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Controllability for the heat equation with memory: a recent approach

Abstract. We present some recent ideas and new results in the study of controllability of a distributed system with persistent memory, which is encountered in several applications, most noticeably thermodynamics of systems with memory and viscoelasticity.

Keywords. Controllability, systems with persistent memory, thermodynamics of materials with memory, viscoelasticity

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Contents

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1 - Introduction

We outline some recent results on the *heat equation with memory*

$$w' = \int_0^t N(t-s)\Delta w(s)\,\mathrm{d}s \tag{1}$$

where $w = w(x,t)$ with $t > 0$ and $x \in \Omega$, a region with C^2 boundary and $N \in H^3(0,T)$ for every $T > 0$. We associate the initial and boundary conditions

$$\begin{cases} w(0) = w_0 \in L^2(\Omega)\,, \\ w(t) = f(t) \text{ if } x \in \Gamma \subseteq \Omega\,, \quad w(t) = 0 \text{ if } x \in \partial\Omega \setminus \Gamma \end{cases}$$

(Γ is relatively open in $\partial\Omega$. The case $\Gamma = \partial\Omega$ is not excluded).

Remark 1.1. Eq. (1) can be written in the equivalent form

$$w'' = \Delta w + \int_0^t M(t-s)\Delta w(x)\,\mathrm{d}s\,, \quad M(t) = N'(t)$$

and in this form the equation is encountered in the study of viscoelasticity for Maxwell-type materials, w being the displacement. But here for definiteness we consider the expression (1) and we call w the *temperature*.

We assume that the reader is familiar with the properties of the solutions of Eq (1), as presented for example in [**17**] and we recall the following facts:

- when $N(0) > 0$, signals propagate with finite speed. The speed is $c = \sqrt{N(0)}$.
- the (density of the) flux of heat $q(x,t)$ at the position x and time t is not $-\theta_x(x,t)$, as in the case of the standard heat equation $\theta' = \theta_{xx}$, but it is given by

  $$q(x,t) = -\int_0^t N(t-s)\theta_x(x,s)\,\mathrm{d}s\ : \tag{2}$$

  a measure of the flux at a certain time t gives a measure of this integral (in viscoelasticity, the *traction* at position x and time t is $q'(x,t)$).

For the sake of simplicity of exposition, in the following we shall assume that the time scale has been normalized so to have

$$N(0) = 1.$$

In this paper, we outline recent results concerning **controllability:** the function f is a control which we use to steer the initial datum $w_0 \in L^2(\Omega)$ to a target $\xi \in L^2(\Omega)$ at a certain time T.

Note that: 1) controllability of the pair (w, w') has to be studied in viscoelasticity, see for example [**11, 13, 17, 18**] and references therein. 2) also the controllability of the pair of the deformation and the stress (or the flux) has its interest (see [**19**]), and this is a new problem which appears in the case of systems with memory, see [**1, 2, 15**].

We end this introduction with a short comment on important previous works on controllability. The first results seems to be due to Baumeister and Leugering, see for example [**4, 7**], where use have been made of Fourier expansions and moment methods, an approach extended in [**16**] (see also [**17**] and references therein). Extension to (1) of the inverse inequality of the wave equation is in [**10**] while Carlemn estimates are used in [**5**].

An approach using cosine operator theory is in [**12**] and it is used here too, following the ideas in [**18**]. In fact, our goal here is to present the methods of [**18**] in a simple setting.

2 - Cosine operators and the solutions of Eq. (1)

We introduce the operators

$$\mathcal{A} = i(-A)^{1/2} \quad \text{where} \quad \operatorname{dom} A = H^2(\Omega) \cap H_0^1(\Omega)),\ A\phi = \Delta\phi$$

while the operator D, the Dirichlet operator, is defined by

$$u = Df \iff \Delta u = 0 \quad u(x) = f(x) \text{ on } \Gamma\,,\ u = 0 \text{ on } \partial\Omega \setminus \Gamma\,.$$

The operator $\mathcal{A}$ generates a strongly continuous group, so that we can consider the strongly continuous operators $R_+(t)$ and $R_-(t)$ defined by:

$$R_+(t) = \frac{1}{2}\left[e^{\mathcal{A}t} + e^{-\mathcal{A}t}\right], \qquad R_-(t) = \frac{1}{2}\left[e^{\mathcal{A}t} - e^{-\mathcal{A}t}\right].$$

It is known that the solutions of the wave equation

$$\text{(3)} \qquad \begin{cases} u'' = \Delta u + F\,, u(0) = u_0 \in H_0^1(\Omega)\,, u'(0) = u_1 \in L^2(\Omega) \\ u = f \in L^2(0, T; L^2(\Gamma))\,, \quad u = 0 \text{ in } \partial\Omega \setminus \Gamma \end{cases}$$

is given by

$$
(4)\qquad
\begin{aligned}
&u(t) = R_+(t)u_0 + \mathcal{A}^{-1}R_-(t)u_1 + \mathcal{A}^{-1}\int_0^t R_-(t-s)F(s)\,\mathrm{d}s\\
&-\mathcal{A}\int_0^t R_-(t-s)Df(s)\,\mathrm{d}s,\\
&u'(t) = \mathcal{A}R_-(t)u_0 + R_+(t)u_1 + \int_0^t R_+(t-s)F(s)\,\mathrm{d}s\\
&-A\int_0^t R_+(t-s)Df(s)\,\mathrm{d}s.
\end{aligned}
$$

The following result is known (see [8]). Let γ_1 be the exterior normal derivative,

$$\gamma_1\phi(x) = \frac{\partial}{\partial n}\phi(x)\,, \quad x \in \partial\Omega.$$

Theorem 2.1. *The following properties hold for the* memoryless *wave equation* (3).

1. *Let* $f = 0$ *and* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $F \in L^1(0,T;L^2(\Omega))$. *Then* $u(t) \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ *and it is a linear and continuous function of* u_0, u_1, F *in the specified spaces. Furthermore, for every* $T > 0$ *there exists* $M > 0$ *such that*

$$(5)\qquad \int_\Gamma\int_0^T |\gamma_1 u(t)|^2\,\mathrm{d}t\,\mathrm{d}\Gamma \le M\Big[\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|F\|^2_{L^1(0,T;L^2(\Omega))}\Big].$$

2. *If* $f = 0$ *and* $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega)$, $F \in L^1(0,T;L^2(\Omega))$ *then* $u(t) \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$ *and it is a linear and continuous function of* u_0, u_1, F *in the specified spaces.*

3. *If* $f \in L^2(0,T;L^2(\Gamma))$ *and* $u_0 = 0$, $u_1 = 0$, $F = 0$ *then* $u(t) \in C([0,T];\, L^2(\Omega)) \cap C^1([0,T];\, H^{-1}(\Omega))$ *and depends continuously on* f, u_0, u_1, F.

The previous properties justify the following definition, where the control time is called $2T$ for later convenience:

Definition 2.1. The wave equation (3) is *controllable* at time $2T$ if the following map is surjective. The map acts from $L^2(0,T;L^2(\Gamma))$ to $L^2(\Omega) \times H^{-1}(\Omega)$ and it is defined by

$$f \mapsto \Lambda_0(2T)f = \left(\Lambda_0^1(2T), \Lambda_0^2(2T)\right)f$$
$$= \left(\mathcal{A}\int_0^{2T} R_-(2T-s)Df(s)\,\mathrm{d}s,\ A\int_0^{2T} R_+(2T-s)Df(s)\,\mathrm{d}s\right).$$

It turns out that controllability holds if Γ has suitable geometric condition, and T is large enough, see [**3**].

2.1 - *A consequence in terms of bases*

It is known that the operator A is selfadjoint with compact resolvent. Hence, $L^2(\Omega)$ has an orthonormal basis whose elements $\phi_n(x)$ are eigenvectors of A:

$$A\phi_n = -\lambda_n^2\phi_n\,.$$

It is a fact that $\lambda_n^2 > 0$ hence λ_n is real and we can choose $\lambda_n > 0$.

It is known that (see [**20**, Prop. 10.6.1] and note that our operator A is $-A_0$ in [**20**])

$$\int_\Omega \phi_n(x)Df\,\mathrm{d}x = -\frac{1}{\lambda_n^2}\int_\Gamma (\gamma_1\phi_n)f\,\mathrm{d}\Gamma.$$

It is easily seen that the operator $-\Lambda_0(2T)f$ has the following representation:

$$\left(\sum_{n=1}^{+\infty}\phi_n(x)\int_0^{2T}\int_\Gamma\left(\frac{\gamma_1\phi_n}{\lambda_n}\right)(\sin\lambda_n s)f(x,2T-s)\,\mathrm{d}\Gamma\ \mathrm{d}s,\right.$$
$$\left.\sum_{n=1}^{+\infty}\left(\lambda_n\phi_n(x)\right)\int_0^{2T}\int_\Gamma\left(\frac{\gamma_1\phi_n}{\lambda_n}\right)(\cos\lambda_n s)f(x,2T-s)\,\mathrm{d}\Gamma\ \mathrm{d}s\right).$$

The sequence $\{\lambda_n\phi_n\}$ is an orthonormal basis of $H^{-1}(\Omega)$ (in an inner product whose norm is equivalent to the standard norm) so that every target has the representation

$$\xi = \sum_{n=1}^{+\infty}\xi_n\phi_n\,,\quad \eta = \sum_{n=1}^{+\infty}\eta_n\left(\lambda_n\phi_n\right)\quad \{\xi_n\}\in l^2\,,\ \{\eta_n\}\in l^2.$$

Controllability is equivalent to the solvability of the following *moment problem*, in terms of a *real* function f:

$$(6)\qquad \int_0^{2T}\int_\Gamma \Psi_n e^{i\lambda_n s} f(x,2T-s)\,\mathrm{d}\Gamma\ \mathrm{d}s=\eta_n+i\xi_n=c_n\,,\quad n\in\mathbb{N}\qquad \Psi_n=\frac{\gamma_1\phi_n}{\lambda_n}\,.$$

Note that $\{c_n\}$ is an *arbitrary complex valued l^2-sequence.*

As usual with Fourier type expansions, it is convenient to put the problem in "symmetric form", namely let

$$\mathbb{Z}'=\mathbb{Z}\setminus\{0\}\,,\quad \lambda_n=-\lambda_{-n}\,,\ \phi_n=\phi_{-n}\ \text{for}\ n<0.$$

Then, the moment problem (6) is equivalent to

$$(7)\qquad \int_0^{2T}\int_\Gamma \Psi_n e^{i\lambda_n s} h(x,2T-s)\,\mathrm{d}\Gamma\ \mathrm{d}s=c_n\,,\quad n\in\mathbb{Z}'$$

where now $h\in L^2(0,2T;L^2(\Gamma))$ is *complex valued* and $\{c_n\}\in l^2(\mathbb{Z}')$. This implies that controllability is equivalent to surjectivity of the (continuous) *moment operator*

$$\mathbb{M}_0\,h=\left\{\int_0^{2T}\int_\Gamma \Psi_n e^{i\lambda_n s} h(x,2T-s)\,\mathrm{d}\Gamma\ \mathrm{d}s\right\}.$$

This operator is continuous thanks to the inequality (5). So, its domain is $L^2(0,2T;L^2(\Gamma))$ and its restriction to $\mathrm{cl\,span}\{\Psi_n e^{i\lambda_n t}\}$ is invertible. As proved for example in [**17**, Ch. 3]), the following holds when the associated wave equation is controllable in time $2T$:

- the sequence $\{\Psi_n e^{i\lambda_n t}\}_{n\in\mathbb{Z}'}$ is a Riesz sequence in $L^2(0,2T;L^2(\Gamma))$;
- the sequences $\{\Psi_n\cos\lambda_n t\}_{n\in\mathbb{N}}$, $\{\Psi_n\sin\lambda_n t\}_{n\in\mathbb{N}}$ are Riesz sequences in $L^2(0,T;L^2(\Gamma))$;
- the operator $\Lambda_0^1(T)$ is surjective.

We recall that a Riesz sequence is a sequence in a Hilbert space H which can be transformed to an orthonormal basis using a linear bounded and boundedly invertible transformation.

3 - The solutions of System (1)

As proved in [**17**, Ch. 2]), the solutions w of (1) solve the Volterra integral equation in $L^2(\Omega)$

$$(8)\quad w(t)=u(t)+\mathcal{A}^{-1}\int_0^t L(t-s)w(s)\,\mathrm{d}s\,,\quad L(t)w=bR_-(t)w+\int_0^t K(t-r)R_-(r)w\,\mathrm{d}r.$$

Here $u(t)$ solves the associated wave equation and b, $K(t)$ are easily computed and $K(t) \in \mathcal{L}(L^2(\Omega))$ is strongly continuous.

Equation (8) is a Volterra integral equation for $w(t)$ which we solve using Picard iteration:

$$w(t) = u(t) + \int_0^t H(t-s)u(s)\,\mathrm{d}s \tag{9}$$

where

$$H(t) = \sum_{k=1}^{+\infty} \left(\mathcal{A}^{-1}\right)^k (L)^{*k} = \mathcal{A}^{-1}\left(\sum_{k=1}^{+\infty} \left(\mathcal{A}^{-1}\right)^{k-1} \left(L^{*k}\right)\right)$$

(L^{*k} denotes iterated convolution). These formulas can be used to prove:

Theorem 3.1. *Let $F \in L^1(0,T;L^2(\Omega))$, $f \in L^2(0,T;L^2(\Gamma))$, $w_0 \in L^2(\Omega)$ and $w_1 \in H^{-1}(\Omega)$. Then $w \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$. If $f = 0$ and $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$ then $w \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$.*

If $w_0 = 0$, $w_1 = 0$, $F = 0$ and $f \in L^2(0,T;L^2(\Gamma))$ then $w \in C([0,T];\ L^2(\Omega)) \cap C^1([0,T];\ H^{-1}(\Omega))$.

In both the cases w depends continuously on the data in the specified spaces.

These results justify the following definition of controllability:

Definition 3.1. Let $T > 0$ and

$$\Lambda_M(T)f = w(T)\,, \qquad R_M(T) = \operatorname{im} \Lambda_M(T) = \left\{w(T)\,,\, f \in L^2(0,T;L^2(\Gamma))\right\}.$$

System (1) is *controllable* when the map $\Lambda_M(T)$ is surjective, i.e. when $R_M(T) = L^2(\Omega)$.

The result that we shall prove is:

Theorem 3.2. *Let the associated wave equation be controllable at time* $2T$ *and let* $\varepsilon > 0$. *Then system* (1) *is controllable at time* $T + \varepsilon$.

Using the representation (9) it is possible to lift the *direct inequality* (5) to the solution of the system (1) (see [**12**] for a different proof).

Formula (5) shows a “hidden regularity” of the wave equation, and this inequality is called the “direct inequality” of the wave equation. We are going to prove an analogous result for the solution of the equation with memory, i.e. we prove:

Theorem 3.3. *Let $T > 0$. Let $w_0 \in H^1_0(\Omega)$ and $w_1 \in L^2(\Omega)$ and $f = 0$. Then $\gamma_1 w$ belongs to $L^2(0, T; L^2(\Gamma))$ and depends continuously on w_0, w_1, i.e. there exists M such that*

$$|\gamma_1 w|^2_{L^2(0,T;L^2(\Gamma))} \le M\big(|w_0|^2_{H^1_0(\Omega)} + |w_1|^2_{L^2(\Omega)} + |F|^2_{L^1(0,T;L^2(\Omega))}\big). \tag{10}$$

We give the proof in the case $F = 0$ (the proof is easily adapted to $F \neq 0$).

The proof uses this property, that

$$\text{if } \phi \in \operatorname{dom} A \text{ then } \gamma_1 \phi = -D^* A \phi .$$

In order to prove Theorem 3.3 we introduce the notation

$$H_1(t) = \sum_{k=2}^{+\infty} \left(\mathcal{A}^{-1}\right)^k L^{*k} = \mathcal{A}^{-1} \sum_{k=2}^{+\infty} \left(\mathcal{A}^{-1}\right)^{k-2} L^{*k}$$

so that

$$w(t) = u(t) + \mathcal{A}^{-1} \int_0^t L(t-s) u(s)\, \mathrm{d}s + \mathcal{A}^{-1} \int_0^t H_1(t-s) u(s)\, \mathrm{d}s. \tag{11}$$

This shows continuous dependence on of $w_0 \in H^1_0(\Omega)$ and $w_1 \in L^2(\Omega)$ of the following functions:

$$\gamma_1 u(t)\,, \quad \gamma_1 \left(A^{-1} \int_0^t H_1(t-s) u(s)\, \mathrm{d}s \right) = -D^* \left(\int_0^t H_1(t-s) u(s)\, \mathrm{d}s \right).$$

We study the first integral in (11),

$$\mathcal{A}^{-1} \int_0^t L(t-s) u(s)\, \mathrm{d}s = \mathcal{A}^{-1} \int_0^t L(t-s) R_-(s) w_1\, \mathrm{d}s$$
$$+ \mathcal{A}^{-1} \int_0^t L(t-s) R_+(s) w_0\, \mathrm{d}s = \boxed{1} + \boxed{2}.$$

The term $\boxed{1}$ gives

$$\gamma_1 \left(A^{-1} \int_0^t L(t-s) R_-(s) w_1\, \mathrm{d}s \right) = -D^* \left(\int_0^t L(t-s) R_-(s) w_1\, \mathrm{d}s \right),$$

a continuous function of $w_1 \in L^2(\Omega)$. We study $\boxed{2}$ using the expression of $L(t)$ in (8).

We consider first

$$\mathcal{A}^{-1}\int_0^t R_-(t-s)u(s)\,\mathrm{d}s$$
$$=\mathcal{A}^{-1}\int_0^t R_-(t-s)R_+(s)w_0\,\mathrm{d}s+\mathcal{A}^{-1}\int_0^t R_-(t-s)R_-(s)w_1\,\mathrm{d}s.$$

The trace of the second addendum is treated as above. To handle the first addendum, we use

$$R_-(\tau)R_+(r)=\frac{1}{2}(R_-(r+\tau)-R_+(r-\tau))$$

so that

$$\mathcal{A}^{-1}\int_0^t R_-(t-s)R_+(s)w_0\,\mathrm{d}s=\frac{1}{2}t\Big(\mathcal{A}^{-1}R_-(t)w_0\Big)+\frac{1}{2}\int_0^t R_+(t-2s)\mathcal{A}^{-1}w_0\,\mathrm{d}s.$$

The first addendum is the velocity term of the wave equation (with $w_0\in H_0^1(\Omega)$) and the continuity of the trace follows from the properties of the wave equation. The same property holds also for $\gamma_1\big(R_+(t-2s)\mathcal{A}^{-1}w_0\big)$ (say on the interval $(-T,T)$).

The convolution of these terms with K retain the required properties.

4 - The proof of controllability

In this section we prove Theorem 3.2. The proof is in two steps. In the first step we prove that $R_M(T)$ is a closed subspace of $L^2(\Omega)\times H^{-1}(\Omega)$ and that $R_M(T)^{\perp}$ is finite dimensional. In the second step we prove $R_M(T)^{\perp}=0$, hence controllability. The proof of the first step is taken from [**12**] where the existence of a control time was proved, but the control time itself was not identified.

4.1 - *The first step: $R_M(T)$ is closed and $R_M^{\perp}(T)$ is finite dimensional*

Theorem 4.1. *Let the associated wave equation be controllable at time $2T$. Then $R_M(T)$ is closed with finite codimension.*

Proof. In the study of $R_M(T)$ we use the notation

$$u(t)=-\mathcal{A}\int_0^t R_-(t-s)\,Df(s)\,\mathrm{d}s.$$

We fix any $\gamma < 1/4$. It is known that $\operatorname{im} D \subseteq H^{1/2}(\Omega) \subseteq \operatorname{dom}(-A)^{\gamma}$ and $(-A)^{\gamma}$ can be interchanged with $R_+(t)$, $R_-(t)$ and $L(t)$.

We note that

$$\mathcal{A}^{-1}\int_0^T L(t-s)u(s)\,\mathrm{d}s = -\int_0^T L(t-s)\int_0^s R_-(s-r)Df(r)\,\mathrm{d}r\ \mathrm{d}s$$
$$= (-A)^{-\gamma}\int_0^T L(t-s)\int_0^s R_-(s-r)(-A)^{\gamma}Df(r)\,\mathrm{d}r\ \mathrm{d}s.$$

This is the composition of a continuous transformation with the compact transformation $(A)^{-\gamma}$. Hence it is a compact operator. For the same, and stronger, reasons the map

$$f \mapsto K_T f = \mathcal{A}^{-1}\int_0^T L(T-s)u(s)\,\mathrm{d}s + \mathcal{A}^{-1}\left[\sum_{k=2}^{+\infty}\left(\mathcal{A}^{-1}\right)^{k-2}L^{*k}*u\right](T)$$

is compact, from $L^2(0,T;L^2(\Omega))$ to $L^2(\Omega)$.

Then we have

$$R_M(T) = \operatorname{im}\left(\Lambda_0^1(T) + K_T\right).$$

The operator $\Lambda_0^1(T)$ is surjective in $L^2(\Omega)$ by assumption while we proved that K_T is compact.

Hence, $R_M(T)$ is closed with finite codimension, as wanted. □

4.2 - *The space $R_M(T)^{\perp}$*

We characterize $R_M(T)^{\perp} \subseteq L^2(\Omega)$:

$$(R_M(T))^{\perp} = \left\{\xi_0 \in L^2(\Omega)\,, \quad \int_{\Omega}\xi_0(x)w(x,T)\,\mathrm{d}x = 0\right\}.$$

This characterization will be applied also to the elements of $R_M(T+\varepsilon)^{\perp}$ and we note that

$$R_M(T+\varepsilon)^{\perp} \subseteq R_M(T)^{\perp}.$$

The orthogonal can be computed by assuming that the control f is C^{∞}-smooth with compact support. Taking into account continuity of the transformation $f \mapsto u$, a standard computation shows:

Lemma 4.1. *If* $\xi_0 \perp R_M(T)$ *then*

$$D^*A\left(\mathcal{A}^{-1}R_-(r)\xi_0 + \int_0^r H(r-s)\mathcal{A}^{-1}R_-(s)\xi_0 \,\mathrm{d}s\,\mathrm{d}\Gamma\ \mathrm{d}t\right) = 0. \tag{12}$$

The parenthesis is clearly the solution of Eq. (13) below with a suitable initial condition:

Theorem 4.2. *Let* ψ *solve the problem*

$$\psi'' = \Delta\psi + b\psi + \int_0^t K(t-s)\psi(s)\,\mathrm{d}s \qquad \begin{cases} \psi(0)=0\,,\ \psi'(0)=\xi_0, \\ \psi = 0 \text{ on } \partial\Omega. \end{cases} \tag{13}$$

We have $\xi_0 \perp R_M(T)$ *if and only if the solution of* (13) *has the additional property*

$$\gamma_1\psi(t) = 0 \ \text{ on } \ (0,T).$$

Note that in this statement we used that when ξ_0 is "smooth", (for example if $\xi_0 \in \operatorname{dom} A$) then $-D^*A\psi(t) = \gamma_1\psi(t)$ and the direct inequality shows that $\gamma_1\psi(t)$ is the continuous extension of $-D^*A\psi(t)$ to every $\xi \in L^2(\Omega)$.

Clearly, this characterization can be applied to every time T, in particular to a time which we denote $T+\varepsilon$.

Controllability of the associated wave equation has an interesting consequence:

Theorem 4.3. *Let the wave equation be controllable at time* $2T$ *and let* $\varepsilon > 0$. *If* $\xi_0 \in L^2(\Omega)$ *belongs to* $R_M(T+\varepsilon)^\perp$ *then we have* $\xi_0 \in \operatorname{dom} A$, *i.e.*

$$\xi_0(x) = \sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n^2}\,\phi_n(x)\,, \qquad \{\sigma_n\} \in l^2\,.$$

Proof. It is known that

$$\dim\Omega = d \Longrightarrow m_0 n^{2/d} \le \lambda_n^2 \le M n^{2/d}\,, \qquad m_0 > 0\,.$$

In this proof we use the condition $\dim\Omega \le 3$ which implies

$$\{\lambda_n^2\} \in l^2 \text{ i.e. } \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^4} < +\infty \tag{14}$$

but it will be clear that this condition can be easily removed. Furthermore we present the computation in the case $b=0$, only for simplicity of notations. We shall see that this condition has no real effect on the computations.

We use

$$\psi_n(t) = \frac{1}{\lambda_n}\,\xi_n\,\sin\lambda_n t + \int_0^t \left[\frac{1}{\lambda_n}\int_0^{t-s} K(r)\sin\lambda_n(t-s-r)\,\mathrm{d}r\right]\psi_n(s)\,\mathrm{d}s. \tag{15}$$

We introduce the notations

$$S_n(t) = \sin\lambda_n t\,, \qquad C_n(t) = \cos\lambda_n t$$

and $L_n(t)$, the resolvent kernel of the bracket in (15) (with the sign changed) so that

$$\begin{aligned} L_n &= -\frac{1}{\lambda_n} K * S_n + \frac{1}{\lambda_n}(K * S_n) * L_n \\ &= -\frac{1}{\lambda_n} K * S_n - \frac{1}{\lambda_n^2} K^{*2} * S_n^{*2} + \frac{1}{\lambda_n^2}\left(K^{*2} * S_n^{*^2}\right) * L_n. \end{aligned} \tag{16}$$

The first line of (16) shows that

$$|L_n(t)| \le M/\lambda_n \text{ for } t \in (0,T)\,. \tag{17}$$

Due to the fact that the associated wave equation is controllable in time $2T$, hence also in larger times, we know from Sect. 2.1 that both $\{\Psi_n S_n\}$ and $\{\Psi_n C_n\}$ where $\Psi_n = \gamma_1 \phi_n/\lambda_n$ are Riesz sequences in $L^2(0,T;L^2(\Gamma))$ and in $L^2(0,T+\varepsilon;L^2(\Gamma))$ and so the series

$$\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n\,, \qquad \sum_{n=1}^{+\infty} \xi_n \Psi_n C_n$$

converge when $\{\xi_n\} \in l^2$.

Now we use

$$\psi(x,t) = \sum_{n=1}^{+\infty} \phi_n(x)\psi_n(t)\xi_n\,, \qquad \psi_n(t) = \frac{1}{\lambda_n} S_n(t) - \frac{1}{\lambda_n}(L_n * S_n)(t).$$

So, the condition of orthogonality to $R_M(T)$ is

$$\sum_{n=1}^{+\infty} \{\xi_n \Psi_n S_n - \xi_n \Psi_n (L_n * S_n)) = 0.$$

This series converges and the equality holds in $L^2(0,T+\varepsilon;L^2(\Gamma))$ and, as we noted, the series $\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n$ converges too, so that we can write

$$\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n = \sum_{n=1}^{+\infty} \xi_n \Psi_n (L_n * S_n).$$

We prove that this function belongs to $H^1(0,T+\varepsilon;L^2(\Gamma))$. We formally compute termwise the derivative of the series on the right hand side and we prove that the resulting series converges in $L^2(0,T;L^2(\Gamma))$. In fact, the derivative is

$$
\begin{aligned}
\sum_{n=1}^{+\infty} \Psi_n \xi_n (\lambda_n L_n * C_n) &= -\sum_{n=1}^{+\infty} \Psi_n \xi_n K * S_n * C_n \\
&- \sum_{n=1}^{+\infty} \Psi_n \xi_n \frac{1}{\lambda_n} K^{*2} * S_n^{*2} * C_n + \\
&+ \sum_{n=1}^{+\infty} \Psi_n \xi_n \frac{1}{\lambda_n} K^{*2} * S_n^{*^2} * C_n * L_n\,.
\end{aligned} \tag{18}
$$

The first and second series on the right hand side converge since

$$
S_n * C_n = \frac{1}{2} t S_n\,, \quad S_n^{*2} * C_n = -\frac{1}{8}\left[t^2 C_n(t) - \frac{1}{\lambda_n} t S_n(t) \right].
$$

The third series converges (even uniformly) since, using (17),

$$
\left| \frac{1}{\lambda_n} L_n \right| \le \frac{M}{\lambda_n^2}\,. \tag{19}
$$

Hence we have

$$
\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n \in H^1(0, T+\varepsilon; L^2(\Gamma)).
$$

We combine with the fact that $\{\Psi_n S_n\}$, $\{\Psi_n C_n\}$ (and $\{\Psi_n e^{i\lambda_n t}\}$) are Riesz sequences on the *shorter* interval $(0,T)$ and we deduce (see [**17**, Chapt. 3])

$$
\xi_n = \frac{\delta_n}{\lambda_n}\,, \qquad \{\delta_n\} \in l^2.
$$

We replace this expression of ξ_n and we equate the derivatives of both the sides. We get:

$$
\begin{aligned}
\sum_{n=1}^{+\infty} \delta_n \Psi_n C_n = &- \sum_{n=1}^{+\infty} \Psi_n \frac{\delta_n}{\lambda_n} K * S_n * C_n \\
&- \sum_{n=1}^{+\infty} \Psi_n \frac{\delta_n}{\lambda_n} \frac{1}{\lambda_n} K^{*2} * S_n^{*2} * C_n \\
&+ \sum_{n=1}^{+\infty} \Psi_n \frac{\delta_n}{\lambda_n} \frac{1}{\lambda_n} K^{*2} * S_n^{*^2} * C_n * L_n.
\end{aligned}
$$

Now we see that the right hand side belong to $H^1(0,T;L^2(\Gamma))$. In fact, computing the derivatives termwise of the three series we get

$$\sum_{n=1}^{+\infty} \Psi_n \delta_n K * C_n^{*2}\,, \tag{20}$$

$$\sum_{n=1}^{+\infty} \Psi_n \delta_n \frac{1}{\lambda_n} K^{*2} * C_n^{*2} * S_n\,, \tag{21}$$

$$\sum_{n=1}^{+\infty} \Psi_n \delta_n \frac{1}{\lambda_n} K^{*2} * S_n * C_n^{*^2} * L_n\,. \tag{22}$$

The series (20) and (21) converge since

$$C_n^{*2}(t) = \frac{1}{2}\left(tC_n(t) + \frac{1}{\lambda_n} S_n(t)\right),$$

$$S_n * C_n^{*2} = \frac{1}{8}\left[\left(t^2 + \frac{1}{\lambda_n^2}\right) S_n(t) - \frac{1}{\lambda_n} tC_n(t)\right].$$

The series (22) converges, even uniformly, thanks to the inequality (19) (note that also the series (21) converges uniformly, for a similar reason).

Hence we have

$$\sum_{n=1}^{+\infty} \delta_n \Psi_n C_n \in H^1(0,T;L^2(\Omega)) \quad \text{so that} \quad \delta_n = \frac{\sigma_n}{\lambda_n}$$

and so

$$\xi_n = \frac{\sigma_n}{\lambda_n^2}\,,$$

as we wanted to prove. □

Remark 4.1. The condition $\dim \Omega \le 3$ has been used when we replace $L_n(t)$ with its representation in the second line of (16), which has a coefficient $1/\lambda_n^2$. Then we use $\{1/\lambda_n^2\} \in l^2$. If $\dim \Omega > 3$ then we have $\{1/(\lambda_n^{2k})\} \in l^2$ provided k is sufficiently large. And we can get a factor $1/(\lambda_n^{2k})$ in (16) by taking iterates of sufficiently high order. So, the condition $\dim \Omega \le 3$ is easily removed.

Also the condition $b = 0$ it is easily removed: it is sufficient to replace λ_n with $\beta_n = \sqrt{\lambda_n^2 - b}$.

4.3 - *The proof that $R_M(T+\varepsilon)^\perp = 0$*

Let $\xi_0 \perp R_M(T+\varepsilon)^\perp$. We are going to prove $\xi_0 = 0$. We expand

$$\xi_0(x) = \sum_{n=1}^{+\infty} \phi_n(x) \xi_n\,, \qquad \{\xi_n\} \in l^2. \tag{23}$$

The solution ψ of system (13) has the expansion

$$\psi(x,t) = \sum_{n=1}^{+\infty} \phi_n(x)\psi_n(t)\xi_n$$

where $\psi_n(t)$ solves

$$\psi_n'' = -\lambda_n^2\psi_n + b\psi_n(t) + \int_0^t K(t-s)\psi_n(s)\,\mathrm{d}s\,, \qquad \psi_n(0) = 0\,, \quad \psi_n'(0) = 1. \tag{24}$$

The condition $\xi_0 \perp R_M(T+\varepsilon)$ is the condition

$$\gamma_1\psi(t) = \gamma_1\left(\sum_{n=1}^{+\infty} \phi_n(x)\xi_n\psi_n(t)\right) = \sum_{n=1}^{+\infty} (\gamma_1\phi_n)\xi_n\psi_n(t) = 0, \quad 0 < t < T+\varepsilon \tag{25}$$

(we can exchange γ_1 and the series thanks to the direct inequality).

Controllability follows since we can prove:

Theorem 4.4. *Let the associated wave equation be controllable in time* $2T$ *and let* $\varepsilon > 0$. *Equality* (25) *implies* $\xi_0 = 0$.

We first prove:

Theorem 4.5. *Let the associated wave equation be controllable in time* $2T$ *and let* $\xi_0 = \sum_{n=1}^{+\infty} \xi_n\phi_n(x) \perp R_M(T+\varepsilon)$ *with* $\varepsilon > 0$. *Then all but a finite number of coefficients* ξ_n *are equal to zero.*

Proof. In the proof we use Theorem 4.3 and so the condition $\varepsilon > 0$ is crucially used.

We consider $\xi_0 \in R_M(T+\varepsilon)^\perp$ and the orthogonality condition (25) which, using Theorem 4.3 can be written as

$$\sum_{n=1}^{+\infty} (\gamma_1\phi_n)\frac{\sigma_n}{\lambda_n^2}\,\psi_n(t) = 0\,, \quad \{\sigma_n\} \in l^2.$$

Note that

$$\sum_{n=1}^{+\infty} (\gamma_1\phi_n)\sigma_n\psi_n(t)$$

is convergent. And so the following equality holds:

$$0=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_{n=1}^{+\infty}(\gamma_1\phi_n)\xi_n\psi_n(t)=-\sum_{n=1}^{+\infty}(\gamma_1\phi_n)\left(\lambda_n^2\xi_n\right)\psi_n(t)$$

$$+\sum_{n=1}^{+\infty}(\gamma_1\phi_n)\left[b\psi_n(t)+\int_0^t K(t-s)\psi_n(s)\,\mathrm{d}s\right]\xi_n=-\sum_{n=1}^{+\infty}(\gamma_1\phi_n)\sigma_n\psi_n(t).$$

This is the condition that

$$\xi_1=\sum_{n=1}^{+\infty}\phi_n(x)\sigma_n=\sum_{n=1}^{+\infty}\phi_n(x)\left(\lambda_n^2\xi_n\right)\perp R_M(T)\,.$$

So, using $\xi_0\perp R_M(T+\varepsilon)$ we constructed a second element $\xi_1\perp R_M(T+\varepsilon)$ and the two elements ξ_0 and ξ_1 are linearly independent *thanks to the fact that* (*at least*) *two entries of* ξ_0 *which correspond to* different eigenvalues *are nonzero.*

The new element

$$\xi_1=\sum_{n=1}^{+\infty}\phi_n(x)\sigma_n$$

has the same properties as ξ_0 and so the procedure can be repeated. We get a third element $\xi_2\perp R_M(T+\varepsilon)$,

$$\xi_2=\sum_{n=1}^{+\infty}\phi_n(x)\left(\lambda_n^4\xi_n\right)\in L^2(\Omega)$$

and the vectors ξ_0, ξ_1 and ξ_2 are linearly independent *since* (*at least*) *three entries of* ξ_0 *which correspond to* different eigenvalues *are nonzero.*

The procedure can be iterated as many times as we want, because we assumed that ξ_0 has infinitely many non zero entries (while every eigenvalue has finite multiplicity) and we find that $\dim R_M(T+\varepsilon)^\perp=+\infty$. We proved already that this is false and so we get that $\xi_0\perp R_M(T+\varepsilon)$ has *only a finite number of nonzero components* ξ_n in the expansion (23).

This is the result that we wanted to achieve. □

Finally

Theorem 4.6. *Let the associated wave equation be controllable in time* $2T$. *Then* $R_M(T+\varepsilon)=L^2(\Omega)$.

Proof. It is enough that we prove that if $\xi_0 \perp R_M(T+\varepsilon)$ then $\xi_0 = 0$. We proved in Theorem 4.5 that

$$\xi_0 = \sum_{n=1}^{N} \xi_n \phi_n(x), \qquad N \in \mathbb{N} \tag{26}$$

and the condition of orthogonality is

$$\sum_{n=1}^{N} \xi_n \left(\gamma_1 \phi_n\right) \psi_n(t) = 0. \tag{27}$$

The sum cannot have only one addendum, since otherwise we should have

$$\gamma_1 \phi_{n_0} = 0 \text{ on } \Gamma$$

and ϕ_{n_0} is an eigenvector of A and Γ is the active part of $\partial\Omega$. It is known that this is not possible if there exists a time at which the wave equation is controllable. So, the terms with nonzero coefficients ξ_n must belong to different eigenvalues, see [**6, 17**].

In conclusion, the sum must have at least two terms which correspond to different eigenvalues and we can assume $\xi_N \neq 0$.

We compute the second derivatives of both the sides of (27) and we use (24). We get:

$$\sum_{n=1}^{N} \lambda_n^2 \xi_n \left(\gamma_1 \phi_n\right) \psi_n(t) = 0\,. \tag{28}$$

We multiply (27) with λ_N^2 and we subtract from (28). We get

$$\sum_{n=1}^{N-1} \left(\lambda_n^2 - \lambda_N^2\right) \xi_n \left(\gamma_1 \phi_n\right) \psi_n(t) = 0\,.$$

If in this sum the nonzero coefficients $\left(\lambda_n^2 - \lambda_N^2\right)\xi_n$ correspond to the same eigenvalue, this contradicts the previous observation. But, after a finite number of iteration of the procedure surely we obtain this case, which is not possible. Hence, every ξ_n in (26) is equal to zero.

The proof of controllability is now complete.

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