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Various stability estimates for the problem of determining an initial heat distribution from a single measurement

Abstract. We consider the problem of determining the initial heat distribution in the heat equation from a point measurement. We show that this inverse problem is naturally related to the one of recovering the coefficients of Dirichlet series from its sum. Taking the advantage of existing literature on Dirichlet series, in connection with Müntz's theorem, we establish various stability estimates of Hölder and logarithmic type. These stability estimates are then used to derive the corresponding ones for the original inverse problem, mainly in the case of one space dimension.

In higher space dimensions, we are interested to an internal or a boundary measurement. This issue is closely related to the problem of observability arising in Control Theory. We complete and improve the existing results.

Keywords. Heat equation, fractional heat equation, initial heat distribution, Müntz's theorem, point measurement, boundary measurement, stability estimates of Hölder and logarithmic type.

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Contents

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1 - Introduction

Inverse heat source problems appear in many branches of engineering and science. A typical application is for instance an accurate estimation of a pollutant source, which is a crucial environmental safeguard in cities with dense populations (e.g. for instance [**9**] and [**20**]).

These inverse problems are severely ill-posed, involving a strongly time-irreversible parabolic dynamics. Their mathematical analysis is difficult and still a widely open subject.

In the present work, we are concerned with sources located at the initial time.

1.1 - *State of art*

Approximation: In an earlier paper Gilliam and Martin [**17**] considered the problem of recovering the initial data of the heat equation when the output is measured at points discrete in time and space. They observed that this problem is linked to the theory of Dirichlet series and a solution is found in the one dimensional case.

In [**14**], Gilliam, Lund and Martin provide a simple and extremely accurate procedure for approximating the initial temperature for the heat equation on the line using a discrete time and spatial sampling. This procedure in based on "sinc expansion". Later, in [**15**], the same authors give a discrete sampling scheme for the approximate recovery of initial data for one dimensional parabolic initial-boundary value problems on bounded intervals.

The problem of recovering the initial states of distributed parameter systems, governed by linear partial differential equations, from finite approximate data, was studied by Gilliam, Mair and Martin in [**16**].

Li, Osher and Tsai considered in [**19**] the inverse problem of finding sparse initial data from the sparsely sampled solutions of the heat equation. They prove that pointwise values of the heat solution at only a few locations are enough in an ℓ^1 constrained optimization to find the initial data.

In a recent work, De Vore and Zuazua [**7**] studied the problem of approximating accurately the initial data in the one-dimensional Dirichlet problem from finite measurements made at $(x_0, t_1), \ldots, (x_0, t_n)$. They proved that, for suitable choices of the point location of the sensor, x_0, there is a sequence $t_1 < \ldots < t_n$ of time instants that guarantees the approximation with an optimal rate, of the order of $O(n^{-r})$ in the L^2-sense, depending on the Sobolev regularity of the datum being recovered.

It is worth mentioning that the orthogonality method in [**15**] leads to a best approximation in space, while in [**7**] the authors show a best approximation in time.

Uniqueness: The determination of the initial distribution in the heat equation on a flat torus of arbitrary dimension was considered by Danger, Foote and Martin [**8**]. They established that the observation of the solution along a geodesic determines uniquely the initial heat distribution if and only if the geodesic is dense in the torus. Their result was obtained by using a Fourier decomposition together with results from the theory of almost periodic functions. Similar ideas were independently employed in [**5**] in the context of the approximate controllability and unique continuation for the heat equation along oscillating sensor and actuator locations.

Let u be the solution of the heat equation in the whole space $\mathbb{R}^d$. Nakamura, Saitoh and Syarif [**23**] showed that the initial distribution is determined and simply represented by the observations $u(t, x_0, x')$ and $\partial_{x_1} u(t, x_0, x')$, $t \geq 0$, $x' \in \mathbb{R}^{d-1}$, for some fixed $x_0 \in \mathbb{R}$.

Stability: To our knowledge there are only few works dealing with the stability issue. In a series of papers by Saitoh and al. [**2, 25, 26**], based on Reznitskaya transform and some properties of Bergman-Selberg spaces, they obtained Lipschitz stability estimate from a point or a boundary observation.

1.2 - *The relationship with the Dirichlet series*

We consider the initial-boundary value problem, abbreviated to IBVP in the sequel, for the one dimensional heat equation

$$
\begin{cases} (\partial_t - \partial_x^2)u = 0 \ \text{ in } (0, \pi) \times (0, +\infty), \\ u(0, \cdot) = u(\pi, \cdot) = 0, \\ u(\cdot, 0) = f. \end{cases} \tag{1.1}
$$

It has a unique solution $u_f \in C([0,+\infty), L^2((0,\pi)))$ whenever $f \in L^2((0,\pi))$. This solution can be written in term of Fourier series as follows

$$u_f(x,t) = \frac{2}{\pi}\sum_{k\geq 1} \widehat{f}_k e^{-k^2 t}\sin(kx), \tag{1.2}$$

where $\widehat{f}_k$ is the k-th Fourier coefficient of f:

$$\widehat{f}_k = \frac{2}{\pi}\int_0^\pi f(x)\sin(kx)dx.$$

Given a point $x_0 \in (0,\pi)$ for the placement of the sensor, we address the question of reconstructing the initial distribution f from $u_f(x_0,t)$, $t \in (0,T)$. In light of (1.2), setting $a_k = \widehat{f}_k \sin(kx_0)$, we see that the actual problem is reduced to one of recovering the sequence $a = (a_k)$ from the sum of the corresponding Dirichlet series

$$\sum_{k\geq 1} a_k e^{-k^2 t}.$$

For this to be the case x_0 has to be chosen in a strategic way so that $\sin(kx_0) \neq 0$ for all $k \geq 1$.

1.3 - *Outline*

Section 2 is devoted to establish stability estimates for the problem of recovering the coefficients a_k of a general Dirichlet series $\sum_{k\geq 1} a_k e^{-\lambda_k t}$ from its sum. Here (λ_k) is a strictly increasing sequence of non negative real numbers, diverging to infinity. The behavior of this problem depends on whether the series $\sum \frac{1}{\lambda_k}$ converges.

- When $\sum \frac{1}{\lambda_k}$ converges, $(e^{-\lambda_k t})$ admits a biorthogonal family and this simplifies the analysis.
- The case $\sum \frac{1}{\lambda_k} = \infty$ is much harder and the stability estimates we obtain are weaker.

In both cases, we prove stability estimates of logarithmic or Hölder type and to carry out our analysis we need to compare λ_k with k^β, $\beta > 0$ is given. In the case $\sum \frac{1}{\lambda_k} = \infty$, a gap condition on the sequence (λ_k) is also necessary in our analysis (see (2.21) in Subsection 2.2).

In Section 3, we apply the results obtained in Section 2 to the problem of determining the initial heat distribution in a one dimensional heat equation from an overspecified data. We get a various stability estimates of Hölder or logarithmic type. Our results include fractional heat equations of any order. We also establish a boundary observability inequality with an output located at one of the end points. This latter enables us to obtain a logarithmic stability estimate for the inverse problem of recovering the initial condition from a boundary measurement.

In Section 4, we first consider the particular case of the (fractional) heat equation in a d-dimensional rectangle Ω. We show that if the Dirichlet eigenvalues of the laplacian in Ω are simple[1], then, in some cases, the determination of the initial heat distribution can be reduced to the one dimensional case. This is achieved when the measurements consist in the values of the solution of the heat equation on d affine $(d-1)$-dimensional subspaces. Next, we revisit the case when overdetermined datum is an internal or a boundary observation. We comment the existing results and show that these latter can be improved using recent observability inequalities.

1.4 - *Notations*

The unit ball of a Banach space X is denoted by B_X.

$\ell^p = \ell^p(\mathbb{C})$, $1 \le p < \infty$, is the usual Banach space of complex-valued sequences $a = (a_n)$ such that the series $\sum |a_n|^p$ is convergent. We equip ℓ^p with its natural norm

$$\|a\|_{\ell^p} = \left(\sum_{k\ge 1} |a_k|^p\right)^{1/p}, \quad a = (a_k) \in \ell^p.$$

$\ell^\infty = \ell^\infty(\mathbb{C})$ denotes the usual Banach space of bounded complex-valued sequences $a = (a_n)$, normed by

$$\|a\|_\infty = \sup_k |a_k|, \quad a = (a_k) \in \ell^\infty.$$

For $\theta > 0$, the space $h^\theta = h^\theta(\mathbb{C})$ is defined as follows

$$h^\theta = \left\{ b = (b_k) \in \ell^2;\ \sum_{k\ge 1} \langle k\rangle^{2\theta} |b_k|^2 < \infty \right\},$$

where $\langle k\rangle = (1+k^2)^{1/2}$.

h^θ is a Hilbert space when it is equipped with the norm

$$\|b\|_{h^\theta} = \left(\sum_{k\ge 1} \langle k\rangle^{2\theta} |b_k|^2\right)^{1/2}.$$

[1] This is a generic property among open bounded subsets of $\mathbb{R}^d$.

2 - Determining the coefficients of a Dirichlet series from its sum

We limit our study to Dirichlet series whose coefficients consist in sequences from ℓ^p, $1 \le p \le \infty$.

We pick a real-valued sequence (λ_k) satisfying $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k < \ldots$ and $\lambda_k \to +\infty$ as k goes to $+\infty$. We additionally assume that $\sum_{n\ge 1} e^{-\lambda_n t} < \infty$ for any $t > 0$.

To $a = (a_n) \in \ell^p$, $1 \le p \le \infty$, we associate the Dirichlet series

$$F_a(t) = \sum_{n\ge 1} a_n e^{-\lambda_n t}.$$

Let $1 < p < \infty$ and p' be the conjugate exponent of p. Observing that

$$a_n e^{-\lambda_n t} = a_n e^{-\lambda_n t/p} e^{-\lambda_n t/p'},$$

we get by applying Hölder's inequality

$$\begin{aligned} |F_a(t)| &\le \left(\sum_{n\ge 1} e^{-\lambda_n t}\right)^{1/p'} \sum_{n\ge 1} |a_n|^p e^{-\lambda_n t} \\ &\le \left(\sum_{n\ge 1} e^{-\lambda_n t}\right)^{1/p'} \sum_{n\ge 1} |a_n|^p, \quad t > 0. \end{aligned}$$

Also,

$$|F_a(t)| \le \sum_{n\ge 1} |a_n|, \quad t \ge 0 \ (p = 1),$$

$$|F_a(t)| \le \left(\sum_{n\ge 1} e^{-\lambda_n t}\right) \sup_n |a_n|, \quad t > 0 \ (p = \infty).$$

Therefore, the series $F_a(t)$ converges for $t > 0$, for any $a \in \ell^p$, $1 \le p \le \infty$.

From the classical theory of Dirichlet series (see for instance [28]), we know that F_a has an analytic extension to the half plane $\Re z > 0$. Since a Dirichlet series is zero if and only if its coefficients are identically equal to zero, we conclude that the knowledge of F_a in a subset of $\Re z > 0$ having an accumulation point determines uniquely a. In other words, if $D \subset \{\Re z > 0\}$ has an accumulation point and F_a vanishes on D, then $a = 0$.

The most interesting case is when $p = 1$. We can define in that case the operator $\mathcal{U}$ by

$$\begin{aligned} &\mathcal{U} : \ell^1 \longrightarrow C_b([0, +\infty)) \\ &\qquad a \longmapsto \mathcal{U}(a) := F_a, \ F_a(s) = \sum_{k\ge 1} a_k e^{-\lambda_k s}. \end{aligned}$$

Here, $C_b([0,+\infty))$ is the Banach space of bounded continuous function on $[0,+\infty)$, equipped with the supremum norm

$$\|F\|_\infty = \sup\{|F(s)|;\ s \in [0,+\infty)\}.$$

Then $\mathcal{U}$ is an injective linear contractive operator.

Similarly to entire series, we address the question to know whether it is possible to reconstruct the coefficients of a Dirichlet series from its sum. This is always possible if the values of the sum is known in the half plane $\Re z > 0$. More specifically, we have the following formula (see a proof in [**28**]): for $\lambda_n < \lambda < \lambda_{n+1}$ and $\gamma > 0$,

$$\sum_{k=1}^n a_k = \frac{1}{2i\pi}\,\mathrm{pv}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F_a(z)}{z}\, e^{\lambda z}dz.$$

In the present section we aim to establish the modulus of continuity, at the origin, of the inverse of the mapping $a \in \ell^p \to F_{a|D}$, where D is a subset of $(0,+\infty)$. Roughly speaking, we seek an estimate of the form

$$\|a\|_{\ell^p} \le \Psi\big(\|F_a\|_{L^\infty(D)}\big),$$

for a in some appropriate subset of ℓ^p, $p = 1$ or 2, where Ψ is a continuous, non decreasing and non negative real-valued function satisfying $\Psi(0) = 0$.

We discuss separately the cases $\Lambda := \sum_{k\ge 1}\dfrac{1}{\lambda_k} < \infty$ and $\Lambda := \sum_{k\ge 1}\dfrac{1}{\lambda_k} = \infty$.

It is worthwhile recalling that the nature of the series $\sum_{k\ge 1}\dfrac{1}{\lambda_k}$ is related to Müntz's theorem saying that the closure of the vector space spanned by $\{e^{-\lambda_k t};\ k \ge 1\}$ is dense in $C_0([0,+\infty)) = \{\varphi \in C_0([0,+\infty));\ \varphi(+\infty) = 0\}$ if and only if $\Lambda = \infty$. Usually this theorem is stated in the following equivalent form: $\{x^{\lambda_k};\ \ k \ge 1\}$ is dense in $C_0([0,1]) = \{\varphi \in C_0([0,1]);\ \varphi(0) = 0\}$ if and only if $\Lambda = \infty$.

Let (λ_k) be the sequence of eigenvalues of the Laplace operator, on a bounded domain of $\mathbb{R}^d$, with Dirichlet boundary condition. For simplicity, let us assume that these eigenvalues are simple. From [**18**, Lemma 3.1, page 229], $\lambda_k = O(k^{2/d})$. Therefore, in that case $\Lambda < \infty$ holds if and only if $d = 1$.

2.1 - *The case $\Lambda < \infty$*

The following lemma will be useful in the sequel. Henceforth,

$$F_a^N(t) = \sum_{k=1}^N a_k e^{-\lambda_k t},\quad t \ge 0,\quad N \ge 1.$$

Lemma 2.1. *Let $a \in \ell^2$. Then $F_a \in L^2((0,T))$ and F_a^N converges to F_a in $L^2((0,T))$ as $N \to \infty$.*

Proof. From Cauchy-Schwarz's inequality

$$\left|F_a^N(t)\right|^2,\ |F_a(t)|^2 \le \left(\sum_{k=1}^{\infty}|a_k|^2\right)\left(\sum_{k=1}^{\infty}e^{-2\lambda_k t}dt\right) = G(t),\quad N \ge 1.$$

But

$$\sum_{k=1}^{\infty}\int_0^T e^{-2\lambda_k t}dt = \sum_{k=1}^{\infty}\frac{1-e^{-2\lambda_k T}}{2\lambda_k} \le \frac{\Lambda}{2}.$$

Hence $F_a \in L^2((0,T))$ and

$$\|F_a\|_{L^2((0,T))} \le \frac{\Lambda}{2}\|a\|_{\ell^2}.$$

Similarly

$$\left|F_a(t) - F_a^N(t)\right|^2 \le \left(\sum_{k\ge 1}e^{-2\lambda_k t}\right)\sum_{k\ge N+1}|a_k|^2,\quad N \ge 1,\ t > 0.$$

Thus $F_a^N(t) \to F_a(t)$ as $N \to \infty$, for any $t \in (0,T]$. As $\left|F_a^N\right|^2 \le G$, $N \ge 1$, we apply Lebesgue's dominated convergence theorem in order to get that F_a^N converges to F_a in $L^2((0,T))$. □

Let $\mathcal{E}$ be the closure in $L^2((0,T))$ of the vector space spanned by $\{e^{-\lambda_k t};\ k \ge 1\}$. By [**27**, theorem in page 24], $\mathcal{E}$ is a proper subspace of $L^2((0,T))$. Additionally, $\{e^{-\lambda_k t}\}$ possesses a biorthogonal set $\{\psi_k\}$ in $L^2((0,T))$:

$$\int_0^T \psi_k(t)e^{-\lambda_n t}dt = \delta_{nk}. \tag{2.1}$$

We assume that there are constants $\beta > 1$, $K > 0$ and $\alpha > 0$ such that

$$\lambda_n = K(n+\alpha)^\beta + o(n^{\beta-1}),\quad n \to \infty. \tag{2.2}$$

From in [**10**, formula (3.25)],

$$\|\psi_n\|_{L^2((0,T))} \le Ce^{C\lambda_n^{1/\beta}},\quad n \ge 1, \tag{2.3}$$

for some constant $C > 0$ that can depend only on (λ_n).

In light of (2.1), we get

$$a_n = \int_0^T F_a^N(t)\psi_n(t)dt,\quad n \le N.$$

Therefore, by Cauchy-Schwarz's inequality,

$$|a_n| \le \|F_a^N\|_{L^2((0,T))}\|\psi_n\|_{L^2((0,T))}, \quad n \le N. \tag{2.4}$$

By Lemma 2.1, we can pass to the limit, as $N \to \infty$, in (2.4). We get

$$|a_n| \le \|F_a\|_{L^2((0,T))}\|\psi_n\|_{L^2((0,T))}, \quad n \ge 1.$$

Then, (2.3) implies

$$|a_n| \le Ce^{C\lambda_n^{1/\beta}}\|F_a\|_{L^2((0,T))}. \tag{2.5}$$

Hence, for any $N \ge 1$,

$$\sum_{n=1}^{N} |a_n|^2 \le CNe^{C\lambda_N^{1/\beta}}\|F_a\|^2_{L^2((0,T))}.$$

But $\lambda_N^{1/\beta} = O(N)$ by (2.2). Consequently,

$$\sum_{n=1}^{N} |a_n|^2 \le e^{CN}\|F_a\|^2_{L^2((0,T))}. \tag{2.6}$$

Let $\theta > 0$ and $m > 0$ be two given constants. By inequality (2.6) we have, for any $a \in mB_{h^\theta}$,

$$\begin{aligned}\|a\|^2_{\ell^2} &= \sum_{k=1}^{N} |a_k|^2 + \sum_{k>N} |a_k|^2 \\ &\le \sum_{k=1}^{N} |a_k|^2 + \frac{1}{\langle N+1\rangle^{2\theta}}\sum_{k>N}\langle k\rangle^{2\theta}|a_k|^2 \\ &\le e^{CN}\|F_a\|^2_{L^2((0,T))} + \frac{m^2}{N^{2\theta}}.\end{aligned}$$

That is,

$$\|a\|^2_{\ell^2} \le e^{CN}\|F_a\|^2_{L^2((0,T))} + \frac{m^2}{N^{2\theta}}. \tag{2.7}$$

We need the following result to pursue our analysis. It is stated as Theorem 5.1 in [4].

Theorem 2.1. *Let* $0 < \tau < 1$. *There is a constant* c *depending only on* (λ_k) (*and not on* ρ, A *and the length of* p) *so that*

$$\|p\|_{L^\infty(0,\rho)} \le c\|p\|_{L^\infty(A)},$$

for every $p \in \operatorname{span}\{x^{\lambda_k};\ k \geq 1\}$ *and every Lebesgue-measurable set* $A \subset [\rho, 1]$ *of Lebesgue measure at least* τ.

Corollary 2.1. *Let* B *be a Lebesgue-measurable set of* $[0, T]$ *of positive Lebesgue measure. There is a constant* d*, that can depend on* (λ_k)*,* B *and* T*, so that, for any* $a \in \ell^1$*,*

$$\|F_a\|_{L^\infty((0,T))} \leq d\|F_a\|_{L^\infty(B)}. \tag{2.8}$$

Proof. We proceed similarly as in the beginning of the proof of [**22**, Corollary 5.2]. Let $\rho = e^{-T}$ and

$$A = \{x = e^{-t};\ t \in B\} \subset [\rho, 1].$$

Then

$$|A| = \int_B e^{-t} dt \geq e^{-T}|B|.$$

We recall that

$$F_a^N(t) = \sum_{k=1}^N a_k e^{-\lambda_k t},\ \ t \geq 0,\ N \geq 1.$$

We get by applying Theorem 2.1

$$\|F_a^N\|_{L^\infty((0,T))} \leq d\|F_a^N\|_{L^\infty(B)},\ \ \text{for any } N \geq 1. \tag{2.9}$$

On the other hand, F_a^N converges uniformly to F_a in $[0, +\infty)$. This is an immediate consequence of the following estimate

$$\left|F_a(t) - F_a^N(t)\right| \leq \sum_{k \geq N+1} |a_k|,\ \ N \geq 1,\ t \geq 0.$$

Therefore, (2.8) is obtained by passing to the limit, as $N \to \infty$, in (2.9). □

Now, estimate (2.8) in (2.7) yields, where B is a given Lebesgue-measurable set of $[0, T]$ of positive Lebesgue measure,

$$\begin{aligned} \|a\|_{\ell^2}^2 &\leq e^{CN}\|F_a\|_{L^\infty(B)}^2 + \frac{m^2}{N^{2\theta}},\ \ a \in mB_{h^\theta} \cap \ell^1 \\ &\leq \max(1, m^2)\left(e^{CN}\|F_a\|_{L^\infty(B)}^2 + \frac{1}{N^{2\theta}}\right). \end{aligned} \tag{2.10}$$

Let $\widetilde{N}$ be the greatest integer satisfying

$$e^{C\widetilde{N}}\|F_a\|^2_{L^\infty(B)} \le \frac{1}{\widetilde{N}^{2\theta}}.$$

Such an $\widetilde{N}$ exists provided that $\|F_a\|_{L^\infty(B)}$ is sufficiently small. A straightforward computation shows that

$$\widetilde{N} > C\big|\ln\|F_a\|_{L^\infty(B)}\big|,$$

By taking $N = \widetilde{N}$ in (2.10), we get that there exists $\delta > 0$ so that

$$\|a\|_{\ell^2} \le C\big|\ln\|F_a\|_{L^\infty(B)}\big|^{-\theta}, \ \text{ if } \ \|F_a\|_{L^\infty(B)} \le \delta. \tag{2.11}$$

When $\|F_a\|_{L^\infty(B)} > \delta$,

$$\|a\|_{\ell^2} \le \frac{m}{\delta}\|F_a\|_{L^\infty(B)}. \tag{2.12}$$

A combination of (2.11) and (2.12) implies

$$\|a\|_{\ell^2} \le C\Big\{\big|\ln\|F_a\|_{L^\infty(B)}\big|^{-\theta} + \|F_a\|_{L^\infty(B)}\Big\}.$$

We sum up our analysis in the following theorem.

Theorem 2.2. *We assume that assumption* (2.2) *is satisfied. Let B a Lebesgue-measurable set of* $[0,T]$ *of positive Lebesgue measure,* $m > 0$ *and* $\theta > 0$. *There exists a constant* $C > 0$, *that can depend on* B, (λ_n), m *and* θ, *so that, for any* $a \in mB_{h^\theta} \cap \ell^1$,

$$\|a\|_{\ell^2} \le C\Big\{\big|\ln\|F_a\|_{L^\infty(B)}\big|^{-\theta} + \|F_a\|_{L^\infty(B)}\Big\}. \tag{2.13}$$

We observe that $h^\theta \subset \ell^1$ when $\theta > 1/2$. Therefore, $mB_{h^\theta} \cap \ell^1 = mB_{h^\theta}$ if $\theta > 1/2$.

Remark 2.1. 1) In light of (2.2) and (2.5), we have the following Lipschitz stability estimate, where c is a constant depending only on (λ_n),

$$\sum_{n\ge 1} e^{-cn}|a_n|^2 \le C\|F_a\|_{L^2((0,T))}, \ \ a \in \ell^2.$$

Here the left hand side of this inequality is seen as an ℓ^2-weighted norm of a.

2) Starting from (2.7), we can prove the following estimate

$$\|a\|_{\ell^2} \le C\Big\{\Big|\ln\|F_a\|_{L^2((0,T))}\Big|^{-\theta} + \|F_a\|_{L^2((0,T))}\Big\}, \ \ a \in mB_{h^\theta}, \tag{2.14}$$

for any $\theta > 0$.

3) It is possible to establish a Hölder stability estimate. This is can be done by substituting in Theorem 2.2 h^θ by the following subspace

$$h_{c,\gamma} = \left\{ b = (b_n);\ \sum_{n\geq 1} e^{cn^\gamma} |b_n|^2 < \infty \right\},$$

with $c > 0$ and $\gamma > 1$. A proof of a similar result will be detailed in the next subsection.

4) A Lipschitz or a Hölder stability estimate is not true in general. Indeed, let us assume that we have an estimate of the form, where $0 < \mu \leq 1$,

$$\|a\|_{\ell^2} \leq C\left(\|F_a\|^\mu_{L^2((0,T))} + \|F_a\|_{L^2((0,T))} \right),\quad a \in B_{h^\theta}. \tag{2.15}$$

Let (e_k) be the usual orthonormal basis of ℓ^2. That is $e_k = (\delta_{kn})$, where δ_{kn} is the Kronecker symbol. Letting $f_k = \langle k\rangle^{-\theta} e_k$, we get by a straightforward computation

$$\|F_{f_k}\|_{L^2((0,T))} \leq \frac{1}{\langle k\rangle^\theta} \frac{1}{\sqrt{2\lambda_k}},\quad k \geq 1. \tag{2.16}$$

Since $f_k \in B_{h^\theta}$, if (2.15) were true then we would have from (2.16)

$$\frac{1}{\langle k\rangle^\theta} \leq C\left(\frac{1}{\langle k\rangle^{\theta\mu} \lambda_k^{\mu/2}} + \frac{1}{\langle k\rangle^\theta \lambda_k^{1/2}} \right),\quad k \geq k_0. \tag{2.17}$$

The particular choice of $\lambda_k = k^\beta$, $k \geq 1$ in (2.15) yields

$$1 \leq C\left(\frac{1}{k^{\mu\beta/2-\theta(1-\mu)}} + \frac{1}{k^{\beta/2}} \right),\quad k \geq 1.$$

But this inequality cannot be true if $\mu\beta/2 - \theta(1-\mu) > 0$.

2.2 - *The case $\Lambda = \infty$*

We pick $a = (a_k) \in B_{\ell^1}$ and we set

$$\varrho = \|F_a\|_\infty\ (\leq \|a\|_{\ell^1} \leq 1).$$

Since $\varrho \geq |F_a(s)| \geq |a_1|e^{-\lambda_1 s} - e^{-\lambda_2 s}$, we have

$$|a_1| \leq \varrho e^{\lambda_1 s} + e^{-(\lambda_2-\lambda_1)s},\ \text{for any } s \geq 0.$$

The choice of $s = \dfrac{1}{\lambda_2}\ln(1/\varrho)$ gives

$$|a_1| \leq 2\varrho^{1-\frac{\lambda_1}{\lambda_2}}.$$

More generally, we have

$$|a_k| \le (\varrho + |a_1| + \ldots + |a_{k-1}|)e^{-\lambda_k s} + e^{-\lambda_{k+1} s}$$

and then

$$|a_k| \le (\varrho + |a_1| + \ldots + |a_{k-1}|)^{1-\frac{\lambda_k}{\lambda_{k+1}}}.$$

So an induction argument leads to the following estimate

$$|a_1| + \ldots + |a_k| \le C_k \varrho^{(1-\frac{\lambda_1}{\lambda_2})\ldots(1-\frac{\lambda_k}{\lambda_{k+1}})}, \tag{2.18}$$

with $C_1 = 2$ and $C_{k+1} = 3C_k + 2$. Therefore $C_k = 2\sum_{i=1}^{k-1} 3^i \le 3^k$, $k \ge 2$.

If

$$p_k = \prod_{i=1}^{k} \left(1 - \frac{\lambda_i}{\lambda_{i+1}}\right),$$

then (2.18) implies

$$\sum_{i=1}^{k} |a_i| \le 3^k \varrho^{p_k}. \tag{2.19}$$

We introduce the weighted ℓ^1 space, where $\theta > 0$,

$$\ell^{1,\theta} = \left\{ a = (a_i);\ \sum_{i\ge 1} i^\theta |a_i| < \infty \right\}.$$

We equip $\ell^{1,\theta}$ with its natural norm

$$\|a\|_{\ell^{1,\theta}} = \sum_{i\ge 1} i^\theta |a_i|.$$

Let $a \in B_{\ell^{1,\theta}}$. In light of (2.19), we have

$$\begin{aligned} \|a\|_{\ell^1} &= \sum_{i=1}^{k} |a_i| + \sum_{i\ge k+1} |a_i| \\ &\le 3^k \varrho^{p_k} + \frac{1}{k^\theta} \sum_{i\ge k+1} i^\theta |a_i|. \end{aligned}$$

Hence

$$\|a\|_{\ell^1} \le 3^k \varrho^{p_k} + \frac{1}{k^\theta}, \quad k \ge 1. \tag{2.20}$$

Let us assume that the sequence (λ_k) obeys to the following assumptions: there are four constants $\beta_0 \ge 0$, $\beta_1 > 0$, $c > 0$ and $d > 0$ so that

$$\lambda_{i+1} - \lambda_i \ge \frac{d}{(i+1)^{\beta_0}} \quad \text{and} \quad \lambda_i \le c i^{\beta_1} \quad i \ge 1. \tag{2.21}$$

Under these assumptions,

$$p_k \geq q_k = \frac{c_*^k}{(k+1)^{\beta k}}, \quad \text{with } \beta = \beta_0 + \beta_1 \text{ and } c_* = \min(d/c, 1).$$

Therefore, (2.20) yields

$$\|a\|_{\ell^1} \leq 3^k \varrho^{q_k} + \frac{1}{k^\theta}, \quad k \geq 1. \tag{2.22}$$

If ϱ is sufficiently small, we denote by $\widetilde{k}$ the greatest positive integer such that

$$3^{\tilde{k}} \varrho^{q_{\tilde{k}}} \leq \frac{1}{\widetilde{k}^\theta}.$$

Let $c_\star = (\ln 3 + \theta + \beta)^{1/2}$. Since

$$3^{\tilde{k}+1} \varrho^{q_{\tilde{k}+1}} > \frac{1}{(\widetilde{k}+1)^\theta},$$

we have

$$\widetilde{k} > \frac{1}{2c_\star} (\ln |\ln \varrho|)^{1/2}$$

by a straightforward computation.

We end up getting

$$\|a\|_{\ell^1} \leq \left(\frac{1}{2c_\star}\right)^\theta \left(\ln |\ln \|F_a\|_\infty|\right)^{-\theta/2}, \quad \|F_a\|_\infty = \varrho \leq \varrho_0, \tag{2.23}$$

for some $\varrho_0 > 0$.

When $\|F_a\|_\infty \geq \varrho_0$, we have

$$\|a\|_{\ell^1} \leq \|a\|_{\ell^{1,\theta}} \leq 1 \leq \frac{\|F_a\|_\infty}{\varrho_0}. \tag{2.24}$$

Hereafter we used that $F_{\lambda a} = \lambda F_a$, $\lambda \in \mathbb{C}$, which a consequence of the linearity of $\mathcal{U}$. In light of (2.23) and (2.24), we can state the following result.

Theorem 2.3. *We assume that assumption* (2.21) *is satisfied. Let* $m > 0$. *There exists a constant* $C > 0$, *that can depend only on* (λ_n) *and* m, *so that, for any* $a \in mB_{\ell^{1,\theta}}$,

$$\|a\|_{\ell^1} \leq C\Big(\left|\ln |\ln (m^{-1}\|F_a\|_\infty)|\right|^{-\theta/2} + \|F_a\|_\infty\Big).$$

We are now going to show that, even in the present case ($\Lambda = \infty$), it is possible to establish a Hölder stability estimate.

We pick $a \in \ell^1$, N a non negative integer and we recall that F_a^N is given by

$$F_a^N(s) = \sum_{n=1}^{N} e^{-\lambda_n s} a_n, \quad s \geq 0.$$

Let $x_n = e^{-\lambda_n}$, $n = 1, \dots N$. We introduce the following Vandermonde matrix

$$V_N = \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_N \\ \vdots & \dots & \vdots \\ x_1^{N-1} & \dots & x_N^{N-1} \end{pmatrix}.$$

By setting $A_N = (a_1, \dots, a_N)^t$ and $B_N = (F_a^N(0), \dots, F_a^N(N-1))^t$, we get in a straightforward manner that $V_N A_N = B_N$ or equivalently $A_N = V_N^{-1} B_N$.

If $V_N^{-1} = (w_{ij})$ and $\|V_N^{-1}\| = \sum_{1 \leq i,j \leq N} |w_{ij}|$, then

$$\|A_N\|_1 \leq \|V_N^{-1}\| \, \|B_N\|_\infty. \tag{2.25}$$

From the proof of [**13**, Theorem 1], we obtain

$$\|V_N^{-1}\| \leq \sum_{1 \leq j \leq N} \prod_{i \neq j} \frac{1 + |x_j|}{|x_i - x_j|}.$$

Therefore, under assumption (2.21), we get after some technical calculations

$$\|V_N^{-1}\| \leq Ce^{CN^{\beta_1}},$$

where β_1 is the same as in (2.21). Hence, (2.25) entails

$$\sum_{n=1}^{N} |a_i| \leq Ce^{CN^{\beta_1}} \|F_N\|_\infty \leq Ce^{CN^{\beta_1}} \left(\|F_a\|_\infty + \sum_{n>N} |a_n| \right). \tag{2.26}$$

For $\alpha > 0$ and $\beta > 0$, we introduce the following weighted ℓ^1-space:

$$\ell^1_{\alpha,\beta} = \left\{ a = (a_i);\ \sum_{n \geq 1} e^{\alpha n^\beta} |a_i| < \infty \right\}.$$

We equip this space with its natural norm

$$\|u\|_{\ell^1_{\alpha,\beta}} = \sum_{n \geq 1} e^{\alpha n^\beta} |a_i|.$$

Let m be a non negative constant and $\beta > \beta_1$ (β_1 is the same as in (2.21)). Assuming that $a \in mB_{\ell^1_{\alpha,\beta}}$, we obtain in light of (2.26)

$$\|a\|_{\ell^1} \leq Ce^{CN^{\beta_1}} \left(\|F_a\|_\infty + me^{-cN^\beta} \right) + me^{-cN^\beta}.$$

Therefore, we find an integer N_0 so that for any $N \geq N_0$,

$$\|a\|_{\ell^1} \leq C\left(e^{CN^{\beta_1}}\|F_a\|_\infty + e^{-\widetilde{C}N^\beta}\right).$$

We derive by minimizing with respect to N the following Hölder stability estimate.

Theorem 2.4. *We assume that* (2.21) *is satisfied. Let $m > 0$, $\alpha > 0$ and $\beta > \beta_1$. There exist two constants $C > 0$ and $\gamma > 0$, that can depend only on (λ_n), m, α and β, so that, for any $a \in mB_{\ell^1_{\alpha,\beta}}$,*

$$\|a\|_{\ell^1} \leq C\big(\|F_a\|_\infty^\gamma + \|F_a\|_\infty\big).$$

3 - Determining the initial heat distribution in one the dimensional heat equation

3.1 - *Point measurement*

We come back to the one dimensional heat equation. We consider again the IBVP

$$\begin{cases} (\partial_t - \partial_x^2)u = 0 \ \text{ in } (0,\pi)\times(0,+\infty), \\ u(0,\cdot) = u(\pi,\cdot) = 0, \\ u(\cdot,0) = f. \end{cases} \tag{3.1}$$

The solution of the IBVP (3.1) is given by

$$u_f(x,t) = \frac{2}{\pi}\sum_{k\geq 1}\widehat{f}_k e^{-k^2t}\sin(kx), \tag{3.2}$$

where $\widehat{f}_k$ is the Fourier coefficient of $f \in L^2((0,\pi))$:

$$\widehat{f}_k = \frac{2}{\pi}\int_0^\pi f(x)\sin(kx)dx.$$

From [**1**, Lemma 1.1.4, page 30], there exists $x_0 \in (0,\pi)$ satisfying

$$|\sin(kx_0)| \geq d_0k^{-1}, \quad k \geq 1, \tag{3.3}$$

where d_0 a constant depending on x_0.

Let $\theta \geq 0$. It is known that the Sobolev space $H^\theta((0,\pi))$ can be constructed by using Fourier series. Precisely, we have

$$H^\theta((0,\pi)) = \{h \in L^2((0,\pi));\ \sum_{k\geq 1}\langle k\rangle^{2\theta}|\widehat{h}_k|^2 < \infty\}.$$

$H^\theta((0,\pi))$ is equipped with its natural norm

$$\|h\|_{H^\theta((0,\pi))} = \left(\sum_{k\geq 1}\langle k\rangle^{2\theta}|\widehat{h}_k|^2\right)^{1/2}.$$

We take $f \in H^2((0,\pi))$ and we set

$$a_k = \frac{2}{\pi}\sin kx_0 \widehat{f}_k, \quad k \geq 1.$$

In light of (3.3), we get

$$\frac{\pi}{2}|a_k| \leq |\widehat{f}_k| \leq c_0 k|a_k|. \tag{3.4}$$

Here c_0 is a constant depending on x_0. Therefore

$$\sum_{k\geq 1}|\widehat{f}_k|^2 \leq c_0^2\sum_{k\geq 1}\langle k\rangle^2|a_k|^2.$$

Hence

$$\sum_{k\geq 1}|\widehat{f}_k|^2 \leq c_0^2\left(\sum_{k\geq 1}\langle k\rangle^4|a_k|^2\right)^{1/2}\left(\sum_{k\geq 1}|a_k|^2\right)^{1/2} \tag{3.5}$$

by Cauchy-Schwarz's inequality.

But

$$\sum_{k\geq 1}\langle k\rangle^4|a_k|^2 \leq \frac{4}{\pi^2}\sum_{k\geq 1}\langle k\rangle^4|\widehat{f}_k|^2 = \frac{4}{\pi^2}\|f\|^2_{H^2((0,\pi))}.$$

This estimate in (3.5) gives

$$\|f\|_{L^2((0,\pi))} \leq \widetilde{c}_0\|f\|^{1/2}_{H^2((0,\pi))}\|a\|^{1/2}_{\ell^2}.$$

Here $\widetilde{c}_0 = \sqrt{2}c_0/\sqrt{\pi}$.

Then a consequence of Theorem 2.2 is

Theorem 3.1. *Let B a measurable set of $[0,T]$ of positive Lebesgue measure and $m>0$. There exists a constant $C>0$, that can depend only on B, x_0 and m, so that, for any $f \in mB_{H^2((0,\pi))}$,*

$$\|f\|_{L^2((0,\pi))} \leq C\Big(\big|\ln\|u_f(x_0,\cdot)\|_{L^\infty(B)}\big|^{-1} + \|u_f(x_0,\cdot)\|_{L^\infty(B)}\Big).$$

We extend the previous result to a fractional one dimensional heat equation. To this end, for $\alpha>0$, we define A^α, the fractional power of the operator $A = -\partial_x^2$ under

Dirichlet boundary condition, as follows

$$A^\alpha f = \frac{2}{\pi}\sum_{k\geq 1} k^{2\alpha}\widehat{f}_k \sin(kx),$$

$$D(A^\alpha) = \Big\{f \in L^2((0,\pi));\ \sum_{k\geq 1} k^{4\alpha}|\widehat{f}_k|^2 < \infty\Big\}.$$

The IBVP for the heat equation for the fractional one dimensional Laplacian is represented by the Cauchy problem

$$\begin{cases} (\partial_t + A^\alpha)u = 0 \ \text{ in } (0,+\infty), \\ u(\cdot\,,0) = f. \end{cases} \tag{3.6}$$

The solution of this Cauchy problem is given by, where $f \in L^2((0,\pi))$,

$$u_f^\alpha(x,t) = \frac{2}{\pi}\sum_{k\geq 1} e^{-k^{2\alpha}t}\widehat{f}_k \sin(kx).$$

If $1/2 < \alpha < 1$, we can apply again Theorem 2.2. We get

Theorem 3.2. *We assume* $1/2 < \alpha < 1$*. Let* B *a Lebesgue-measurable set of* $[0,T]$ *of positive Lebesgue measure and* $m > 0$*. There exists a constant* $C > 0$*, that can depend only on* B*,* x_0*,* α *and* m*, so that, for any* $f \in mB_{H^2((0,\pi))}$*,*

$$\|f\|_{L^2((0,\pi))} \leq C\Big(\big|\ln \|u_f^\alpha(x_0,\cdot)\|_{L^\infty(B)}\big|^{-1} + \|u_f^\alpha(x_0,\cdot)\|_{L^\infty(B)}\Big).$$

Next we consider the case $0 < \alpha \leq 1/2$. Since $\sum_{k\geq 1} k^{-2\alpha} = \infty$, Theorem 3.2 is no longer valid in the present case. We are going to apply Theorem 2.3 instead of Theorem 2.2.

We pick $f \in H^{\theta+1}((0,\pi))$ for some $\theta > 1/2$. Let

$$a_k = \frac{2}{\pi}\sin(kx_0)\widehat{f}_k, \quad k \geq 1.$$

In light of (3.4), we get by using Cauchy-Schwarz's inequality

$$\sum_{k\geq 1}|\widehat{f}_k| \leq c_0\left(\sum_{k\geq 1}k^2|a_k|\right)^{1/2}\left(\sum_{k\geq 1}|a_k|\right)^{1/2}. \tag{3.7}$$

But

$$\sum_{k\geq 1}k^2|a_k| \leq \left(\sum_{k\geq 1}\langle k\rangle^{-2\theta}\right)^{1/2}\left(\frac{4}{\pi^2}\sum_{k\geq 1}\langle k\rangle^{2(\theta+1)}|\widehat{f}|^2\right)^{1/2}. \tag{3.8}$$

This and the first inequality in (3.4) imply

$$\sum_{k\geq 1} k^2|a_k| \leq c_\theta \|f\|_{H^{\theta+1}((0,\pi))}. \tag{3.9}$$

Here and henceforth c_θ is a constant that can depend only on θ.

Now a combination of (3.7) and (3.9) entails

$$\sum_{k\geq 1} |\widehat{f}_k| \leq c_\theta \|f\|^{1/2}_{H^{\theta+1}((0,\pi))} \|a\|^{1/2}_{\ell^1}. \tag{3.10}$$

As $H^{\theta+1}((0,\pi))$ is continuously embedded in $C([0,\pi])$,

$$\sum_{k\geq 1} |\widehat{f}_k|^2 \leq \sup_k |\widehat{f}_k| \sum_{k\geq 1} |\widehat{f}_k| \leq 2\|f\|_\infty \sum_{k\geq 1} |\widehat{f}_k| \leq c_\theta \|f\|_{H^{\theta+1}((0,\pi))} \sum_{k\geq 1} |\widehat{f}_k|. \tag{3.11}$$

Hence, it follows from (3.10) and (3.11) that

$$\|f\|_{L^2((0,\pi))} \leq c_\theta \|f\|^{3/4}_{H^{\theta+1}((0,\pi))} \|a\|^{1/4}_{\ell^1}. \tag{3.12}$$

Similarly to (3.8), we prove

$$\|(\widehat{f}_k)\|_{\ell^1} \leq c_\theta \|f\|_{H^\theta((0,\pi))}.$$

Thus

$$\|a\|_{\ell^1} \leq c_\theta \|f\|_{H^\theta((0,\pi))} \tag{3.13}$$

by (3.4).

On the other hand, we have from (3.9)

$$\|a\|_{\ell^{1,2}} \leq c_\theta \|f\|_{H^{\theta+1}((0,\pi))}. \tag{3.14}$$

In light of (3.12), (3.13) and (3.14), we obtain by applying Theorem 2.3.

Theorem 3.3. *Let $m > 0$ and $\theta > 1/2$. There exists a constants $C > 0$, that can depend only on θ, α, x_0 and m, so that, for any $f \in mB_{H^{\theta+1}((0,\pi))}$,*

$$\|f\|_{L^2((0,\pi))} \leq C\left(\left|\ln\left|\ln\left(m^{-1}c_\theta^{-1}\|u_f^\alpha(x_0,\cdot)\|_\infty\right)\right|\right|^{-1/4} + \|u_f^\alpha(x_0,\cdot)\|_\infty \right).$$

Here c_θ is the constant in (3.9).

We observe that (2.21) is satisfied for the sequence $(n^{2\alpha})$ when $0 < \alpha \leq 1/2$. The case $\alpha = 1/2$ is obvious and for the case $0 < \alpha < 1/2$ it is a consequence of the following elementary inequality

$$(n+1)^{2\alpha} - n^{2\alpha} = \frac{1}{2\alpha}\int_n^{n+1} \rho^{2\alpha-1}d\rho \geq \frac{1}{2\alpha}\frac{1}{(n+1)^{1-2\alpha}}.$$

We mention that is possible to get a Hölder stability estimate even when $0 < \alpha \le 1/2$. To do that, we apply Theorem 2.4 instead of Theorem 2.3. We leave to the interested reader to write down the details.

3.2 - *Boundary measurement*

Let $\alpha > 0$. We recall that the solution of the fractional heat equation (3.6) is given by, where $f \in L^2((0,\pi))$,

$$u_f^\alpha(x,t) = \frac{2}{\pi}\sum_{k\ge 1} e^{-k^{2\alpha}t}\widehat{f}_k \sin(kx).$$

Since

$$u_f^\alpha(x,T) = \frac{2}{\pi}\sum_{k\ge 1} e^{-k^{2\alpha}T}\widehat{f}_k \sin(kx),$$

we get by applying Parseval's inequality

$$\|u_f^\alpha(\cdot\,,T)\|^2_{L^2((0,\pi))} = \sum_{k\ge 1} e^{-2k^{2\alpha}T}|\widehat{f}_k|^2. \tag{3.15}$$

On the other hand,

$$\partial_x u_f^\alpha(0,t) = \frac{2}{\pi}\sum_{k\ge 1} k\widehat{f}_k e^{-k^{2\alpha}t}. \tag{3.16}$$

When $\alpha > 1/2$, (ψ_k), the biorthognal set in $L^2(0,T)$ to $(e^{-k^{2\alpha}t})$, satisfies, for some constant $C > 0$ depending on α,

$$\|\psi_n\|_{L^2((0,1))} \le Ce^{Ck}.$$

This inequality is obtained from [**10**, estimate (3.25)].

Therefore we have, similarly to (2.5),

$$k^2|\widehat{f}_k|^2 \le Ce^{Ck}\|\partial_x u_f^\alpha(0,\cdot)\|^2_{L^2((0,T))}$$

and then

$$\begin{aligned} k^2 e^{-2k^{2\alpha}T}|\widehat{f}_k|^2 &\le Ce^{Ck-2k^{2\alpha}T}\|\partial_x u_f^\alpha(0,\cdot)\|^2_{L^2((0,T))} \\ &\le C\|\partial_x u_f^\alpha(0,\cdot)\|^2_{L^2((0,T))}. \end{aligned} \tag{3.17}$$

Estimate (3.17) implies the following observability inequality

$$\begin{aligned} \|u_f^\alpha(\cdot\,,T)\|_{L^2((0,\pi))} &= \left(\sum_{k\ge 1} e^{-2k^{2\alpha}T}|\widehat{f}_k|^2\right)^{1/2} \\ &\le C\|\partial_x u_f^\alpha(0,\cdot)\|_{L^2((0,T))}. \end{aligned} \tag{3.18}$$

Let B a Lebesgue-measurable set of $(0,T)$ of positive Lebesgue measure. As $\partial_x u_f^\alpha(0,\cdot)$ is given by a Dirichlet series, we get from Corollary 2.1.

$$\|u_f^\alpha(\cdot,T)\|_{L^2((0,\pi))} = \left(\sum_{k\geq 1} e^{-2k^{2\alpha}T}|\widehat{f}_k|^2\right)^{1/2} \leq C\|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)}, \tag{3.19}$$

under the condition that $(\widehat{f}_k)\in \ell^1$.

Let $\widehat{g}_k$ be the k-th Fourier coefficient of $u_f^\alpha(\cdot,T)$. Then

$$\widehat{f}_k = e^{k^{2\alpha}T}\widehat{g}_k,\quad k\geq 1.$$

Hence, for any $N\geq 1$,

$$\sum_{k\leq N}|\widehat{f}_k|^2 \leq Ne^{N^{2\alpha}T}\sum_{k\leq N}|\widehat{g}_k|^2 \leq Ne^{N^{2\alpha}T}\sum_{k\leq N}\|u_f^\alpha(\cdot,T)\|^2_{L^2((0,\pi))}.$$

In light of (3.19), this estimate yields

$$\sum_{k\leq N}|\widehat{f}_k|^2 \leq Ce^{CN^{2\alpha}}\|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)}.$$

Assuming in addition that $f\in mB_{H^\beta((0,\pi))}$, for some $\beta>1/2$, we get

$$\|f\|^2_{L^2((0,\pi))} \leq Ce^{CN^{2\alpha}}\|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)} + \frac{m^2}{N^{2\beta}}.$$

Here we used the fact that if $f\in H^\beta((0,\pi))$, with $\beta>1/2$, then $(\widehat{f}_k)\in\ell^1$.

As before, this estimate allows us to prove the following theorem.

Theorem 3.4. *Let B a Lebesgue-measurable set of $[0,T]$ of positive Lebesgue measure, $\beta>1/2$ and $m>0$. There exists a constant $C>0$, that can depend only on B, α, β and m, so that, for any $f\in mB_{H^\beta((0,\pi))}$,*

$$\|f\|_{L^2((0,\pi))} \leq C\left(\left|\ln\|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)}\right|^{-\frac{\beta}{\max(\alpha,\beta)}} + \|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)}\right). \tag{3.20}$$

We observe that if instead of (3.19) we use (3.18), then we get a variant of Theorem 3.4 in which (3.20) is substituted by

$$\|f\|_{L^2((0,\pi))} \leq C\left(\left|\ln\|\partial_x u_f^\alpha(0,\cdot)\|_{L^2((0,T))}\right|^{-\frac{\beta}{\max(\alpha,\beta)}} + \|\partial_x u_f^\alpha(0,\cdot)\|_{L^2((0,T))}\right),$$

without the restriction that $\beta>1/2$. We have only to assume that $\beta>0$.

4 - Multidimensional case

4.1 - *An application of the one dimensional case*

We firstly recall that a finite or infinite sequence of real numbers is said to be non-resonant if every nontrivial rational linear combination of finitely many of its elements is different from zero.

Let $\Omega = \prod_{i=1}^{d}(0, \mu_i \pi)$, where the sequence $(\mu_1, \dots \mu_d)$ is non-resonant. From [**24**, Proposition 5] the Dirichlet-Laplacian on Ω has simple eigenvalues

$$\lambda_K = \prod_{i=1}^{d} \frac{k_i^2}{\mu_i^2}, \quad K = (k_1, \dots, k_d) \in \mathbb{N}^k, \ k_i \geq 1.$$

To each λ_K corresponds the eigenfunction

$$\varphi_K = \left(\frac{2}{\pi}\right)^d \frac{1}{\prod_{i=1}^{d} \mu_i} \prod_{i=1}^{d} \sin(k_i x_i / \mu_i)$$

so that (φ_K) forms an orthonormal basis of $L^2(\Omega)$.

Let $A : L^2(\Omega) \to L^2(\Omega)$ be the unbounded operator given by $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The fractional power A^α, $\alpha > 0$, is defined as follows

$$A^\alpha f = \sum_{K=(k_1,\dots k_d)\in\mathbb{N}^d,\, k_i \geq 1} \lambda_K^\alpha (f, \varphi_K)\varphi_K.$$

$$D(A^\alpha) = \left\{ f \in L^2(\Omega); \sum_{K=(k_1,\dots k_d)\in\mathbb{N}^d,\, k_i \geq 1} \lambda_K^{2\alpha} |(f, \varphi_K)|^2 < \infty \right\}.$$

We consider the Cauchy problem for the fractional heat equation associated to A^α:

$$\begin{cases} (\partial_t + A^\alpha)u = 0 \ \text{ in } (0, +\infty), \\ u(\cdot, 0) = f. \end{cases} \tag{4.1}$$

The solution of this Cauchy problem is given by

$$u_f^\alpha(x, t) = \sum_{K=(k_1,\dots k_d)\in\mathbb{N}^d,\, k_i \geq 1} e^{-t\lambda_K^\alpha} (f, \varphi_K)\varphi_K.$$

To reduce the multidimensional case to the one dimensional case, we need to restrict the initial sources to those of the form $f = f_1 \otimes \dots \otimes f_d \in \bigotimes_{i=1}^{d} C_0^\infty(0, \mu_i \pi)$. In that case,

$$u_f^\alpha(x, t) = \prod_{i=1}^{d} \sum_{k_i \geq 1} e^{-t k_i^{2\alpha}/\mu_i^{2\alpha}} (f_i, \varphi_{k_i})\varphi_{k_i},$$

where

$$\varphi_{k_i} = \frac{2}{\mu_i \pi} \sin\left(\frac{k_i x_i}{\mu_i}\right).$$

In other words,

$$u_f^\alpha(x,t) = \prod_{i=1}^d u_{f_i}^\alpha(x_i,t). \tag{4.2}$$

Here $u_{f_i}^\alpha$ is the solution of the one dimensional fractional heat equation (3.6) when $(0,\pi)$ is substituted by $(0,\mu_i\pi)$.

According to the maximum principle, we have

$$\|u_{f_i}^\alpha\|_{L^\infty((0,\mu_i\pi)\times(0,T))} = \|f_j\|_{L^\infty((0,\mu_i\pi))}. \tag{4.3}$$

Let us assume that

$$\inf_j \|f_j\|_{L^\infty((0,\mu_i\pi))} := \eta > 0. \tag{4.4}$$

Henceforth, x_0 and c_0 are the same as in (3.3). In light of (4.2), (4.3) and (4.4), we get

$$\|u_f^\alpha(\cdot,\ldots,\cdot,\mu_j x_0,\cdot,\ldots,\cdot)\|_{L^\infty\left(\prod_{i\neq j}(0,\mu_i\pi)\times(0,T)\right)} \geq \eta^{d-1}\|u_{f_j}^\alpha(\mu_j x_0,\cdot)\|_{L^\infty((0,T))}$$

and then

$$\begin{aligned}\Lambda(f) := \max_{1\leq j\leq d} \|u_f^\alpha(\cdot,\ldots,\cdot,\mu_j x_0,\cdot,\ldots,\cdot)\|_{L^\infty\left(\prod_{i\neq j}(0,\mu_i\pi)\times(0,T)\right)} \\ \geq \eta^{d-1}\|u_{f_j}^\alpha(\mu_j x_0,\cdot)\|_{L^\infty((0,T))}.\end{aligned} \tag{4.5}$$

We fix $m > 0$ and $\alpha \geq 1$. We prove similarly to Theorem 3.1 that there exists a constant $C > 0$, that can depend only on m and α, so that, for any $f_i \in B_{H^2((0,\mu_i\pi))}$, $1 \leq i \leq d$,

$$\|f_i\|_{L^2((0,\mu_i\pi))} \leq C\left|\ln \|u_{f_i}^\alpha(\mu_i x_0,\cdot)\|_{L^\infty((0,T))}\right|^{-1}$$

if $\Gamma(f)$ is sufficiently small. Hence, in light of (4.5), there is $\Lambda_0 > 0$ such that

$$\|f_i\|_{L^2((0,\mu_i\pi))} \leq C\left|\ln\left(\eta^{d-1}\Lambda(f)\right)\right|^{-1}, \quad \Lambda(f) \leq \Lambda_0. \tag{4.6}$$

From estimate (4.6) we get in a straighforward manner that

$$\|f\|_{L^2(\Omega)} = \prod_{i=1}^d \|f_i\|_{L^2((0,\mu_i\pi))} \leq dC\left(\left|\ln\left(\eta^{d-1}\Lambda(f)\right)\right|^{-1} + \Lambda(f)\right). \tag{4.7}$$

A continuity argument enables us to extend estimate (4.7) the closure of $\bigotimes_{i=1}^d C_0^\infty(0,\mu_i\pi)$ in $H^{2+(d-1)/2}(\Omega)$.

The case $\alpha < 1$ can be treated similarly by using Theorems 2.3 and 2.4 instead of Theorem 2.2.

4.2 - *Boundary or internal measurement*

As we said in the introduction, there are only few results in the literature dealing with the problem of determining the initial heat distribution in a multidimensional heat equation from an overdermined data. The usual overspecified data consists in an internal or a boundary measurement. We describe and comment briefly the main existing results and show the possible improvements.

Let Ω be a bounded domain of $\mathbb{R}^d$ with C^2-smooth boundary Γ. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_k \leq \dots$ be the sequence of eigenvalues, counted according to their multiplicity, of the unbounded operator defined on $L^2(\Omega)$ by $A = -\Delta$ and $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Let (ϕ_k) the corresponding sequence of eigenfunctions, chosen so that it forms an orthonormal basis of $L^2(\Omega)$.

By [**6**, Theorem 1.43, page 27], for any $f \in H_0^1(\Omega)$, the IBVP for the heat equation

$$\begin{cases} (\partial_t - \Delta)u = 0 \text{ in } Q = \Omega \times (0, T), \\ u = 0 \text{ on } \Sigma = \Gamma \times (0, T), \\ u(\cdot, 0) = f, \end{cases} \tag{4.8}$$

has a unique solution $u_f \in H^{2,1}(Q) = L^2(0, T, H^2(\Omega)) \cap H^1(0, T, L^2(\Omega))$. Moreover, it follows from [**6**, Theorem 1.42, page 26] that $\partial_\nu u_f \in L^2(\Sigma)$.

Let γ be a non empty open subset of Γ and ω be a non empty open subset of Ω. We set $\Sigma_\gamma = \gamma \times (0, T)$ and $Q_\omega = \omega \times (0, T)$.

When $f \in H_0^1(\Omega)$, we have the following two final observability inequalities

$$\|u_f(\cdot, T)\|_{H_0^1(\Omega)} \leq C\|\partial_\nu u_f\|_{L^2(\Sigma_\gamma)} \tag{4.9}$$

and

$$\|u_f(\cdot, T)\|_{H_0^1(\Omega)} \leq C\|u_f\|_{L^2(Q_\omega)}. \tag{4.10}$$

Here u_f is the solution of the IBVP (4.8) and C is a constant independent on f.

Inequality (4.9) follows from [**6**, Proposition 3.5, page 170] and (4.10) is proved similarly (a variant of estimate (4.10) with less regularity assumption was given in [**21**, Corollary 6.3]).

Let, for $s > 1/2$,

$$H_0^s(\Omega) = \{w \in H^s(\Omega);\ u = 0 \text{ on } \Gamma \text{ (in the trace sense)}\}.$$

Following [**12**]

$$H_0^{2\theta}(\Omega) = \left\{ w \in L^2(\Omega);\ \sum_{k\geq 1} \lambda_k^{2\theta} |(w,\phi_k)|^2 < \infty \right\},\ \text{ if }\ 1/4 < \theta < 3/4.$$

From the proof of [**6**, Theorem 3.6, page 173], we deduce in a straightforward manner the following result

Theorem 4.1. *Let* $1/2 \leq \theta < 3/4$ *and* $m > 0$. *Then there exists a constant* $C > 0$, *depending on* Ω, γ *(resp.* ω*),* θ *and* m, *so that, for any* $f \in mB_{H_0^{2\theta}(\Omega)}$,

$$\|f\|_{L^2(\Omega)} \leq C\left(\left| \ln \|\partial_\nu u_f\|_{L^2(\Sigma_\gamma)} \right|^{-\theta} + \|\partial_\nu u_f\|_{L^2(\Sigma_\gamma)} \right) \tag{4.11}$$

and

$$\|f\|_{L^2(\Omega)} \leq C\left(\left| \ln \|u_f\|_{L^2(Q_\omega)} \right|^{-\theta} + \|u_f\|_{L^2(Q_\omega)} \right). \tag{4.12}$$

Remark 4.1. 1) According to [**3**, Theorem 1], we can replace in (4.12), $\|u_f\|_{L^2(Q_\omega)}$ by $\|u_f\|_{L^2(D)}$, where D is any Lebsegue-measurable set contained in $\Omega \times (0,T)$, having a non zero Lebesgue measure. We can also improve the estimate (4.11) when $\partial\Omega$ contains a real-analytic open sub-manifold, that we denote by Γ_a. In light of [**3**, Theorem 2], estimate (4.11) holds true if Σ_γ is substituted by any Lebesgue-measurable subset of $\Gamma_a \times (0,T)$ with non zero Lebesgue measure.

We note that the observability inequalities appearing in [**3**, Theorems 1 and 2] hold for bounded domains Ω which are Lipschitz and locally star-shaped.

2) From [**11**, Remark 6.1], there exist two constants $C_1 > 0$ and $C_2 > 0$, depending only on Ω, ω and T, so that

$$\sum_{n\geq 1} e^{-C_1\sqrt{\lambda_n}} |\widehat{f}_n|^2 \leq C_2 \int_{Q_\omega} |u_f|^2 dxdt,\ \ f \in L^2(\Omega). \tag{4.13}$$

Here $\widehat{f}_n = \int_\Omega f\phi_n dx$.

On the other hand, since there is a constant $c > 1$ such that $c^{-1}n^{2/d} \leq \lambda_n \leq cn^{2/d}$, $n \geq 1$, the inequality (4.13) is equivalent to the following one

$$\sum_{n\geq 1} e^{-C_1 n^{1/d}} |\widehat{f}_n|^2 \leq C_2 \int_{Q_\omega} |u_f|^2 dxdt,\ \ f \in L^2(\Omega). \tag{4.14}$$

Clearly, the mapping

$$f \to \|f\|_{L^2_w(\Omega)} = \left(\sum_{n\geq 1} e^{-C_1 n^{1/d}} |\widehat{f}_n|^2\right)^{1/2}$$

defines a norm on $L^2(\Omega)$, weaker than the usual norm on $L^2(\Omega)$. Therefore, (4.14) can be reinterpreted as a Lipschitz stability estimate of determining f from $u_{f|Q_\omega}$:

$$\|f\|_{L^2_w(\Omega)} \leq C_2 \|u_f\|_{L^2(Q_\omega)}, \ \ f \in L^2(\Omega).$$

A consequence of (4.14) is

$$|\widehat{f}_n|^2 \leq C_2 e^{C_1 n^{1/d}} \int_{Q_\omega} |u_f|^2 dxdt, \ \ f \in L^2(\Omega), \ \ n \geq 1.$$

This estimate allows us to retrieve the estimate (4.12).

3) Let us show that, in the case of an internal measurement, we can directly get a stability estimate without using the observability inequality (4.10). The key of this direct proof relies on Lebeau-Robbiano type inequality for the eigenfunctions ϕ_n. For $f \in L^2(\Omega)$, we set

$$u_N(\cdot\,,t) = \sum_{n=1}^{N} e^{-\lambda_n t} \widehat{f}_n \phi_n.$$

Let ω be a Lebesgue-measurable subset of Ω of positive Lebesgue measure. From [**3**, Theorem 5], we have

$$\sum_{n=1}^{N} e^{-2\lambda_n t} |\widehat{f}_n|^2 \leq C e^{C\lambda_N} \int_{\omega} |u_N(\cdot\,,t)|^2 dx.$$

But

$$\int_{\omega} |u_N(\cdot\,,t)|^2 dx \leq \int_{\omega} |u_f(\cdot\,,t)|^2 dx + \sum_{n\geq N+1} |\widehat{f}_n|^2.$$

Hence

$$\sum_{n=1}^{N} e^{-2\lambda_n t} |\widehat{f}_n|^2 \leq C e^{C\lambda_N} \left[\int_{\omega} |u_f(\cdot\,,t)|^2 dx + \sum_{n\geq N+1} |\widehat{f}_n|^2 \right].$$

We integrate, with respect to t, between 0 and T. We get in a straightforward manner that

$$\sum_{n=1}^{N} |\widehat{f}_n|^2 \leq C e^{C\lambda_N} \left[\int_{Q_\omega} |u_f|^2 dx + \sum_{n\geq N+1} |\widehat{f}_n|^2 \right],$$

implying that

$$\sum_{n\geq 1} |\widehat{f}_n|^2 \leq Ce^{C\lambda_N}\left[\int_{Q_\omega} |u_f|^2 dx + \sum_{n\geq N+1} |\widehat{f}_n|^2\right]$$
$$\leq Ce^{CN^{2/d}}\left[\int_{Q_\omega} |u_f|^2 dx + \sum_{n\geq N+1} |\widehat{f}_n|^2\right].$$

Therefore, under the assumption

$$\sum_{n\geq 1} e^{cn^\gamma} |\widehat{f}_n|^2 \leq m,$$

for some $c > 0$, $m > 0$ and $\gamma > d/2$, we obtain similarly to Theorem 2.4 the following Hölder stability estimate

$$\|f\|_{L^2(\Omega)} \leq C\Big(\|u_f\|_{Q_\omega}^\theta + \|u_f\|_{Q_\omega}\Big).$$

We end this subsection by mentioning that a Lipschitz stability estimate was established in [**25**] when the space of the initial heat distribution is given by a Banach space, that we denote by B_μ in the sequel, built on the Bergman-Selberg space H_μ, where $\mu > 1/2$ is a parameter.

Theorem 4.2. *We fix $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$, $\mu \in (1, 5/4)$ and we set*

$$\gamma_0 = \{x \in \Gamma;\ (x - x_0)\cdot \nu(x) > 0\} \text{ and } \Sigma_0 = \gamma_0 \times (0, T).$$

There is a constant C, that can depend only on Ω, x_0 and μ, so that

$$C^{-1}\|f\|_{L^2(\Omega)} \leq \|\partial_\nu u_f\|_{B_\mu(\Sigma_0)} \leq C\|f\|_{H^2(\Omega)},\ \ f \in H^2(\Omega)\cap H_0^1(\Omega).$$

A detailed proof of this theorem is given in [**6**].

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References

[**1**] K. Ammari and S. Nicaise, *Stabilization of elastic systems by collocated feedback*, Springer, Cham, 2015.

[**2**] M. Asaduzzaman and S. Saitoh, *Inversion formulas for the Reznitskaya transform and stability of Lipschitz type in determination of initial heat distribution*, Appl. Anal. **77** (2001), no. 3-4, 343-350.

[3] J. APRAIZ, L. ESCAURIAZA, G. WANG and C. ZHANG, *Observability inequalities and measurable sets*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 11, 2433-2475.

[4] P. BORWEIN and T. ERDÉLYI, *Generalizations of Müntz's theorem via a Remez-type inequality for Müntz spaces*, J. Amer. Math. Soc. **10** (1997), 327-349.

[5] C. CASTRO and E. ZUAZUA, *Unique continuation and control for the heat equation from an oscillating lower dimensional manifold*, SIAM J. Control Optim. **43** (2004/05), no. 4, 1400-1434.

[6] M. CHOULLI, *Une introduction aux problèmes inverses elliptiques et paraboliques* (French), Mathématiques & Applications, **65**, Springer-Verlag, Berlin 2009.

[7] R. DEVORE and E. ZUAZUA, *Recovery of an initial temperature from discrete sampling*, Math. Models Methods Appl. Sci. **24** (2014), no. 12, 2487-2501.

[8] L. D. DRAGER, R. L. FOOTE and C. F. MARTIN, *Observing the heat equation on a torus along a dense geodesic*, Systems Sci. Math. Sci. **4** (1991), no. 2, 186-192.

[9] A. EL BADIA, T. HA-DUONG and A. HAMDI, *Identification of a point source in a linear advection-dispersion-reaction equation: application to a pollution source problem*, Inverse Problems **21** (2005), 1121-1136.

[10] H. O. FATTORINI and D. L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. **43** (1971), 272-292.

[11] E. FERNÁNDEZ-CARA and E. ZUAZUA, *The cost of approximate controllability for heat equations: the linear case*, Adv. Differential Equations **5** (2000), no. 4-6, 465-514.

[12] D. FUJIWARA, *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*, Proc. Japan Acad. **43** (1967), 82-86.

[13] W. GAUTSCHI, *On inverse of Vandermonde and confluent Vandermonde matrices*, Numer. Math. **4** (1962), 117-123.

[14] D. S. GILLIAM, J. R. LUND and C. F. MARTIN, *A Discrete sampling inversion scheme for the heat equation*, Numer. Math. **54** (1989), 493-506.

[15] D. S. GILLIAM, J. R. LUND and C. F. MARTIN, *Inverse parabolic problems and discrete orthogonality*, Numer. Math. **59** (1991), 361-383.

[16] D. S. GILLIAM, B. A. MAIR and C. F. MARTIN, *Determination of initial states of parabolic systems from discrete data*, Inverse Problems **6** (1990), 737-747.

[17] D. S. GILLIAM and C. F. MARTIN, *Discrete observability and Dirichlet series*, Systems Control Lett. **9** (1987), 345-348.

[18] O. KAVIAN, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques* (French), Mathématiques & Applications, **13**, Springer-Verlag, Paris 1993.

[19] Y. LI, S. OSHER and R. TSAI, *Heat source identification based on ℓ_1 constrained minimization*, Inverse Probl. Imaging 8 (2014), no. 1, 199-221.

[20] G. S. LI, Y. J. TAN, J. CHENG and X. Q. WANG, *Determining magnitude of groundwater pollution sources by data compatibility analysis*, Inverse Probl. Sci. Eng. **14** (2006), no. 3, 287-300.

[21] J. LE ROUSSEAU and G. LEBEAU, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM Control Optim. Calc. Var. **18** (2012), no. 3, 712-747.

[22] S. Micu, I. Roventa and M. Tucsnak, *Time optimal boundary controls for the heat equation*, J. Funct. Anal. **263** (2012), 25-49.

[23] G. Nakamura, S. Saitoh and A. Syarif, *Representations of initial heat distributions by means of their heat distributions as functions of time*, Inverse Problems **15** (1999), 1255-1261.

[24] Y. Privat and M. Sigalotti, *The squares of the Laplacian-Dirichlet eigenfunctions are generically linearly independent*, ESAIM Control Optim. Calc. Var. **16** (2010), no. 3, 794-805.

[25] S. Saitoh and M. Yamamoto, *Stability of Lipschitz type in determination of initial heat distribution*, J. Inequal. Appl. **1** (1997), no. 1, 73-83.

[26] V. K. Tuan, S. Saitoh and M. Saigo, *Size of support of initial heat distribution in the 1D heat equation*, Appl. Anal. **74** (2000), no. 3-4, 439-446.

[27] L. Schwartz, *Étude des sommes d'exponentielles* (French), Hermann, Paris 1959.

[28] D. V. Widder, *An introduction to transform theory*, Academic Press, New York 1971.

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