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Angelo Favini, Noboru Okazawa and Jan Prüss

Singular perturbation approach to Legendre type operators

Abstract. Let Ω be a bounded domain in $\mathbb{R}^N$ with compact smooth boundary ($N \in \mathbb{N}$). Then this paper is concerned with the nonnegative selfadjointness in $L^2(\Omega)$ of the maximal realization T_2 of N-dimensional second-order differential operators in divergence form with diffusion coefficients vanishing on the boundary $\Gamma = \partial\Omega$. The operators may be called Legendre type operators over Ω. The key to the proof is a singular perturbation argument developed in [**9**]. In particular, the resolvent of T_2 is given as the uniform limit of $(\xi + n^{-1}(-\Delta) + T_2)^{-1}$ as $n \to \infty$, for every $\xi > 0$, where $-\Delta$ is the Neumann-Laplacian in $L^2(\Omega)$. It should be noted that if $N = 1$ then $(\xi + n^{-1}(-\Delta) + T_p)^{-1}$ converges strongly to $(\xi + T_p)^{-1}$ in $L^p(I)$, where T_p is the one-dimensional analog constructed by Campiti, Metafune and Pallara [**2**].

Keywords. Selfadjointness, Legendre type operators, degeneration at the boundary, singular perturbation.

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1 - Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with compact smooth boundary $\Gamma = \partial\Omega$. Let $\phi \in C^2(\bar{\Omega})$, with $\phi > 0$ on Ω and $\phi = 0$ on Γ. Then the operator

$$(T_2 u)(x) = -\operatorname{div}[\phi(x)\nabla u(x)] = -\frac{\partial}{\partial x_j}\left[\phi(x)\frac{\partial u}{\partial x_j}\right] \tag{1.1}$$

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in $L^2(\Omega)$ with its maximal domain

$$D(T_2) = \{u \in H^1(\Omega);\ \phi\,\nabla u \in H_0^1(\Omega)^N\} \tag{1.2}$$

may be called a Legendre type operator. Since T_2 is (densely defined and) symmetric, it follows that T_2 is closable in $L^2(\Omega)$. Moreover, it is not so difficult to prove the essential selfadjointness of T_2 in $L^2(\Omega)$ (see, e.g., Okazawa [**9**, Section 5.3]).

The purpose of this short note is to show that T_2 itself is m-accretive (and hence selfadjoint) in $L^2(\Omega)$ under the following two conditions on the coefficient $\phi \in C^2(\bar{\Omega})$:

(**i**) $\phi > 0$ on Ω and $\phi = 0$ on Γ;

(**ii**) $\partial\phi/\partial\nu < 0$ on Γ, with non-positivity of the Hessian matrix of ϕ on Ω:

$$\sum_{j,k=1}^{N} \frac{\partial^2\phi}{\partial x_j \partial x_k}(x)\,\xi_j\,\xi_k \le 0 \quad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N. \tag{1.3}$$

Actually, T_2 is implicitly obtained as the uniform-resolvent limit of the approximate sequence $\{n^{-1}S + T_2\}$:

$$(T_2 + \xi)^{-1} = \lim_{n\to\infty} \left(n^{-1}S + T_2 + \xi\right)^{-1} \quad \forall\, \xi > 0, \tag{1.4}$$

with rate of convergence:

$$\|(T_2 + \xi)^{-1} - \left(n^{-1}S + T_2 + \xi\right)^{-1}\| = O(n^{-1/2}) \quad \text{as}\ \ n \to \infty,$$

where S is the so-called Neumann-Laplacian:

$$S := -\Delta \quad \text{with}\ \ D(S) := \{u \in H^2(\Omega);\ \partial u/\partial\nu = \nu\cdot\nabla u = 0\ \ \text{on}\ \ \Gamma\}$$

satisfying the domain inclusion

$$D(S) \subset D(T_2) \subset D(S^{1/2}) = H^1(\Omega).$$

In other words, the domain of degenerate elliptic operator T_2 is completely characterized as in (1.2). In view of (1.2) $n^{-1}S$ is regarded as a singular perturbation with respect to T_2 because

$$D(n^{-1}S + T_2) = D(S) \subset D(T_2) \quad \forall\, n \in \mathbb{N}$$

(see Kato [**7**] for singular perturbation after the publication of the book [**6**]). Besides, (1.4) is equivalent to the following condition:

$$\forall\, u \in D(T_2)\ \exists\, \{v_n\} \subset D(S);\ \ v_n \to u,\ (n^{-1}S + T_2)v_n \to T_2 u\ \ (n \to \infty). \tag{1.5}$$

That is, the Neumann boundary condition vanishes in the limit as $n \to \infty$.

Now let $1 < p < \infty$ ($p \neq 2$). Then it is desirable to develop the L^p-theory for

$$(T_p u)(x) := -\operatorname{div}[\phi(x)\nabla u(x)] \quad \text{with } D(T_p) := \{u \in W^{1,p}(\Omega);\ \phi\nabla u \in W_0^{1,p}(\Omega)^N\}$$

or more general degenerate elliptic operators. About twenty years ago the one-dimensional case of T_p is settled by Campiti, Metafune and Pallara [**2**]. It should be noted that some related problems are recently dealt with by Fornaro, Metafune, Pallara, Prüss and Schnaubelt in [**3**] and [**5**]. In particular, the interplay between diffusion and drift terms is discussed in these papers. The investigation in [**5**] is based on the one-dimensional case developed in [**4**].

2 - Preliminaries (abstract lemmas in Hilbert space)

Let X be a Hilbert space.

Lemma 2.1 ([**9**, Theorems 4.1 and 7.1]). *Let T be a linear accretive operator in X. Let S be a nonnegative and selfadjoint operator in X. Assume that $D(S) \subset D(T)$ and there exist nonnegative constants α and β such that*

$$\operatorname{Re}(Tu, Su) \geq -\alpha\|u\|^2 - \beta(u, Su) \quad \forall\, u \in D(S). \tag{2.1}$$

Then one has the following assertions:

(a) *for every $n \in \mathbb{N}$, $n^{-1}S + T$ is m-accretive in X.*

(b) *$\tilde{T}$, the closure of T, is m-accretive in X and $D(S)$ is a core for $\tilde{T}$. Furthermore, the resolvent of $\tilde{T}$ is given by the limit of that of $n^{-1}S + T$:*

$$(\tilde{T} + \xi)^{-1} = \operatorname*{s-lim}_{n\to\infty}\Big(\frac{1}{n}S + T + \xi\Big)^{-1} \quad \forall\, \xi > 0. \tag{2.2}$$

Lemma 2.2. *The condition* (2.1) *yields that*

$$\operatorname{Re}(Tu, (\alpha^{1/2} + S)u) \geq -(\alpha^{1/2} + \beta)(u, (\alpha^{1/2} + S)u) \quad \forall\, u \in D(S). \tag{2.3}$$

Let $v \in D(S^{1/2}) = D((\alpha^{1/2} + S)^{1/2})$ and $\xi > \alpha^{1/2} + \beta$. Then (2.3) *implies that $D(S^{1/2})$ is invariant under $(\tilde{T} + \xi)^{-1}$, with*

$$\|(\alpha^{1/2} + S)^{1/2}(\tilde{T} + \xi)^{-1}v\| \leq (\xi - (\alpha^{1/2} + \beta))^{-1}\|(\alpha^{1/2} + S)^{1/2}v\|, \tag{2.4}$$

where $Q^{1/2}$ denotes the square root of Q.

Note that the inequality (2.4) with $\alpha = 0$ is contained in [**9**].

Lemma 2.3 ([**9**, Theorem 4.4]). *Let T and S be the same as defined in* Lemma 2.1, *satisfying* (2.1). *Assume further that $D(\tilde{T}) \subset D(S^{1/2})$. Then, in addition to the assertions of* Lemma 2.1, *for every $\zeta \in \mathbb{C}$ with* $\mathrm{Re}\,\zeta > 0$ *there is a constant* $c(\zeta) > 0$ *such that*

$$\left\|(\tilde{T} + \zeta)^{-1} - (n^{-1}S + T + \zeta)^{-1}\right\| \le c(\zeta)/\sqrt{n}, \quad n \in \mathbb{N};$$

hence the compactness of $(n^{-1}S + T + \zeta)^{-1}$ implies that of $(\tilde{T} + \zeta)^{-1}$.

Applications of these lemmas are stated in the next section. For other restricted applications of these lemmas (in which $S = T^*T$) see [8] and [**11**].

3 - Legendre type operators in $L^2(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^N$ with compact smooth boundary $\Gamma = \partial\Omega$. Put $X := L^2(\Omega)$ and $Y := L^2(\Omega)^N$. Let $A := \nabla$ with $D(A) := H^1(\Omega)^N$. Then A is closed and densely defined from X to Y and its adjoint A^* is given by

$$A^* := -\operatorname{div}, \;\; D(A^*) := \{v \in Y;\; \operatorname{div} v \in X,\; \nu \cdot v = 0 \;\text{ on }\; \Gamma\},$$

where ν denotes the unit outward normal on Γ. In terms of A and A^* we obtain the so-called Neumann-Laplacian in X:

$$\begin{aligned} A^*A = -\Delta, \quad D(A^*A) &= \{u \in D(A);\; Au \in D(A^*)\} \\ &= \{u \in H^2(\Omega);\; \partial u/\partial\nu = \nu \cdot \nabla u = 0 \;\text{ on }\; \Gamma\}. \end{aligned} \tag{3.1}$$

Next let B be a bounded operator of multiplication by the square root of a non-negative function $\phi \in C^2(\bar{\Omega})$, satisfying

(**i**) $\phi(x) > 0$ on Ω and $\phi(x) = 0$ on Γ;

(**ii**) $(\partial\phi/\partial\nu)(x) = \nu(x) \cdot \nabla\phi(x) < 0$ on Γ (and hence $|\nabla\phi(x)| > 0$ on Γ); additionally, the symmetric Hessian matrix

$$D^2\phi(x) = \left(\frac{\partial^2\phi}{\partial x_j\,\partial x_k}(x)\right)_{jk}$$

of ϕ is non-positive everywhere on Ω:

$$-\sum_{j,k=1}^{N} \frac{\partial^2\phi}{\partial x_j\,\partial x_k}(x)\,\xi_j\,\xi_k \ge 0 \quad \forall\, x \in \Omega,\; \forall\, \xi \in \mathbb{R}^N. \tag{3.2}$$

EXAMPLE 3.1. (a) Put $\phi(x) := 1 - |x|^2$ over $\Omega := \{|x| < 1\}$ with $\nu(x) = |x|^{-1}x$ for $x \in \Gamma$. Then $\nabla\phi(x) = -2x$ and $\dfrac{\partial^2\phi}{\partial x_j \partial x_k}(x) = -2\,\delta_{jk}$ (in terms of the Kronecker delta). Hence $\dfrac{\partial\phi}{\partial\nu}(x) = -2|x| = -2$ on Γ and, moreover, we have

$$-\sum_{j,k=1}^{N} \frac{\partial^2\phi}{\partial x_j\,\partial x_k}(x)\,\xi_j\,\xi_k = 2\,|\xi|^2 \geq 0 \quad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N$$

with $\operatorname{tr}(D^2\phi)(x) = \Delta\phi(x) = -2N < 0$ on Ω.

(b) Put $\phi(x) := \cos\Big(\dfrac{\pi}{2}|x|^2\Big)$ over $\Omega := \{|x| < 1\}$ (Ω is the same as in (a)). Then

$$\nabla\phi(x) = -\pi x \sin\Big(\frac{\pi}{2}|x|^2\Big) \text{ and}$$

$$\frac{\partial^2\phi}{\partial x_j \partial x_k}(x) = -\pi\,\delta_{jk} \sin\Big(\frac{\pi}{2}|x|^2\Big) - \pi^2 x_j x_k \cos\Big(\frac{\pi}{2}|x|^2\Big).$$

Hence $\dfrac{\partial\phi}{\partial\nu}(x) = -\pi|x| \sin\Big(\dfrac{\pi}{2}|x|^2\Big) = -\pi$ on Γ and, moreover, we have

$$-\sum_{j,k=1}^{N} \frac{\partial^2\phi}{\partial x_j\,\partial x_k}(x)\,\xi_j\,\xi_k = \pi|\xi|^2 \sin\Big(\frac{\pi}{2}|x|^2\Big) + \pi^2(x\cdot\xi)^2 \cos\Big(\frac{\pi}{2}|x|^2\Big) \geq 0$$

$\forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N$ with

$$\operatorname{tr}(D^2\phi)(x) = \Delta\phi(x) = -N\pi \sin\Big(\frac{\pi}{2}|x|^2\Big) - \pi^2|x|^2 \cos\Big(\frac{\pi}{2}|x|^2\Big) \leq 0 \text{ on } \Omega.$$

Then we consider

$$(BAu)(x) = \sqrt{\phi(x)}\,\nabla u(x),\quad u \in D(A),\ \text{ with } (BA)^* = A^*B^* = A^*B$$

and hence **Legendre type operators** in $L^2(\Omega)$ are defined in terms of A, A^* and B:

$$((BA)^*BA)u(x) = (A^*B^2A)u(x) = -\operatorname{div}\big(\phi(x)\nabla u(x)\big), \tag{3.3}$$

$$D(A^*B^2A) = \{u \in H^1(\Omega);\ \phi\nabla u \in H^1_0(\Omega)\}.$$

First we show that A^*B^2A is closed under conditions (**i**) and (**ii**).

LEMMA 3.1. *Let $T_2 := A^*B^2A$ be the Legendre type operator in $L^2(\Omega)$ as defined above.*

(a) *Under condition* (**i**) *one has*

$$\big\|\sqrt{\phi}|\nabla u|\big\|^2_{L^2} = \|BAu\|^2_{L^2} = (T_2u, u)_{L^2} \leq \|T_2u\|_{L^2}\|u\|_{L^2} \quad \forall\, u \in D(T_2). \tag{3.4}$$

(b) *Put* $(T_2)_{\mathrm{II}}u = -\phi\Delta u$ *and* $(T_2)_{\mathrm{I}}u = -\nabla\phi\cdot\nabla u$. *Then under condition* (**i**) *and* (**ii**) *one has*

$$\begin{aligned}(3.5)\qquad \|(T_2)_{\mathrm{I}}u\|_{L^2} &\le \big\||\nabla\phi|\cdot|\nabla u|\big\|_{L^2}\\ &\le 2\|T_2u\|_{L^2} + \sqrt{M}(T_2u,u)_{L^2}^{1/2} \quad \forall\, u\in D(T_2);\end{aligned}$$

hence T_2 *has the separation property*:

$$\|(T_2)_{\mathrm{I}}u\|_{L^2} + \|(T_2)_{\mathrm{II}}u\|_{L^2} \le 5\|T_2u\|_{L^2} + 2\sqrt{M}(T_2u,u)_{L^2}^{1/2},$$

where $M := \|\Delta\phi\|_{L^\infty} = \max\{|\Delta\phi(x)|;\ x\in\bar{\Omega}\}$.

(c) *The inequalities in* (a) *and* (b) *imply* (*under conditions* (**i**) *and* (**ii**)) that

$$(3.6)\qquad \|\nabla u\|_{L^2} \le c_1\|T_2u\|_{L^2} + c_2\|u\|_{L^2} \quad \forall\, u\in D(T)$$

and hence $T_2 = A^*B^2A$ *is closed in* $L^2(\Omega)$.

Proof. (a) Let $u\in D(T_2)$. Then under condition (**i**) we have

$$\begin{aligned}(3.7)\qquad (T_2u,u)_{L^2} &= -\int_\Omega \overline{u(x)}\,\mathrm{div}\big(\phi(x)\nabla u(x)\big)\,dx\\ &= -\int_\Gamma \phi(x)\frac{\partial u}{\partial\nu}\overline{u(x)}\,dS + \int_\Omega \phi(x)|\nabla u(x)|^2\,dx\\ &= \big\|\sqrt{\phi}|\nabla u|\big\|_{L^2}^2.\end{aligned}$$

(b) Here we shall use the symbol $(T_2)_{\mathrm{I}}$ as in the statement. Then under condition (**i**) we have

$$\begin{aligned}(T_2u,(T_2)_{\mathrm{I}}u)_{L^2} &= \int_\Omega \mathrm{div}(\phi(x)\nabla u(x))\big(\nabla\phi(x)\cdot\overline{\nabla u(x)}\big)\,dx\\ &= -\int_\Omega \phi(x)\sum_{j,k=1}^N \frac{\partial\phi}{\partial x_k}\frac{\partial u}{\partial x_j}\overline{\frac{\partial^2 u}{\partial x_j\partial x_k}}\,dx - \int_\Omega \phi(x)\sum_{j,k=1}^N \frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}}\,dx.\end{aligned}$$

Applying (3.2) in condition (**ii**) we see that

$$\begin{aligned}\mathrm{Re}\,(T_2u,(T_2)_{\mathrm{I}}u)_{L^2} &\ge -\frac12\int_\Omega \big(\phi(x)\nabla\phi(x)\big)\cdot\nabla\big(|\nabla u(x)|^2\big)\,dx\\ &= \frac12\int_\Omega |\nabla\phi(x)|^2|\nabla u(x)|^2\,dx + \frac12\int_\Omega \phi(x)(\Delta\phi)(x)|\nabla u(x)|^2\,dx.\end{aligned}$$

Now let M be as in (3.5). Then we see from (3.7) that

$$\Big|\int_\Omega \phi(x)(\Delta\phi)(x)|\nabla u(x)|^2\,dx\Big| \le M(T_2u,u)_{L^2}.$$

Hence we obtain

$$\begin{aligned}\big\||\nabla\phi|\cdot|\nabla u|\big\|_{L^2}^2 &\le 2\mathrm{Re}\,(T_2u,(T_2)_{\mathrm{I}}u)_{L^2} + M(T_2u,u)_{L^2}\\ &\le 2\|T_2u\|_{L^2}\big\||\nabla\phi|\cdot|\nabla u|\big\|_{L^2} + M(T_2u,u)_{L^2}.\end{aligned}$$

Setting $t := \big\||\nabla\phi|\cdot|\nabla u|\big\|_{L^2}$, we have a quadratic inequality with respect to t. Completing the square, we obtain

$$(t-\|T_2u\|_{L^2})^2 \le \|T_2u\|_{L^2}^2 + M(T_2u,u)_{L^2}.$$

It turns out that $t \le 2\|T_2u\|_{L^2} + \sqrt{M}(T_2u,u)_{L^2}^{1/2}$. This is nothing but (3.5).

(c) Suppose that conditions (**i**) and (**ii**) are satisfied. Then, since $\bar\Omega$ is compact, it follows that $\sqrt{\phi}$ and $|\nabla\phi|^2$ are uniformly continuous on $\bar\Omega$. This implies that for (small) $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{cases} \sqrt{\phi(x)} \ge \varepsilon \ \text{ for } \ x \ \text{ with } \ \mathrm{dist}\,(x,\Gamma) \ge \delta,\\ |\nabla\phi(x)|^2 \ge \varepsilon \ \text{ for } \ x \ \text{ with } \ \mathrm{dist}\,(x,\Gamma) \le \delta.\end{cases} \tag{3.8}$$

Thus we see from (3.4) and (3.5) that

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2\,dx &\le \frac{1}{\varepsilon}\int_{\mathrm{dist}(x,\Gamma)\le\delta} |\nabla\phi(x)|^2|\nabla u(x)|^2\,dx + \frac{1}{\varepsilon}\int_{\mathrm{dist}(x,\Gamma)\ge\delta}\sqrt{\phi(x)}|\nabla u(x)|^2\,dx\\ &\le \frac{1}{\varepsilon}\int_\Omega |\nabla\phi(x)|^2|\nabla u(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega \sqrt{\phi(x)}|\nabla u(x)|^2\,dx\\ &\le \frac{1}{\varepsilon}\big(8\|T_2u\|_{L^2}^2 + (2M+1)(T_2u,u)_{L^2}\big).\end{aligned}$$

This yields that $\|\nabla u\|_{L^2} \le \varepsilon^{-1/2}\big(2\sqrt{2}\|T_2u\|_{L^2} + \sqrt{2M+1}\,(T_2u,u)_{L^2}^{1/2}\big)$. □

Now we are in a position to state the main theorem in this section.

Theorem 3.2. *Let A and B be as stated above. Then there exists a non-negative constant β such that*

$$\mathrm{Re}\,((A^*B^2A)u, A^*Au)_{L^2} \ge -\beta\,(u,A^*Au)_{L^2} \quad \forall\, u \in D(A^*A). \tag{3.9}$$

Therefore the Legendre type operator A^*B^2A is m-accretive (and selfadjoint) in X and $D(A^*A)$ is a core for A^*B^2A, with inclusion relation

$$D(A^*A) \subset D(A^*B^2A) \subset D(A) = D((A^*A)^{1/2}).$$

Here $D(A)$ is invariant under $(A^*B^2A+\xi)^{-1}$ $(\xi > 0)$. Consequently, one has

$$(A^*B^2A+\xi)^{-1} = \lim_{n\to\infty}\Big(\frac{1}{n}A^*A + A^*B^2A+\xi\Big)^{-1} \quad \forall\, \xi > 0,$$

where the convergence is uniform in the sense of operator topology.

Proof. Reminding that A^*B^2A is closed (see Lemma 3.1), we can apply Lemmas 2.1 – 2.3 with $T := A^*B^2A$ and $S := A^*A$, respectively. It remains to prove (3.9). Put

$$\left\{\begin{aligned} D(C) &:= D(A^*) = \{v\in Y;\ \operatorname{div} v\in X,\ \nu\cdot v = 0 \ \text{ on } \ \Gamma\},\\ C &:= A^*B^2 - B^2A^*. \end{aligned}\right. \tag{3.10}$$

Then, since B is a bounded symmetric operator on X, it follows that

$$\begin{aligned} \operatorname{Re}((A^*B^2A)u, A^*Au)_{L^2} &= \|B(A^*A)u\|_{L^2}^2 + \operatorname{Re}(CAu, A^*Au)_{L^2}\\ &\ge \operatorname{Re}(CAu, A^*Au)_{L^2}. \end{aligned} \tag{3.11}$$

Noting that $(Cv)(x) = -(\nabla\phi(x))\cdot v(x)$ for $v := Au \in D(A^*)$, we see from the boundary condition $\partial u/\partial\nu = 0$ on Γ that

$$\begin{aligned} \operatorname{Re}(CAu, A^*Au)_{L^2} &= \operatorname{Re}\int_\Omega \big((\nabla\phi(x))\cdot\nabla u(x)\big)\operatorname{div}(\nabla u(x))\,dx \\ &= -\operatorname{Re}\int_\Omega \sum_{j,k=1}^N \frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}}\,dx - \operatorname{Re}\int_\Omega \sum_{j,k=1}^N \frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial \phi}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}}\,dx. \end{aligned} \tag{3.12}$$

Applying (3.2) in condition (**ii**) to the first term on the right-hand side, we have

$$\begin{aligned} \operatorname{Re}(CAu, A^*Au)_{L^2} &\ge -\frac{1}{2}\int_\Omega \nabla\phi(x)\cdot\nabla\big(|\nabla u(x)|^2\big)\,dx \\ &= -\frac{1}{2}\int_\Gamma \frac{\partial\phi}{\partial\nu}(x)|\nabla u(x)|^2\,dS + \frac{1}{2}\int_\Omega |\nabla u(x)|^2\Delta\phi(x)\,dx. \end{aligned}$$

Noting that $(\partial\phi/\partial\nu)(x) < 0$ on Γ as assumed in condition (**ii**), we obtain

$$\operatorname{Re}(CAu, A^*Au)_{L^2} \ge -(M/2)\|Au\|_{L^2}^2 = -(M/2)(u, A^*Au)_{L^2} \quad \forall\, u\in D(A^*A),$$

where M is the same as in Lemma 3.1. This implies by (3.11) that (3.9) with $\beta := M/2$ is satisfied. □

4 - Concluding remarks

Finally, we want to mention the strategy to an L^p-theory $(1 < p < \infty)$.

Step 1. Assume that conditions (**i**), (**ii**) in Section 3 are satisfied. For $p \in (1, \infty)$ define Legendre type operators in $L^p(\Omega)$:

$$(T_p u)(x) := -\operatorname{div}\big(\phi(x)\nabla u(x)\big), \quad D(T_p) := \{u \in W^{1,p}(\Omega);\ \phi\nabla u \in W_0^{1,p}(\Omega)^N\}.$$

Then T_p is closed in $L^p(\Omega)$. In fact, one has

$$\|\nabla u\|_{L^p} \le c\|T_p u\|_{L^p} \quad \forall\, u \in D(T_p).$$

Step 2. Let A and B be as stated in Theorem 3.2. Put $T_2 = A^*B^2A$ and $S_2 = A^*A$. Then for every $\xi > 0$, $(\xi + S_2 + T_2)^{-1}$ maps $H^k(\Omega) \subset C(\bar{\Omega})$ to $C^2(\bar{\Omega})$ if k is an integer larger than $N/2$ (see [**1**, Section IX.6]). Since $H^k(\Omega)$ is dense in $L^2(\Omega)$,

$$D_0 := \{u \in C^2(\bar{\Omega});\ \partial u/\partial v = 0 \text{ on } \Gamma\} = (\xi + S_2 + T_2)^{-1}H^k(\Omega) \tag{4.1}$$

forms a common core for $n^{-1}S_2 + T_2$ $(n \in \mathbb{N})$.

Step 3. Let $u_n := (\xi + n^{-1}S_2 + T_2)^{-1}v$ for $v \in H^k(\Omega)$ and $\xi > 0$. Then for the sequence $\{u_n\}$ in $D_0 \subset W^{2,p}(\Omega)$ one has the estimate

$$\|u_n - u_m\|_{L^p} \le \text{const.}\Big(\frac{1}{n} + \frac{1}{m}\Big)^{1/2}\big(\|\nabla u_n\|_{L^p}^2 + \|\nabla u_m\|_{L^p}^2\big)^{1/2}.$$

Step 4. Consider $u_n = (\xi + n^{-1}S_2 + T_2)^{-1}v$ for $v \in H^k(\Omega) \subset C(\bar{\Omega})$ and $\xi > M/p'$, where $M := \|\Delta\phi\|_{L^\infty}$ and $(p')^{-1} + p^{-1} = 1$. Then

$$\|\nabla u_n\|_{L^p} \le (\xi - M/p')^{-1}\|\nabla v\|_{L^p}. \tag{4.2}$$

Hence there exists $u \in W^{1,p}(\Omega)$ such that

$$u = \lim_{n\to\infty} u_n \text{ in } L^p(\Omega), \qquad \nabla u = \text{w-}\lim_{n\to\infty} \nabla u_n \text{ in } L^p(\Omega)^N,$$

with the property $\|\nabla u\|_{L^p} \le (\xi - M/p')^{-1}\|\nabla v\|_{L^p}$.

Step 5. Let $(\xi + n^{-1}S_2 + T_2)u_n = v \in H^k(\Omega)$ as in Step 4. Then $v_n := -(n^{-1} + \phi)\Delta u_n$ is bounded in $L^p(\Omega)$. Hence $\{v_n\}$ converges weakly to $-\phi\Delta u \in L^p(\Omega)$, satisfying

$$-\phi\Delta u - \nabla\phi\cdot\nabla u + \xi u = v.$$

That is, $u \in D(T_p)$ and $(T_p u)(x) = -\operatorname{div}\big(\phi(x)\nabla u(x)\big) = v(x) - \xi u(x)$ which implies that $R(T_p + \xi) \supset H^k(\Omega)$, with

$$u = (\xi + T_p)^{-1}v = \lim_{n\to\infty}(\xi + n^{-1}S_2 + T_2)^{-1}v \quad \forall\, v \in H^k(\Omega).$$

Since T_p is closed, T_p is m-accretive in $L^p(\Omega)$ (actually, m-sectorial as shown in [**10**]).

Up to now the above-mentioned program works well only if $N = 1$. In fact, we can derive (4.2) only in the one-dimensional case. We can explain this within a half page.

For $u \in C^2(\bar{I})$ we consider the p-Laplacian $\Delta_p : L^p(I) \to L^{p'}(I)$:

$$(\Delta_p u)(x) = \left[|u'(x)|^{p-2}u'(x)\right]'$$
$$= |u'(x)|^{p-2}u''(x) + (p-2)u'(x)|u'(x)|^{p-4}\,\mathrm{Re}\left\{\overline{u''(x)}u'(x)\right\} \in C(\bar{I}).$$

In this one-dimensional case we have simple relations:

$$\mathrm{Re}\left\{u''(x)\overline{(\Delta_p u)(x)}\right\} \tag{4.3}$$
$$= |u'(x)|^{p-4}\left[|u'(x)|^2|u''(x)|^2 + (p-2)\left|\mathrm{Re}\left\{u''(x)\overline{u'(x)}\right\}\right|^2\right]$$
$$\geq (p-1)|u'(x)|^{p-4}\left|\mathrm{Re}\left\{u''(x)\overline{u'(x)}\right\}\right|^2 \geq 0,$$

$$\mathrm{Re}\left\{u'(x)\overline{(\Delta_p u)(x)}\right\} \tag{4.4}$$
$$= (p-1)|u'(x)|^{p-2}\,\mathrm{Re}\left\{u''(x)\overline{u'(x)}\right\} = \frac{1}{p'}\frac{d}{dx}\left(|u'(x)|^p\right).$$

Let $u \in D_0$ be as in (4.1). Making the inner product of

$$(n^{-1}S_2 + T_2)u = -(\phi(x) + n^{-1})u''(x) - \phi'(x)u'(x) \in C(\bar{I})$$

with $-\Delta_p u \in C(\bar{I}) \subset L^2(I)$, we see from (4.3) and (4.4) that

$$\mathrm{Re}\left((n^{-1}S_2 + T_2)u, (-\Delta_p)u\right)_{L^2} \geq -\frac{1}{p'}\int_0^1 \phi''(x)|u'(x)|^p\,dx. \tag{4.5}$$

Now let $v \in W^{1,p}(I) \subset C(\bar{I})$. Then $u_n = (\xi + n^{-1}S_2 + T_2)^{-1}v \in D_0$. Therefore it follows from (4.5) that $\mathrm{Re}\left((n^{-1}S_2 + T_2)u_n, (-\Delta_p)u_n\right)_{L^2} \geq -(M/p')\|u_n'\|_{L^p}^p$, where $M = \|\phi''\|_{L^\infty}$. So we can obtain (4.2) by noting further that

$$\mathrm{Re}\,(v - \xi u_n, (-\Delta_p)u_n)_{L^2} \leq \|v'\|_{L^p}\|u_n'\|_{L^p}^{p-1} - \xi\|u_n'\|_{L^p}^p.$$

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Angelo Favini
Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta S. Donato 5
40126 Bologna, Italy
e-mail: angelo.favini@unibo.it

Noboru Okazawa
Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku
Tokyo 162-8601, Japan
e-mail: okazawa@ma.kagu.tus.ac.jp

Jan Prüss
Fachbereich Mathematik und Informatik
Institut für Analysis
Martin-Luther Universtät Halle-Wittenberg
Theodor-Lieser-Strasse 506120
D-06099 Halle (Saale), Germany
e-mail: jan.pruess@mathematik.uni-halle.de