Riv. Mat. Univ. Parma, Vol. **7** (2016), 321-339

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A new age-dependent population model with diffusion and gestation processes

Abstract. In this paper, we introduce a new age-structured population model with diffusion and gestation processes and make a complete study of the qualitative properties of its solutions. The model is in the spirit of a model introduced in [**13, 15**] and studied in [**10**]. We aim here to correct some weakness of the model that was pointed out in [**10**].

Keywords. Population dynamics, coupled equations, qualitative properties, asymptotic behaviour.

Mathematics Subject Classification (2010): 35Q92, 35B40, 35B51.

Contents

Received: January 15, 2016; accepted: June 27, 2016.

1 - Introduction

The goal of this paper is to introduce a new age-structured population model with diffusion and gestation processes and to make a complete study of the qualitative properties of its solutions. The model we suggest here is inspired by a previous one that has been introduced in [**13, 15**] and studied in [**10, 13, 15**].

Classically, most of the population models proposed before (see for example [**5, 8, 16, 17, 18**] and the references therein) consider only a delay due to pregnancy but they do not take into account that a lot of things can happen during the time of gestation (for example the pregnant individuals may move, die or bear before the fixed time). Those "classical" models of population dynamics with diffusion may in general be viewed as *delay equations* and studied using this theory.

The model introduced in [**13, 15**] is more realistic since it takes into account for example the fact that, in general, pregnant individuals move during the period of gestation and therefore can bear in a place different from that they were fecundated. It is a system of two partial differential equations whose unknowns are the size of the total population and the size of the population of individuals in gestation. Of course these two populations are connected, and the novelty of the present paper will be precisely a new coupling relation between these populations. The model introduced in [**13, 15**] has been studied in several papers, with different assumptions and methods:

- using the theory of delay equations with nonautonomous past, see [**13, 15**] and also [**9, 11, 12**];
- in [**10**], we completed the results of [**13, 15**], providing a complete analysis of the following questions: existence, positivity, asymptotic behavior of the solutions,

under rather natural biological assumptions (and moreover with direct methods: Fourier decomposition, maximum principles).

However, in [**10**] we also pointed out some weakness of the model. Indeed we investigated another natural qualitative property that was not studied before in [**13, 15**] : is the number of individuals in gestation less than the total number of individuals (assuming of course that this property is satisfied at the initial time)? Surprisingly, the answer was not obvious, the result was even false in general, and the natural and expected property was true when the initial data and parameters satisfy some additional, quite strong, assumption meaning, roughly speaking, that the property is true (and thus the model is valid) if the population of pregnant individuals is "small" with respect to the total population. The goal of the present work is to introduce a new model that improves the previous one in the sense that the property is true without extra assumption.

In Section 2 we introduce the new model we are interested in. In Section 3, we give our main results on it. Section 4 contains some comments and comparison with other connected models. Section 5 contains the proofs of our main results. Section 6 is devoted to some remarks and perspectives.

2 - The new population model

We consider a spatially distributed population where individuals are characterized by their position. (No sex or age differences are allowed). A special attention is paid to the mechanism of pregnancy, that leads to a delay in the replacement of the population. The model wants to take into account the events that may happen during the gestation : the pregnant individuals may move, die or bear before the term. For this reason, within the total population, we distinguish pregnant individuals and we refer to them by considering their 'age of gestation' a, ranging in $[0, r]$ where $r > 0$ is fixed.

In general, individuals are supposed to die at a given death rate d whereas the pregnant individuals are supposed to die at a death rate d', with $d' \geq d$. Pregnant individuals are also supposed to bear according to a rate $b = b(a)$. Moreover, we assume that the dispersal of the population through the environment is realized by the Laplace operator.

Summing up, let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded with smooth boundary, and denote, respectively, by $u(t, x)$ and $v(t, a, x)$ the total population at time t and position x and the subpopulation of pregnant individuals at time t and position x with a time of gestation a; then the dynamics of the two populations is governed by the following equations :

$$
\begin{cases} u_t(t,x) - \Delta u(t,x) + du(t,x) = \displaystyle\int_0^r b(a)v(t,a,x)\,da, \\ v_t(t,a,x) + v_a(t,a,x) - \Delta v(t,a,x) + d'v(t,a,x) = -b(a)v(t,a,x), \end{cases} \tag{1}
$$

where $(t,a,x) \in \mathbb{R}_+^* \times (0,r) \times \Omega$. In the second equation, the term $b(a)v(t,a,x)$ represents the density of pregnant individuals that bear at time t, at place x and after a time of gestation denoted by a. The minus sign comes from the fact that the pregnant individual that bears is no more pregnant, and hence this term behaves like a mortality one. In the first equation, the coupling term $\int_0^r b(a)v(t,a,x)\,da$ represents the related increase of the total population (i.e. the newborn individuals) at time t and place x.

We consider here the following homogeneous Dirichlet boundary conditions

$$
\begin{cases} u(t,x)_{|\partial\Omega} = 0, & t>0,\ x\in\partial\Omega, \\ v(t,a,x)_{|\partial\Omega} = 0, & t>0,\ (a,x)\in(0,r)\times\partial\Omega, \end{cases} \tag{2}
$$

which means that no individual reaches the borderline. (Notice that we also could consider homogeneous Neumann boundary conditions which means that no individual crosses the borderline). Moreover the initial population at time $t=0$ is given by

$$
\begin{cases} u(0,x) = u_0(x) \geq 0, & x\in\Omega, \\ v(0,a,x) = v_0(a,x) \geq 0, & (a,x)\in(0,r)\times\Omega. \end{cases} \tag{3}
$$

Finally, we need to describe how the number of fecundated individuals depends on the total population, which gives another coupling condition. In [**10, 13, 15**], the assumption is that the number of fecundated individuals at time t and place x is a fixed proportion of the whole population present at that place and time :

$$
v(t,0,x) = f_0 u(t,x), \qquad t>0,\ x\in\Omega. \tag{4}
$$

In the present paper, we are going to assume that the population $v(t,0,x)$ that is fecundated at time t and place x is a fraction of the total population that *has not yet been fecuntaded* at time t and place x. To that purpose, we consider

$$
V(t,x) := \int_0^r v(t,a,x)\,da, \tag{5}
$$

which represents the total population of pregnant individuals at time t and place x, and we replace (4) by

$$
v(t,0,x) = f_0(u(t,x) - V(t,x)), \qquad t>0,\ x\in\Omega, \tag{6}
$$

which means that the population that becomes pregnant at time t and place x, which is $v(t,0,x)$, is a fraction of the total population that has not yet been fecuntaded at time t and place x, which is $u(t,x) - V(t,x)$. This lead us to consider the following new model :

$$
\text{(7)} \qquad \begin{cases} u_t(t,x) - \Delta u(t,x) + du(t,x) = \displaystyle\int_0^r b(a)v(t,a,x)\,da, \\ v_t(t,a,x) + v_a(t,a,x) - \Delta v(t,a,x) + d'v(t,a,x) = -\,b(a)v(t,a,x), \\ v(t,0,x) = f_0\left(u(t,x) - \displaystyle\int_0^r v(t,a,x)\,da\right), \\ u(t,x)_{|\partial\Omega} = 0, \qquad v(t,a,x)_{|\partial\Omega} = 0, \\ u(0,x) = u_0(x), \qquad v(0,a,x) = v_0(a,x), \end{cases}
$$

where $(t,a,x) \in \mathbb{R}_+^* \times (0,r) \times \Omega$. Note that there is a nonlocal local term in the first equation and now also in the (coupling) boundary condition.

3 - Main results

3.1 - *Assumptions and well-posedness*

Let us make the following assumptions:

$$
\text{(8)} \qquad r > 0, \quad d' \geq d > 0 \;\text{ and }\; f_0 > 0,
$$

$$
\text{(9)} \qquad b \in L^\infty_{\mathrm{loc}}([0,r)) \text{ such that } b \geq 0, \quad b \text{ nondecreasing}, \quad \int_0^r b(a)\,da = +\infty,
$$

$$
\text{(10)} \qquad \begin{cases} u_0 \in L^2(\Omega), \quad u_0 \geq 0, \\ v_0 \in L^2((0,r)\times\Omega), \quad v_0 \geq 0 \text{ such that } \sqrt{b}v_0 \in L^2((0,r)\times\Omega). \end{cases}
$$

In the following, we set

$$
\forall s \in (0,r), \quad \tilde{b}(s) := \int_0^s b(\sigma)\,d\sigma.
$$

For any characteristic line

$$
S := \{(t,a) \in (0,T)\times(0,r) \mid a - t = a_0 - t_0\} = \{(t_0+s, a_0+s) \mid s \in (0, r-a_0)\},
$$

with $(t_0,a_0) \in (0,T)\times\{0\} \cup \{0\}\times(0,r)$, we denote by $W^{1,1}(S;L^2(\Omega))$ the space of functions $v : S \to L^2(\Omega)$ such that $v(t_0+\cdot, a_0+\cdot) : (0,r-a_0) \to L^2(\Omega)$ is in $W^{1,1}((0,r-a_0);L^2(\Omega))$. Then we have the following

Theorem 3.1 (**Well-posedness).** *Assume* (8), (9), (10). *Then for any* $T > 0$, *problem* (7) *has a unique solution* (u, v) *on* $(0, T)$ *such that*

$$u \in \mathcal{C}([0,T]; L^2(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^2_{\rm loc}(0,T; H^2(\Omega)), \tag{11}$$

$$v \in \mathcal{C}(\bar S; L^2(\Omega)) \cap W^{1,1}(S; L^2(\Omega)) \cap L^2(S; H_0^1(\Omega)) \cap L^2_{\rm loc}(S; H^2(\Omega)), \tag{12}$$

for almost any characteristic line S *of the equation* $a - t = a_0 - t_0$,

$$\sqrt{b}v \in L^2((0,T) \times (0,r) \times \Omega) \quad \text{and} \quad \int_0^r b(a) v(\cdot, a, \cdot)\, da \in L^2((0,T) \times \Omega). \tag{13}$$

Moreover,

$$\forall\, t, \quad v(t - \varepsilon, r - \varepsilon, \cdot) \to 0 \;\; in \;\; L^2(\Omega) \text{ as } \varepsilon \to 0. \tag{14}$$

3.2 - *Qualitative properties*

Theorem 3.2 (**Qualitative properties).** *Assume* (8), (9), (10) *are satisfied.*

a) *Assume also that*

$$V_0(x) \le u_0(x) \;\; in \;\; \Omega. \tag{15}$$

Then the solution (u, v) *of* (7) *satisfies:*

$$u \ge 0 \;\; in \;\; (0,T) \times \Omega \;\; and \;\; v \ge 0 \;\; in \;\; (0,T) \times (0,r) \times \Omega, \tag{16}$$

and

$$\forall\, t \ge 0, \quad V(t, \cdot) \le u(t, \cdot) \;\; in \;\; \Omega. \tag{17}$$

b) *Assume that there is some* $\theta \in [0, 1)$ *such that*

$$V_0(x) \le (1 - \theta) u_0(x) \;\; in \;\; \Omega \qquad and \qquad d' - d \ge f_0 \, \frac{\theta}{1-\theta} \, . \tag{18}$$

Then the solution (u, v) *of* (7) *satisfies* (16) *and*

$$\forall\, t \ge 0, \quad V(t, \cdot) \le (1 - \theta) u(t, \cdot) \;\; in \;\; \Omega. \tag{19}$$

3.3 - *Asymptotic behaviour*

Finally we set

$$\begin{cases} \hat d = d + f_0, \quad \hat d' = \dfrac{1}{2}(d + d'), \\ \hat f_0 = 2 f_0, \qquad \hat b = b + \dfrac{1}{2}(d' - d). \end{cases} \tag{20}$$

We also denote by λ_0 the smallest eigenvalue of the Laplace operator with Dirichlet boundary conditions and

$$\hat{R}_0 := \hat{f}_0 \int_0^r \hat{b}(s) e^{-(\tilde{\hat{b}}(s) + (\hat{d}' + \lambda_0)s)}\, ds \tag{21}$$

where

$$\forall s \in (0, r), \quad \tilde{\hat{b}}(s) := \int_0^s \hat{b}(\sigma)\, d\sigma.$$

Then we prove the following

Theorem 3.3 (**Asymptotic behaviour).** *Assume* (8), (9), (10). *Assume that* $(u_0, v_0) \neq (0, 0)$ (*otherwise the solution is identically equal to zero*). *Then the solution of* (7) *satisfies*

- **(i)** *if* $\hat{R}_0 < \hat{d} + \lambda_0$, *then* u *goes to zero exponentially fast in* $L^2(\Omega)$ *as* $t \to \infty$, *and* v *goes to zero exponentially fast in* $L^2((0, r) \times \Omega)$;
- **(ii)** *if* $\hat{R}_0 = \hat{d} + \lambda_0$, *then* u *converges exponentially fast* (*in* $L^2(\Omega)$) *to some stationary state* $u^* \in L^2(\Omega)$ *as* $t \to \infty$, *and* v *goes to some stationary state* $v^* \in L^2((0, r) \times \Omega)$;
- **(iii)** *if* $\hat{R}_0 > \hat{d} + \lambda_0$, *then* u *goes exponentially fast to infinity in* $L^2(\Omega)$*-norm as* $t \to \infty$, *and* v *goes exponentially fast to infinity in* $L^2((0, r) \times \Omega)$*-norm.*

4 - Comparison with the existing literature

In this section, we compare the new model (7) to the existing literature. First let us recall the results obtained in [**10**] concerning the model

$$\begin{cases} u_t(t, x) - \Delta u(t, x) + du(t, x) = \displaystyle\int_0^r b(a) v(t, a, x)\, da, \\ v_t(t, a, x) + v_a(t, a, x) - \Delta v(t, a, x) + d' v(t, a, x) = -\, b(a) v(t, a, x), \\ v(t, 0, x) = f_0 u(t, x), \\ u(t, x)_{|\partial\Omega} = 0, \qquad v(t, a, x)_{|\partial\Omega} = 0, \\ u(0, x) = u_0(x), \qquad v(0, a, x) = v_0(a, x), \end{cases} \tag{22}$$

where $(t, a, x) \in \mathbb{R}_+^* \times (0, r) \times \Omega$. This will be useful to understand the differences with (7), and also to prove Theorems 3.1, 3.2 and 3.3.

4.1 - *Well-posedness of the model* (22)

Theorem 4.1 (Well-posedness, [**10**, Theorem 2.1, Theorem 2.2]). *Assume* (8), (9) *and* (10). *For any* $T > 0$, *problem* (22) *has a unique solution* (u, v) *on* $(0, T)$ *such that*

$$u \in \mathcal{C}([0,T]; L^2(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)) \cap L^2_{\text{loc}}(0,T; H^2(\Omega)), \tag{23}$$

$$v \in \mathcal{C}(\bar{S}; L^2(\Omega)) \cap W^{1,1}(S; L^2(\Omega)) \cap L^2(S; H^1_0(\Omega)) \cap L^2_{\text{loc}}(S; H^2(\Omega)), \tag{24}$$

for almost any characteristic line S *of equation* $a - t = a_0 - t_0$,

$$\sqrt{b}v \in L^2((0,T) \times (0,r) \times \Omega) \quad \text{and} \quad \int_0^r b(a)v(a)\, da \in L^2((0,T) \times \Omega). \tag{25}$$

Moreover,

$$\forall\, t, \quad v(t - \varepsilon, r - \varepsilon, \cdot) = 0 \;\text{ in }\; L^2(\Omega) \;\text{ as }\; \varepsilon \to 0. \tag{26}$$

Remark. Note that, in [**13**] and [**15**], b is assumed to belong to $L^\infty(0, r)$ or to $L^1(0, r)$ so that some associated delay operator is well-defined. In [**10**], we prefered to assume

$$\int_0^r b(a)\, da = +\infty. \tag{27}$$

Indeed it is a more "natural" assumption that ensures $v(\cdot, r, \cdot) = 0$, which means that r is the maximal time of gestation. But this induces several technical difficulties, and the well-posedness of (22) was proved when the initial conditions belong to some suitable weighted space (see assumption (10)).

4.2 - *Qualitative properties and weakness of the model* (22)

Theorem 4.2 (Qualitative properties, [**10**, Theorem 2.2]). *Assume* (8), (9) *and* (10). *Then the solution* (u, v) *of* (22) *satisfies:*

$$u \ge 0 \text{ a.e. on } (0,T) \times \Omega \;\text{ and }\; v \ge 0 \text{ a.e. on } (0,T) \times (0,r) \times \Omega. \tag{28}$$

Moreover, if

$$f_0 u_0(x) \int_0^r e^{-\tilde{b}(\tau)} e^{-(d'-d)\tau}\, d\tau \;\le\; u_0(x) - V_0(x) \tag{29}$$

for almost all $x \in \Omega$. *Then the solution satisfies*

$$\forall\, t \ge 0, \forall\, x \in \Omega, \quad u(t,x) \ge V(t,x). \tag{30}$$

The main novelty in [**10**] was the study of property (30), that means that the total population $u(t)$ is larger than the total population of pregnant individuals $V(t)$. Of course, in order to ensure such a property, it is necessary to assume that it is satisfied at time $t = 0$. The natural assumption in order to get (30) would simply be the following one: $u_0 \geq V_0$. Observe that our assumption (29) implies $u_0 \geq V_0$ but is unfortunately stronger. Note also that, even if the result is very natural, it does not follow easily from the equations and the strong assumption (29) is really needed to prove it. Indeed it is possible to construct a counter-example to this property (see [**10**]). Moreover, even under condition (29), the proof of (30) was not obvious since we had to compare u with V whereas we did not have a system of equations in (u, V) but in (u, v).

Hence the previous model (22) is realistic only when the pregnant individuals population is "small enough". This comes from the fact that the model (22) assumes that the number of fecundated individuals at time t and place x is a fixed proportion of the whole population present at that place and time (remember (4)). This coupling condition is natural when the pregnant individuals population is "small". On the contrary, if the pregnant individuals population is "large", this cannot be a realistic model. This is the motivation for the more realistic condition (6). And the gain is that the new model effectively corrects the weakness of the first one since property (17) is now valid without restriction on the parameters and initial data.

4.3 - *Asymptotic behaviour for the model* (22)

Finally, in [**10**], we also completely studied the asymptotic behavior of the solutions of (22): denote λ_0 the smallest eigenvalue of the Laplace operator with Dirichlet boundary conditions, φ_0 the associated eigenfunction, and

$$R_0 := f_0 \int_0^r b(s) e^{-(\tilde{b}(s)+(d'+\lambda_0)s)}\, ds.$$

Then we proved the following

Theorem 4.3 (Asymptotic behaviour, [**10**, Theorem 2.3]). *Assume* (8), (9) *and* (10). *Assume that* $(u_0, v_0) \neq (0,0)$ (*otherwise the solution is identically equal to zero*). *Then the solution of* (22) *satisfies*

(i) *if* $R_0 < d + \lambda_0$, *then u goes to zero exponentially fast as* $t \to \infty$;

(ii) *if* $R_0 = d + \lambda_0$, *then u converges exponentially fast to a stationary state;*

(iii) *if* $R_0 > d + \lambda_0$, *then u goes exponentially fast to infinity as* $t \to \infty$.

Remarks. The stationary state in case (ii) and the rates of convergence in each cases may be explicitly determined by the proof. (We refer to [**10**, Theorem 2.3] for the detailed statement of Theorem 4.3). Concerning the behavior of v, it is then easy to see that $v(t,\cdot,\cdot)$ goes exponentially fast (in $L^2((0,r)\times\Omega)$) to zero, to a stationary state or to infinity as $t\to\infty$ respectively if $R_0 < d+\lambda_0$, $R_0 = d+\lambda_0$ or $R_0 > d+\lambda_0$.

5 - Proofs of Theorems 3.1, 3.2 and 3.3

To prove Theorems 3.1, 3.2 and 3.3, we will make analogies with (22). In particular, there is a change of unknowns that allows us to transform (7) into (22). This is not sufficient to conclude, as we will explain in the following, but it helps a lot.

As in [**10**] we work under the assumption (27) on b. This is natural from a biological point of view, since it implies that the solutions of problem (22) satisfy the property (26), and will imply a similar property for the solutions of (7).

5.1 - *Connection between the problems* (7) *and* (22)

5.1.1 - From (7) to (22)

Assume (u,v) is a solution of (7) such that $v(a=r)=0$. Then, it is easy to see that

$$(\hat u,\hat v)=\left(\frac{u-V}{2},v\right) \tag{31}$$

solves a system having the form of (22). More precisely, we have

$$\begin{cases} \hat u_t(t,x)-\Delta\hat u(t,x)+\hat d\hat u(t,x)=\displaystyle\int_0^r \hat b(a)\hat v(t,a,x)\,da,\\ \hat v_t(t,a,x)+\hat v_a(t,a,x)-\Delta\hat v(t,a,x)+\hat d'\hat v(t,a,x)=-\hat b(a)\hat v(t,a,x),\\ \hat v(t,0,x)=\hat f_0\hat u(t,x),\\ \hat u(t,x)_{|\partial\Omega}=0, \qquad \hat v(t,a,x)_{|\partial\Omega}=0,\\ \hat u(0,x)=\hat u_0(x), \qquad \hat v(0,a,x)=\hat v_0(a,x), \end{cases} \tag{32}$$

with

$$\begin{cases} \hat d=d+f_0, & \hat d'=\dfrac12(d+d'),\\ \hat f_0=2f_0, & \hat b=b+\dfrac12(d'-d),\\ \hat u_0=\dfrac12(u_0-V_0), & \hat v_0=v_0. \end{cases} \tag{33}$$

5.1.2 - From (22) to (7)

Reciprocally, assume $(\hat{u}, \hat{v})$ is a solution of (32) such that $\hat{v}(a = r) = 0$. Then, it is easy to see that

$$(u, v) = \left(2\hat{u} + \int_0^r \hat{v} da, \hat{v}\right) \tag{34}$$

solves (7), with

$$\begin{cases} d = \hat{d} - \frac{1}{2}\hat{f}_0, & d' = 2\hat{d}' - \hat{d} + \frac{1}{2}\hat{f}_0, \\ f_0 = \frac{1}{2}\hat{f}_0, & b = \hat{b} - \left(\hat{d}' - \hat{d} + \frac{1}{2}\hat{f}_0\right), \\ u_0 = 2\hat{u}_0 + \int_0^r \hat{v}_0 da, & v_0 = \hat{v}_0. \end{cases} \tag{35}$$

5.2 - *Proof of Theorem* 3.1

Assume that the parameters d, d', f_0, b, u_0, v_0 satisfy (8), (9), (10) and (15). Assume that we have a solution (u, v) of (7) such that $v(a = r) = 0$. Then apply the transform (31). This transforms problem (7) into (32) with the parameters (33). The assumptions of Theorem 4.1 are all satisfied, except the hypothesis $\hat{d}' \geq \hat{d}$ that is not always satisfied. But this is not essential, since looking to the proof of Theorem 4.1 one can check that the result of Theorem 4.1 remains true without this assumption. Indeed, the existence of the solution of (32) follows from a fixed point argument: consider

$$L^2_+((0, T) \times \Omega) := \{\hat{h} \in L^2((0, T) \times \Omega) \mid \hat{h} \text{ is nonnegative}\};$$

let $\hat{h}$ be given in $L^2_+((0, T) \times \Omega)$ and consider the problem

$$\begin{cases} \hat{u}_t(t, x) - \Delta\hat{u}(t, x) + \hat{d}\hat{u}(t, x) = \int_0^r \hat{b}(a)\hat{v}(t, a, x)\, da, \\ \hat{v}_t(t, a, x) + \hat{v}_a(t, a, x) - \Delta\hat{v}(t, a, x) + \hat{d}'\hat{v}(t, a, x) = -\hat{b}(a)\hat{v}(t, a, x), \\ \hat{v}(t, 0, x) = \hat{f}_0\hat{h}(t, x), \\ \hat{u}(t, x)_{|\partial\Omega} = 0, \\ \hat{v}(t, a, x)_{|\partial\Omega} = 0, \\ \hat{u}(0, x) = \hat{u}_0(x), \\ \hat{v}(0, a, x) = \hat{v}_0(a, x). \end{cases} \tag{36}$$

From the classical results concerning the heat equation, there exists a unique solution $(\hat{u}^{(h)}, \hat{v}^{(h)})$ of (36). Moreover, the map

$$\mathcal{F} : \begin{cases} L^2_+((0,T)\times\Omega) & \to L^2_+((0,T)\times\Omega) \\ \hat{h} & \mapsto \hat{u}^{(h)}, \end{cases}$$

where $(\hat{u}^{(h)}, \hat{v}^{(h)})$ is the solution of (36) associated to $\hat{h}$, is a contraction on the space $L^2_+((0,T)\times\Omega)$, when $L^2((0,T)\times\Omega)$ is endowed with the norm

$$\forall \hat{q} \in L^2((0,T)\times\Omega), \quad \|\hat{q}\|_\lambda := \|e^{-\lambda t/2}\hat{q}\|_{L^2((0,T)\times\Omega)},$$

where the constant $\lambda > 0$ is large enough, and this does not depend on the sign of $\hat{d}' - \hat{d}$ (we refer to [**10**] for the detailed proof). Then the proof of Theorem 4.1 holds true without any assumption on the sign of $\hat{d}' - \hat{d}$, and (32) has a unique solution $(\hat{u}, \hat{v})$ on $(0,T)$ such that

$$\hat{u} \in \mathcal{C}([0,T]; L^2(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)) \cap L^2_{\text{loc}}(0,T; H^2(\Omega)),$$
$$\hat{v} \in \mathcal{C}(\bar{S}; L^2(\Omega)) \cap W^{1,1}(S; L^2(\Omega)) \cap L^2(S; H^1_0(\Omega)) \cap L^2_{\text{loc}}(S; H^2(\Omega)),$$

for almost any characteristic line S of equation $a - t = a_0 - t_0$,

$$\sqrt{\hat{b}}\hat{v} \in L^2((0,T)\times(0,r)\times\Omega) \quad \text{and} \quad \int_0^r \hat{b}(a)\hat{v}(a)\, da \in L^2((0,T)\times\Omega).$$

And (26) remains also true: thanks to (8)-(9), we have that $\hat{b} \in L^\infty_{\text{loc}}([0,r))$, $\hat{b} \geq 0$, $\hat{b}$ is nondecreasing, and $\int_0^r \hat{b}(a)\, da = +\infty$. This implies that

$$\forall\, t, \quad \hat{v}(t-\varepsilon, r-\varepsilon, \cdot) = 0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

Now, using the inverse transform (34), we obtain (u, v) that satisfy (11)-(14). This proves the existence part of Theorem 3.1. The uniqueness part follows immediately from the uniqueness of the solution of (32) (as given in Theorem 4.1), and the transforms (31) and (34). Hence Theorem 3.1 is proved. □

5.3 - *Proof of Theorem* 3.2

The proof of Theorem 3.2 is based on the same idea: consider (u, v) the solution (7), and $(\hat{u}, \hat{v})$ given by (31) the solution of (32) with the parameters (33).

a) First assume that (8), (9), (10) and (15) are satisfied. The iterative procedure described below gives that

$$\hat{f}_0 \geq 0, \hat{u}_0 \geq 0, \hat{v}_0 \geq 0 \quad \Longrightarrow \quad \hat{v} \geq 0, \hat{u} \geq 0. \tag{37}$$

Indeed, take $h_0 = 0$ and solve (36): the weak maximum principle implies that $\hat{v}^{(0)} \geq 0$, and then that $\hat{u}^{(0)} \geq 0$. Then, in a second step, take $h_1 = \hat{u}^{(0)}$. Then one again, the weak maximum principle implies that $\hat{v}^{(h_1)} \geq 0$, and then that $\hat{u}^{(h_1)} \geq 0$. Repeating the procedure, we obtain that $\hat{u} \geq 0$ and $\hat{v} \geq 0$.

Now we go back to (7): clearly (37) and (31) immediately imply that

$$v \geq 0, \qquad \text{and} \qquad u - V \geq 0,$$

which of course imply (16) and (17).

Remark. We could obtain (16) and (17) directly, without the transformation (31), as we explain in the following.

b) Now assume that (8), (9), (10) and (18) are satisfied. Then of course we can apply what we have just proved and (16) and (17) are true. It remains to prove that (19) is satisfied. Consider

$$w(t,x) := (1-\theta)u(t,x) - V(t,x).$$

Denote

$$\mathcal{V}(t,x) := \int_0^r b(a)v(t,a,x)\,da.$$

Then we see that (u, V) satisfies

$$\begin{cases} u_t - \Delta u + du = \mathcal{V} & \text{for } t > 0, \\ V_t - \Delta V + dV = -\mathcal{V} + f_0(u - V) - (d' - d)V & \text{for } t > 0, \\ u(t,x)_{|\partial\Omega} = 0 = V(t,x)_{|\partial\Omega} & \text{for } t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad V(0,x) = V_0(x) & \text{for } x \in \Omega. \end{cases}$$

Hence w satisfies

$$\begin{cases} w_t - \Delta w + dw = (2-\theta)\mathcal{V} - f_0(u - V) + (d' - d)V & \text{for } t > 0, \\ w(t,x)_{|\partial\Omega} = 0 & \text{for } t > 0,\ x \in \partial\Omega, \\ w(0,x) = (1-\theta)u_0(x) - V_0(x) & \text{for } x \in \Omega. \end{cases}$$

Since

$$w = (1-\theta)(u - V) - \theta V,$$

we have

$$w_t - \Delta w + \Big(d + f_0\,\frac{1}{1-\theta}\Big)w = (2-\theta)\mathcal{V} + \Big(d' - d - f_0\,\frac{\theta}{1-\theta}\Big)V,$$

and the weak maximum principle implies that (19) is satisfied. The proof of Theorem 3.2 is completed. □

5.4 - *Proof of Theorem* 3.3

The proof of Theorem 3.2 is based on the same idea: consider (u, v) the solution (7), and $(\hat{u}, \hat{v})$ given by (31) the solution of (32) with the parameters (33). Then we follow the steps of the proof of Theorem 4.3:

- First we decompose $\hat{u}_0$, $\hat{v}_0$ and $\hat{u}$, $\hat{v}$ in Fourier series: let $(\varphi_j)_{j\in\mathbb{N}}$ be the orthonormal basis of $L^2(\Omega)$ composed by the eigenfunctions of the Laplace operator with Dirichlet boundary conditions:

$$\begin{cases} -\Delta\varphi_j(x) = \lambda_j\varphi_j(x) & x \in \Omega, \\ \varphi_j(x) = 0 & x \in \partial\Omega, \end{cases} \tag{38}$$

with $(\lambda_j)_{j\in\mathbb{N}} \subset \mathbb{R}_+^*$ such that $\lambda_j \nearrow +\infty$ as $j \to +\infty$. If we write

$$\hat{u}_0(x) = \sum_{j=0}^{\infty} \hat{u}_0^j \varphi_j(x), \qquad \text{and} \quad \hat{v}_0(a, x) = \sum_{j=0}^{\infty} \hat{v}_0^j(a)\varphi_j(x),$$

then

$$\hat{u}(t, x) = \sum_{j=0}^{\infty} \hat{u}^j(t)\varphi_j(x), \qquad \text{and} \quad \hat{v}(t, a, x) = \sum_{j=0}^{\infty} \hat{v}^j(t, a)\varphi_j(x),$$

where, for each $j \geq 0$, $(\hat{u}^j, \hat{v}^j)$ is solution of the problem without diffusion

$$\begin{cases} \hat{u}_t^j(t) + (\hat{d} + \lambda_j)\hat{u}^j(t, x) = \int_0^r \hat{b}(a)\hat{v}^j(t, a)\, da & t > 0, \\ \hat{v}_t^j(t, a) + \hat{v}_a^j(t, a) + (\hat{d}' + \lambda_j)\hat{v}^j(t, a) = -\hat{b}(a)\hat{v}^j(t, a) & t > 0,\ a \in (0, r) \\ \hat{v}^j(t, 0) = \hat{f}_0\hat{u}^j(t) & t > 0, \\ \hat{u}^j(0) = \hat{u}_0^j & \\ \hat{v}^j(0, a) = \hat{v}_0^j(a) & a \in (0, r). \end{cases}$$

Of course

$$u = 2\hat{u} + V,$$

and

$$\|u(t, \cdot)\|_{L^2(\Omega)}^2 = \sum_{j=0}^{\infty} \Big(2\hat{u}^j(t) + \int_0^r \hat{v}^j(t, a)\, da\Big)^2.$$

Hence we have to study the behavior of $\hat{u}^j$ and $\hat{v}^j$ with respect to t and j.

- Note that since φ_0 is positive in Ω, then

$$\hat{u}_0^0 := \int_\Omega \hat{u}_0(x)\varphi_0(x)\,dx$$

is nonnegative, and $\hat{u}_0^0 = 0$ if and only if $\hat{u}_0 = 0$ in Ω. The same remark holds true for $\hat{v}_0$. In the same spirit, since $\hat{u}$ and $\hat{v}$ are nonnegative, $\hat{u}^0(t) \geq 0$ and $\hat{v}^0(t,a) \geq 0$.

- Define $\hat{R}_0$ by (21); since the function

$$\psi : \alpha \mapsto \hat{f}_0 \int_0^r \hat{b}(a)e^{-\tilde{\hat{b}}(a)}e^{-(\hat{d}'-\hat{d})a}e^{-\alpha a}\,da - \alpha$$

is strictly decreasing, there exists one and only one $\alpha^* > 0$ such that

$$\hat{f}_0 \int_0^r \hat{b}(a)e^{-\tilde{\hat{b}}(a)}e^{-(\hat{d}'-\hat{d})a}e^{-\alpha^* a}\,da = \alpha^*;$$

moreover, since $\psi(\hat{d}+\lambda_0) = \hat{R}_0 - (\hat{d}+\lambda_0)$, we have that

$$\begin{aligned}
\hat{R}_0 < \hat{d}+\lambda_0 \quad &\Longrightarrow \quad \alpha^* < \hat{d}+\lambda_0\,,\\
\hat{R}_0 = \hat{d}+\lambda_0 \quad &\Longrightarrow \quad \alpha^* = \hat{d}+\lambda_0\,,\\
\hat{R}_0 > \hat{d}+\lambda_0 \quad &\Longrightarrow \quad \alpha^* > \hat{d}+\lambda_0\,.
\end{aligned}$$

- Using classical properties of the Laplace transform (see Theorem 3.3 in [**10**]), the following holds:

$$\hat{u}^0(t) \sim c_0^0 e^{(\alpha^*-\hat{d}-\lambda_0)t}, \tag{39}$$

where $c_0^0 > 0$ (and is given in (3.17) in [**10**]).

- Now we are in position to distinguish the three cases of Theorem 3.3:
  - Case (iii): $\hat{R}_0 > \hat{d}+\lambda_0$. Then, as noted, we have $\alpha^* > \hat{d}+\lambda_0$, and (39) implies that $\hat{u}^0(t)$ goes exponentially fast to $+\infty$ as $t \to +\infty$. Since $\hat{v}^0 \geq 0$, and

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} \geq \left(2\hat{u}^0(t) + \int_0^r \hat{v}^0(t,a)\,da\right)^2,$$

we obtain what is claimed in Theorem 3.3, (iii).

– Case (i): $\hat{R}_0 < \hat{d} + \lambda_0$. Then as noted we have $\alpha^* < \hat{d} + \lambda_0$, and (39) implies that $\hat{u}^0(t)$ goes exponentially fast to 0 as $t \to +\infty$. In the same way, all the components $\hat{u}^j$ go exponentially fast to 0, and more precisely, there exists some C such that

$$\|\hat{u}(t,\cdot)\|^2_{L^2(\Omega)} \le Ce^{2(\alpha^* - d - \lambda_0)t}(\|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2((0,r)\times\Omega)}).$$

Since for $t > a$, we have

$$\hat{v}^j(t,a) = \hat{f}_0\hat{u}^j(t-a)e^{-(\hat{d}'+\lambda_j)a - \tilde{\hat{b}}(a)},$$

we obtain a similar uniform exponential decay for all the $\hat{v}^j$, and hence for $\|u(t,\cdot)\|^2_{L^2(\Omega)}$ and $\|v\|^2_{L^2((0,r)\times\Omega)}$ as claimed in Theorem 3.3, (i).

– Case (ii). We decompose

$$\hat{u}(t,x) = \hat{u}^0(t)\varphi_0(x) + \sum_{j=1}^{\infty}\hat{u}^j(t)\varphi_j(x),$$

and

$$\hat{v}(t,a,x) = \hat{v}^0(t,a)\varphi_0(x) + \sum_{j=1}^{\infty}\hat{v}^j(t,a)\varphi_j(x);$$

the previous argument of Case (i) gives that all the components $\hat{u}^j$ and $\hat{v}^j$ go exponentially fast to 0 uniformly with respect to $j \ge 1$, and the first components $\hat{u}^0$ and $\hat{v}^0$ converge to positive constants. Hence we obtain Theorem 3.3, (ii).

6 - Conclusion and perspectives

Finally, we corrected the first model (22) introduced in [**13, 15**] to obtain the more realistic one (7). Now we are ready to turn to some other questions, like for example the introduction of more general models or the study of controllability properties of such models.

6.1 - *Introduction of more general models*

Let us recall the Mc Kendrick-Von Foerster model with diffusion which is a classical population dynamics model studied in particular in [**5**]. In this model, the density of population $u(A,t,x)$ of age A at time t and located at space x satisfies the following equations (where another nonlocal birth process arises in a boundary condition) :

$$
(40) \qquad \begin{cases} u_t + u_A - \Delta u + d(A)u = 0, \\ u(A,t,x)_{|\partial\Omega} = 0, \\ u(0,t,x) = \int_0^{A_+} \beta(A)u(A,t,x)\,dA, \\ u(A,0,x) = u_0(A,x). \end{cases}
$$

Here $A \in (0, A_+)$ denotes the age of the individuals and A_+ represents the maximal life expectancy of an individual, $d(A)$ is the natural death rate, whereas $\beta(A)$ denotes the natural fertility rate. In summary, (40) is a population dynamics model with diffusion and with age dependence, i.e. *structured in space and age*.

On the other hand, in models (22) and (7) introduced here, we recall that no age difference is allowed whereas the time of gestation is considered. In other words, models (22) and (7) are models with diffusion and with time-gestation dependence, i.e. *structured in space and time-gestation*.

A natural question is now to write such a model where age difference will also be considered i.e. where the density of population $u(A,t,x)$ and the density of pregnant individuals $v(A,t,a,x)$ will depend on age A, time t, space x and age of gestation a. At this stage, combining the results and the methods of [**5**] and of [**10**], it should be possible to write and to study such a dynamics population model with diffusion and with age and time-gestation dependence, i.e. *structured in space, age and time-gestation*.

6.2 - *Study of controllability properties*

Controllability of diffusive population dynamics models is a subject that is still mainly open. But there are some recent works concerning the Mc Kendrick-Von Foerster model with diffusion : see e.g. [**1, 2, 3, 4, 5**] for results of approximate, local exact or exact controllability.

We wish now to study the case of diffusive population dynamics models that also take into account the pregnancy. Notice that the study of controllability of such models has a link with the study made in [**14**]. Indeed, in models (22) or (7), the second equation in v has the same structure than the Crocco equation (with constant coefficients) : it is a degenerate parabolic equation that couples a transport phenomenon with a diffusion phenomenon. Of course, the situation will be complicated here by the fact that we deal now with a system of two equations with nonlocal coupling terms...

On the other hand, let us mention that there is a rather large literature on the subject of optimal control problems for models like (40). See for example [**5, 6, 7**] and the references therein. It would also be interesting to study those kinds of problems for models like (22) or (7).

Acknowledgments. First author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and acknowledges the GNAMPA project 2015 "Analisi e controllo di equazioni a derivate parziali nonlineari" for its financial support.

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