogous bounds when $K = \mathbb{F}_p(t)$.

Theorem 1.2. *Let $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ of degree $d \geq 2$ defined over $\mathbb{F}_p(t)$ with good reduction at every finite place. If $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ is a periodic point for ϕ with minimal period n, then*

- $n \leq 3$ *if $p = 2$;*
- $n \leq 72$ *if $p = 3$;*
- $n \leq (p^2 - 1)p$ *if $p \geq 5$.*

More generally if $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ is a preperiodic point for ϕ we have

- $|O_\phi(P)| \leq 9$ *if $p = 2$;*
- $|O_\phi(P)| \leq 288$ *if $p = 3$;*
- $|O_\phi(P)| \leq (p + 1)(p^2 - 1)p$ *if $p \geq 5$.*

Observe that the bounds do not depend on the degree of ϕ but they depend only on the characteristic p. In the proof we will use some ideas already written in [**2**], [**3**] and [**4**]. The original idea of using S–unit theorems in the context of the arithmetic of dynamical systems is due to Narkiewicz [**9**].

2 - Valuations, S-integers and S-units

We adopt the present notation: let K be a global field and $v_{\mathfrak{p}}$ a valuation on K associated to a non archimedean place $\mathfrak{p}$. Let $R_{\mathfrak{p}} = \{x \in K \mid v_{\mathfrak{p}}(x) \geq 1\}$ be the local ring of K at $\mathfrak{p}$.

Recall that we can associate an absolute value to any valuation v_p, or more precisely a place $\mathfrak{p}$ that is a class of absolute values (see [**5**] and [**12**] for a reference about this topic). With $K = \mathbb{F}_p(t)$, all places are exactly the ones associated either to a monic irreducible polynomial in $\mathbb{F}_p[t]$ or to the place at infinity given by the valuation $v_\infty(f(x)/g(x)) = \deg(g(x)) - \deg(f(x))$, that is the valuation associated to $1/t$. In an arbitrary finite extension K of $\mathbb{F}_p(t)$ the valuations of K are the ones that extend the valuations of $\mathbb{F}_p(t)$. We shall call places at infinity the ones that extend the above valuation v_∞ on $\mathbb{F}_p(t)$. The other ones will be called finite places. The situation is similar to the one in number fields. The non archimedean places in $\mathbb{Q}$ are the ones associated to the valuations at any prime p of $\mathbb{Z}$. But there is also a place that is archimedean, the one associated to the usual absolute value on $\mathbb{Q}$. With an arbitrary number field K we call archimedean places all the ones that extend to K the place given by the absolute value on $\mathbb{Q}$.

From now on S will be a finite fixed set of places of K. We shall denote by

$$R_S := \{x \in K \mid v_{\mathfrak{p}}(x) \geq 0 \text{ for every prime } \mathfrak{p} \notin S\}$$

the ring of S-integers and by

$$R_S^* := \{x \in K^* \mid v_{\mathfrak{p}}(x) = 0 \text{ for every prime } \mathfrak{p} \notin S\}$$

the group of S-units.

As usual let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$. The case when $S = \emptyset$ is trivial because if so, then the ring of S–integers is already finite; more precisely $R_S = R_S^* = K^* \cap \overline{\mathbb{F}}_p$. Therefore in what follows we consider $S \neq \emptyset$.

In any case we have that $K^* \cap \overline{\mathbb{F}}_p$ is contained in R_S^*. Recall that the group $R_S^*/K^* \cap \overline{\mathbb{F}}_p$ has finite rank equal to $|S| - 1$ (see [**10**, Proposition 14.2 p. 243]). Thus, since $K \cap \overline{\mathbb{F}}_p$ is a finite field, we have that R_S^* has rank equal to $|S|$.

3 - Good reduction

We shall state the notion of good reduction following the presentation given in [**11**] and in [4].

Definition 3.1. Let $\Phi : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational map defined over K, of the form

$$\Phi([X : Y]) = [F(X, Y) : G(X, Y)]$$

where $F, G \in K[X, Y]$ are coprime homogeneous polynomials of the same degree. We say that Φ is in $\mathfrak{p}$*–reduced form* if the coefficients of F and G are in $R_{\mathfrak{p}}[X, Y]$ and at least one of them is a $\mathfrak{p}$-unit (i.e. a unit in $R_{\mathfrak{p}}$).

Recall that R_p is a principal local ring. Hence, up to multiplying the polynomials F and G by a suitable non-zero element of K, we can always find a $\mathfrak{p}$–reduced form for each rational map. We may now give the following definition.

Definition 3.2. Let $\Phi : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational map defined over K. Suppose that the morphism $\Phi([X : Y]) = [F(X, Y) : G(X, Y)]$ is written in $\mathfrak{p}$-reduced form. The *reduced map* $\Phi_{\mathfrak{p}} : \mathbb{P}_{1,k(\mathfrak{p})} \to \mathbb{P}_{1,k(\mathfrak{p})}$ is defined by $[F_{\mathfrak{p}}(X, Y) : G_{\mathfrak{p}}(X, Y)]$, where $F_{\mathfrak{p}}$ and $G_{\mathfrak{p}}$ are the polynomials obtained from F and G by reducing their coefficients modulo $\mathfrak{p}$.

With the above definitions we give the following one:

Definition 3.3. A rational map $\Phi: \mathbb{P}_1 \to \mathbb{P}_1$, defined over K, has *good reduction* at $\mathfrak{p}$ if $\deg \Phi = \deg \Phi_{\mathfrak{p}}$. Otherwise we say that it has bad reduction at $\mathfrak{p}$. Given a set S of places of K containing all the archimedean ones, we say that Φ has good reduction outside S if it has good reduction at any place $\mathfrak{p} \notin S$.

Note that the above definition of good reduction is equivalent to ask that the homogeneous resultant of the polynomial F and G is invertible in $R_{\mathfrak{p}}$, where we are assuming that $\Phi([X : Y]) = [F(X, Y) : G(X, Y)]$ is written in $\mathfrak{p}$-reduced form.

4 - Divisibility arguments

We define the $\mathfrak{p}$-adic logarithmic distance as follows (see also [8]). The definition is independent of the choice of the homogeneous coordinates.

Definition 4.1. Let $P_1 = [x_1 : y_1], P_2 = [x_2 : y_2]$ be two distinct points in $\mathbb{P}_1(K)$. We denote by

$$(2) \qquad \delta_{\mathfrak{p}}(P_1, P_2) = v_{\mathfrak{p}}(x_1 y_2 - x_2 y_1) - \min\{v_{\mathfrak{p}}(x_1), v_{\mathfrak{p}}(y_1)\} - \min\{v_{\mathfrak{p}}(x_2), v_{\mathfrak{p}}(y_2)\}$$

the $\mathfrak{p}$-adic logarithmic distance.

The divisibility arguments, that we shall use to produce the S–unit equation helpful to prove our bounds, are obtained starting from the following two facts:

Proposition 4.1 [8, Proposition 5.1]

$$\delta_{\mathfrak{p}}(P_1, P_3) \geq \min\{\delta_{\mathfrak{p}}(P_1, P_2), \delta_{\mathfrak{p}}(P_2, P_3)\}$$

for all $P_1, P_2, P_3 \in \mathbb{P}_1(K)$.

Proposition 4.2 [8, Proposition 5.2]. *Let $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ be a morphism defined over K with good reduction at a place $\mathfrak{p}$. Then for any $P, Q \in \mathbb{P}(K)$ we have*

$$\delta_{\mathfrak{p}}(\phi(P), \phi(Q)) \geq \delta_{\mathfrak{p}}(P, Q).$$

As a direct application of the previous propositions we have the following one.

Proposition 4.3 [8, Proposition 6.1]. *Let $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ be a morphism defined over K with good reduction at a place $\mathfrak{p}$. Let $P \in \mathbb{P}(K)$ be a periodic point for ϕ with minimal period n. Then*

- $\delta_{\mathfrak{p}}(\phi^i(P), \phi^j(P)) = \delta_{\mathfrak{p}}(\phi^{i+k}(P), \phi^{j+k}(P))$ *for every $i, j, k \in \mathbb{Z}$.*
- *Let $i, j \in \mathbb{N}$ such that $\gcd(i - j, n) = 1$. Then $\delta_{\mathfrak{p}}(\phi^i(P), \phi^j(P)) = \delta_{\mathfrak{p}}(\phi(P), P)$.*

5 - Proof of Theorem 1.2

We first recall the following statements.

Theorem 5.1 (Morton and Silverman [8], Zieve [**14**]). *Let $K, \mathfrak{p}, p$ be as above. Let Φ be an endomorphism of $\mathbb{P}_1$ of degree at least two defined over K with good reduction at $\mathfrak{p}$. Let $P \in \mathbb{P}_1(K)$ be a periodic point for Φ with minimal period n. Let m be the primitive period of the reduction of P modulo $\mathfrak{p}$ and r the multiplicative period of $(\Phi^m)'(P)$ in $k(\mathfrak{p})^*$. Then one of the following three conditions holds*

(i) $n = m$;

(ii) $n = mr$;

(iii) $n = p^e mr$, *for some $e \geq 1$.*

In the notation of Theorem 5.1, if $(\Phi^m)'(P) = 0$ modulo $\mathfrak{p}$, then we set $r = \infty$. Thus, if P is a periodic point, then the cases (ii) and (iii) are not possible with $r = \infty$.

Proposition 5.1 [8, Proposition 5.2]. *Let* $\phi\colon \mathbb{P}_1 \to \mathbb{P}_1$ *be a morphism defined over* K *with good reduction at a place* $\mathfrak{p}$. *Then for any* $P, Q \in \mathbb{P}(K)$ *we have*

$$\delta_{\mathfrak{p}}(\phi(P), \phi(Q)) \geq \delta_{\mathfrak{p}}(P, Q).$$

Lemma 5.1. *Let*

$$P = P_{-m+1} \mapsto P_{-m+2} \mapsto \ldots \mapsto P_{-1} \mapsto P_0 = [0:1] \mapsto [0:1]. \tag{3}$$

be an orbit for an endomorphism ϕ *defined over* K *with good reduction outside* S. *For any* a, b *integers such that* $0 < a < b \leq m-1$ *and* $\mathfrak{p} \notin S$, *it holds*

a) $\delta_{\mathfrak{p}}(P_{-b}, P_0) \leq \delta_{\mathfrak{p}}(P_{-a}, P_0)$;

b) $\delta_{\mathfrak{p}}(P_{-b}, P_{-a}) = \delta_{\mathfrak{p}}(P_{-b}, P_0)$.

Proof. a) It follows directly from Proposition 5.1.
b) By Proposition 4.1 and part a) we have

$$\delta_{\mathfrak{p}}(P_{-b}, P_{-a}) \geq \min\{\delta_{\mathfrak{p}}(P_{-b}, P_0), \delta_{\mathfrak{p}}(P_{-a}, P_0)\} = \delta_{\mathfrak{p}}(P_{-b}, P_0).$$

Let r be the largest positive integer such that $-b + r(b-a) < 0$. Then

$$\begin{aligned}\delta_{\mathfrak{p}}(P_{-b}, P_0) &\geq \min\{\delta_{\mathfrak{p}}(P_{-b}, P_{-a}), \delta_{\mathfrak{p}}(P_{-a}, P_{b-2a}), \ldots, \delta_{\mathfrak{p}}(P_{-b+r(b-a)}, P_0)\} \\ &= \delta_{\mathfrak{p}}(P_{-b}, P_{-a}).\end{aligned}$$

The inequality is obtained by applying Proposition 4.1 several times. □

Lemma 5.2 (Lemma 3.2 [4]). *Let* K *be a function field of degree* D *over* $\mathbb{F}_p(t)$ *and* S *a non empty finite set of places of* K. *Let* $P_i \in \mathbb{P}_1(K)$ *with* $i \in \{0, \ldots n-1\}$ *be* n *distinct points such that*

$$\delta_{\mathfrak{p}}(P_0, P_1) = \delta_{\mathfrak{p}}(P_i, P_j), \quad \textit{for each distinct } 0 \leq i, j \leq n-1 \textit{ and for each } \mathfrak{p} \notin S. \tag{4}$$

Then $n \leq (p|S|)^{2D}$.

Since $\mathbb{F}_p(t)$ is a principal ideal domain, every point in $\mathbb{P}_1(\mathbb{F}_p(t))$ can be written in S–coprime coordinates, i.e. for each $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ we may write $P = [a:b]$ with $a, b \in R_S$ and $\min\{v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b)\} = 0$, for all $\mathfrak{p} \notin S$. We say that $[a:b]$ are S–coprime coordinates for P.

Proof of Theorem 1.2. We use the same notation of Theorem 5.1. Assume that S contains only the place at infinity.

Case $p = 2$.

Let $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ be a periodic point for ϕ. Without loss of generality we can suppose that $P = [0 : 1]$. Observe that m is bounded by 3 and $r = 1$. By Theorem 5.1, we have $n = m \cdot 2^e$, with e a non negative integral number. Up to considering the m−th iterate of ϕ, we may assume that the minimal periodicity of P is 2^e. So now suppose that $n = 2^e$, with $e \geq 2$. Consider the following 4 points of the cycle:

$$[0 : 1] \mapsto [x_1 : y_1] \mapsto [x_2 : y_2] \mapsto [x_3 : y_3] \ldots$$

where the points $[x_i : y_i]$ are written S−coprime integral coordinates for all $i \in \{1, \ldots, n-1\}$. By applying Proposition 4.3, we have $\delta_\mathfrak{p}([0 : 1], P_1) = \delta_\mathfrak{p}([0 : 1], P_3)$, i. e. $x_3 = x_1$, because of $R_S^* = \{1\}$. From $\delta_\mathfrak{p}([0 : 1], P_1) = \delta_\mathfrak{p}(P_1, P_2)$ we deduce

$$y_2 = \frac{x_2}{x_1}\, y_1 + 1. \tag{5}$$

Furthermore, by Proposition 4.3 we have $\delta_\mathfrak{p}([0 : 1], P_1) = \delta_\mathfrak{p}(P_2, P_3)$. Since $x_3 = x_1$, then

$$y_3 x_2 - x_3 y_2 = x_1. \tag{6}$$

This last equality combined with (5) provides $y_3 = y_1$, implying $[x_1 : y_1] = [x_3 : y_3]$. Thus $e \leq 1$ and $n \in \{1, 2, 3, 6\}$. The next step is to prove that $n \neq 6$. If $n = 6$, with few calculations one sees that the cycle has the following form.

$$[0 : 1] \mapsto [x_1 : y_1] \mapsto [A_2 x_1 : y_2] \mapsto [A_3 x_1 : y_3] \mapsto [A_2 x_1 : y_4] \mapsto [x_1 : y_5] \mapsto [0 : 1], \tag{7}$$

where $A_2, A_3 \in R_S$ and everything is written in S-coprime integral coordinates. We may apply Proposition 4.3, then by considering the $\mathfrak{p}$−adic distances $\delta_\mathfrak{p}(P_1, P_i)$ for all indexes $2 \leq i \leq 5$ for every place $\mathfrak{p}$, we obtain that there exists some S−units u_i such that

$$y_2 = A_2 y_1 + u_2; \quad y_3 = A_3 y_1 + A_2 u_3; \quad y_4 = A_2 y_1 + A_3 u_4; \quad y_5 = y_1 + A_2 u_5. \tag{8}$$

Since $R_S^* = \{1\}$, we have that the identities in (8) become

$$y_2 = A_2 y_1 + 1; \quad y_3 = A_3 y_1 + A_2; \quad y_4 = A_2 y_1 + A_3; \quad y_5 = y_1 + A_2$$

where A_2, A_3 are non zero elements in $\mathbb{F}_p[t]$. By considering the $\mathfrak{p}$−adic distance $\delta_\mathfrak{p}(P_2, P_4)$ for each finite place $\mathfrak{p}$, from Proposition 4.3 we obtain that

$$v_\mathfrak{p}(A_2 x_1) = \delta_\mathfrak{p}(P_2, P_4) = v_\mathfrak{p}(A_2 x_1 (A_2 y_1 + A_3) - A_2 x_1 (A_2 y_1 + 1)) = v_\mathfrak{p}(A_2 A_3 x_1 - A_2 x_1),$$

i. e. $A_2 x_1 = A_2 A_3 x_1 - A_2 x_1$ (because $R_S^* = \{1\}$). Then $A_2 A_3 x_1 = 0$ that contradicts $n = 6$. Thus $n \leq 3$.

Suppose now that P is a preperiodic point. Without loss of generality we can assume that the orbit of P has the following shape:

$$P = P_{-m+1} \mapsto P_{-m+2} \mapsto \ldots \mapsto P_{-1} \mapsto P_0 = [0:1] \mapsto [0:1]. \tag{9}$$

Indeed it is sufficient to take in consideration a suitable iterate ϕ^n (with $n \geq 3$), so that the orbit of the point P, with respect to the iterate ϕ^n, contains a fixed point P_0. By a suitable conjugation by an element of $\mathrm{PGL}_2(R_S)$, we may assume that $P_0 = [0:1]$.

For all $1 \leq j \leq m-1$, let $P_{-j} = [x_j : y_j]$ be written in S–coprime integral coordinates. By Lemma 5.1, for every $1 \leq i < j \leq m-1$ there exists $T_{i,j} \in R_S$ such that $x_i = T_{i,j} x_j$. Consider the $\mathfrak{p}$–adic distance between the points P_{-1} and P_{-j}. Again by Lemma 5.1, we have

$$\delta_{\mathfrak{p}}(P_{-1}, P_{-j}) = v_{\mathfrak{p}}(x_1 y_j - x_1 y_1 / T_{1,j}) = v_{\mathfrak{p}}(x_1 / T_{1,j}),$$

for all $\mathfrak{p} \notin S$. Then, there exists $u_j \in R_S^*$ such that $y_j = \big(y_1 + u_j\big)/T_{1,j}$, for all $\mathfrak{p} \notin S$. Thus, there exists $u_j \in R_S^*$ such that $[x_{-j}, y_{-j}] = [x_1, y_1 + u_j]$. Since $R_S^* = \{1\}$, then $P_{-j} = [x_1 : y_1 + 1]$. So the length of the orbit (9) is at most 3. We get the bound 9 for the cardinality of the orbit of P.

Case $p > 2$.

Since $D = 1$ and $|S| = 1$, then the bound for the number of consecutive points as in Lemma 5.2 can be chosen equal to p^2. By Theorem 5.1 the minimal periodicity n for a periodic point $P \in \mathbb{P}_1(\mathbb{Q})$ for the map ϕ is of the form $n = mrp^e$ where $m \leq p+1$, $r \leq p-1$ and e is a non negative integer.

Let us assume that $e \geq 2$. Since $p > 2$, by Proposition 4.3, for every $k \in \{0, 1, 2, \ldots, p^{e-2}\}$ and $i \in \{2, \ldots, p-1\}$, we have that $\delta_{\mathfrak{p}}(P_0, P_1) = \delta_{\mathfrak{p}}(P_0, P_{k \cdot p+i})$, for any $\mathfrak{p} \notin S$. Then $P_{k \cdot p+i} = [x_1, y_{k \cdot p+i}]$. Furthermore $\delta_{\mathfrak{p}}(P_0, P_1) = \delta_{\mathfrak{p}}(P_0, P_{k \cdot p+i})$ implying that there exists a element $u_{k \cdot p+i} \in R_S^*$ such that

$$P_{k \cdot p+i} = [x_1 : y_1 + u_{k \cdot p+i}]. \tag{10}$$

Since R_S^* has $p-1$ elements and there are $(p^{e-2}+1)(p-2)$ numbers of the shape $k \cdot p + i$ as above, we have $(p^{e-2}+1)(p-2) \leq p-1$. Thus $e = 2$ and $p = 3$.

Then $n \leq 72$ if $p = 3$ and $n \leq (p^2-1)p$ if $p \geq 5$. For the more general case when P is preperiodic, consider the same arguments used in the case when $p = 2$, showing $[x_{-j}, y_{-j}] = [x_1, y_1 + u_j]$, with $u_j \in R_S^*$. Thus, the orbit of a point $P \in \mathbb{P}_1(\mathbb{Q})$ containing $P_0 \in \mathbb{P}_1(\mathbb{Q})$, as in (9), has length at most $|R_S^*| + 2 = p + 1$. The bound in the preperiodic case is then 288 for $p = 3$ and $(p+1)(p^2-1)p$ for $p \geq 5$. □

With similar proofs, we can get analogous bounds for every finite extension K of $\mathbb{F}_p(t)$. The bounds of Theorem 1.2, with $K = \mathbb{F}_p(t)$, are especially interesting, for they are very small.

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