Riv. Mat. Univ. Parma, Vol. **7** (2016), 341-350

Anna Canale and Cristian Tacelli

Kernel estimates for a Schrödinger type operator

Abstract. In this paper the principal result obtained is the estimate for the heat kernel associated to the Schrödinger type operator $(1+|x|^\alpha)\Delta - |x|^\beta$

$$k(t,x,y) \le Ct^{-\frac{\theta}{2}} \frac{\varphi(x)\varphi(y)}{1+|x|^\alpha},$$

where $\varphi = (1+|x|^\alpha)^{\frac{2-\theta}{4}+\frac{1}{\alpha}\frac{\theta-N}{2}}$, $\theta \ge N$ and $0 < t \le 1$, provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. This estimate improves a similar estimate obtained in [3] with respect to the dependence on spatial component.

Keywords. Schrödinger operators, unbounded coefficients, kernel estimates.

Mathematics Subject Classification (2010): 47D07, 35J10, 35K05, 35K10.

1 - Introduction

In this paper we consider the elliptic operator defined by

$$Au(x) = a(x)\Delta u(x) - V(x)u(x), \qquad x \in \mathbb{R}^N, \tag{1}$$

where $a(x) = 1 + |x|^\alpha$, $\alpha > 2$ and $V(x) = |x|^\beta$, $\beta > \alpha - 2$. Our aim is to give a better estimate for the associated heat kernel then those obtained in [3].

Recently elliptic operators with unbounded coefficients have been studied in several paper (see for example [**11**], [**12**], [**13**], [**14**], [**9**], [**6**], [**4**], [**8**], [**7**], [**5**]).

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Received: January 22, 2016; accepted in revised form: October 17, 2016.

tinuous and bounded functions $C_b(\mathbb{R}^N)$ in a standard way (see [**2**]). Solving, indeed, the Cauchy problems of the associate parabolic operator with initial datum $f \in C_b(\mathbb{R}^N)$ on balls of increasing radius, one obtains a sequence of solutions which converges to a function $u(x,t) = T(t)f$. It can be proved that u is a solution to the problem $\partial_t u = Au$, $u(0,\cdot) = f$. If we assume $f \geq 0$ then the function u is nonnegative and it is the minimal among all positive solutions to the above problem. For this reason the semigroup $\{T(t)\}_{t\geq 0}$ of bounded operator (not necessary strongly continuous) is called the minimal semigroup and its generator is denoted by A_{min}. This semigroup can be also extended, in some cases, to the space $L^p(\mathbb{R}^N)$.

In [**9**], for $\beta \geq 0$ and $0 \leq \alpha \leq 2$, and in [**4**], for $\alpha > 2$ and $\beta > \alpha - 2$, it is proved that A endowed with domain

$$D_p(A) = \{u \in W^{2,p}(\mathbb{R}^N) | (1+|x|^\alpha)|D^2u|, (1+|x|^\alpha)^{1/2}\nabla u, |x|^\beta u \in L^p(\mathbb{R}^N)\} \tag{2}$$

generates a strongly continuous and analytic semigroup $T(\cdot)$ in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. This semigroup is also consistent, irreducible and ultracontractive. As regards the case $\beta = 0$ we refer to [**6**] and [**11**].

Due to the regularity of the coefficients of the operator A, the semigroup $T(t)$ can be represented in the following integral form through a heat kernel $k(t,x,y)$

$$T(t)f(x) = \int_{\mathbb{R}^N} k(t,x,y)f(y)dy\,, \qquad t>0,\ x \in \mathbb{R}^N\,,$$

for any $f \in L^p$ (see [**2**], [**10**]).

In [**9**] kernel estimates were proved provided that $N > 2$, $\alpha < 2$ and $\beta > 0$. The estimates was extended in [**3**] to the case $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$ obtaining

$$k(t,x,y) \leq c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \frac{\psi(x)\psi(y)}{1+|y|^\alpha}, \quad t>0,\ x,\ y \in \mathbb{R}^N, \tag{3}$$

where c_1, c_2 are positive constants, $b = \dfrac{\beta-\alpha+2}{\beta+\alpha-2}$ and $\psi(x)$ is the eigenfunction associated to the first eigenvalue, which is equivalent to the function

$$|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}.$$

A better estimate with respect to the time variable was also obtained for small values of t

$$k(t,x,y) \leq Ct^{-\frac{N}{2}}(1+|x|^\alpha)^{\frac{2-N}{4}}(1+|y|^\alpha)^{\frac{2-N}{4}-1}, \quad 0 < t \leq 1\,, \alpha > 2. \tag{4}$$

Estimate (4) is the same obtained in [**12**] for the operator $Lu = a(x)\Delta u$, for $2 < \alpha \leq 4$. Indeed in [**12**] it was obtained the following estimates for the kernel p

associated to L

$$p(x,y,t) \le \frac{C}{t^{\frac{N+\alpha-4}{\alpha-2}}}(1+|x|)^{2-N}(1+|y|)^{2-N-\alpha} \qquad 2<\alpha\le 4, \tag{5}$$

$$p(x,y,t) \le \frac{C}{t^{\frac{N}{2}}}(1+|x|^{\alpha})^{\frac{2-N}{4}}(1+|y|^{\alpha})^{\frac{2-N}{4}-1} \qquad 2<\alpha\le 4.$$

Also in [**13**] it was proved similar kernel estimates for the operator $a(x)\Delta + c|x|^{\alpha-1}\frac{x}{|x|}\nabla$.

Comparing (3) and (4) (and the estimates obtained in [**12**], and [**13**]) ones can see that improving the dependence on t involves worsening in the dependence on the space variables. Conversely, if the space component is improved the other worsens.

In this paper our aim is to quantify how this happens. In order to state the relationship between the dependence on the time and the space components we will state an estimate which depends on a parameter θ. In particular we will prove for $N>2$ and $\alpha>2$ the following estimate for small values of t

$$k(t,x,y) \le Ct^{-\frac{\theta}{2}}\frac{\varphi(x)\varphi(y)}{1+|y|^{\alpha}}, \tag{6}$$

where $\varphi=(1+|x|^{\alpha})^{\frac{2-\theta}{4}+\frac{1}{\alpha}\frac{\theta-N}{2}}$, $\theta\ge N$. We observe that (4) and (5) are a particular case of (6) obtained for $\theta=N$ and $\theta=2\left(\dfrac{N+\alpha-4}{\alpha-2}\right)$.

2 - Weighted spaces and Weighted Nash Inequalities

Let $T(t)$ be the semigroup generated by the operator $(A, D_p(A))$, where A and $D_p(A)$ are defined by (1) and (2). First we show that $T(t)$ can be seen as a suitable semigroup $\mathcal{T}(t)$ on a weighted space. So, we can deduce heat kernel estimates of $T(t)$ by heat kernel estimates of $\mathcal{T}(t)$. Then, in order to obtain kernel estimates of $\mathcal{T}(t)$ we prove a weighted ultracontractivity of the semigroup obtained by a weighted Nash Inequality.

Let us introduce the measure $d\mu(x)=(1+|x|^{\alpha})^{-1}dx$ and the Hilbert space $L^2_\mu := L^2(\mathbb{R}^N, d\mu)$ endowed with the canonical inner product $\langle f,g\rangle = \int_{\mathbb{R}^N} f\overline{g}\,d\mu$ and its associated norm $\|f\|_{L^2_\mu} := \left(\int_{\mathbb{R}^N} f^2 d\mu\right)^{\frac{1}{2}}$. Let

$$H=\{u\in L^2_\mu\cap W^{1,2}_{loc}(\mathbb{R}^N):\ V^{1/2}u\in L^2_\mu\,,\ \nabla u\in (L^2(\mathbb{R}^N))^N\}$$

be the Sobolev space endowed with the inner product

$$(u, v)_H = \int_{\mathbb{R}^N} (1 + V)u\bar{v}\, d\mu + \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v}\, dx.$$

We consider the closed and accretive symmetric form so defined

$$a(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v}\, dx + \int_{\mathbb{R}^N} u\bar{v}\, d\mu \tag{7}$$

for u, v belonging to the closure $\mathcal{V}$ of $C_c^\infty(\mathbb{R}^N)$ in H, with respect to the norm of H. Then we can associate to a the self-adjoint operator

$$\mathcal{A}u = f$$

with domain

$$D(\mathcal{A}) = \{u \in \mathcal{V} : \text{there exists}\, f \in L^2_\mu \quad s.t. \quad a(u, v) = -\int_{\mathbb{R}^N} f\bar{v}\, d\mu \text{ for any } v \in \mathcal{V}\}.$$

By general results since $\mathcal{A}$ is a self-adjoint operator induced by a nonnegative quadratic form on an Hilbert spaces it generates a positive analytic semigroup $\mathcal{T}(t)$ in L^2_μ (see, e.g., [**15**, Prop. 1.51, Thms. 1.52, 2.6, 2.13]), which is also a symmetric Markov semigroup. Indeed using the Beurling-Deny criteria it can be proved that the semigroup is L^∞ contractive.

The Lemma below (see [**3**]) shows that the semigroup $\mathcal{T}(t)$ coincides in $L^p(\mathbb{R}^N) \cap L^2_\mu$ with the semigroup $T(t)$ generated by $(A, D_p(A))$ in $L^p(\mathbb{R}^N)$.

Lemma 2.1. *We get*

$$D(\mathcal{A}) \subset \left\{u \in \mathcal{V} \cap W^{2,2}_{loc}\left(\mathbb{R}^N\right) : (1 + |x|^\alpha)\Delta u - V(x)u \in L^2_\mu\right\}$$

and $\mathcal{A}u = (1 + |x|^\alpha)\Delta u - V(x)u$ *for* $u \in D(\mathcal{A})$. *If* $\lambda > 0$ *and* $f \in L^p(\mathbb{R}^N) \cap L^2_\mu$, *then*

$$(\lambda - \mathcal{A})^{-1} f = (\lambda - A)^{-1} f.$$

Denoting by $k(t, x, y)$ and $k_\mu(t, x, y)$ respectively the heat kernel associated to $T(t)$ and $\mathcal{T}(t)$, by the previous lemma we can deduce that

$$k_\mu(t, x, y) = (1 + |y|^\alpha)k(t, x, y).$$

In the following we describe how estimates of the kernel of a symmetric Markov semigroup can be obtained by using the equivalence between a weighted Nash inequality and a "weighted" ultracontractivity of the semigroup. The equivalence was stated in [**16**, Theorem 3.3] and was reformulated in [**1**, Theorem 2.5]. The equivalence is obtained by means of a suitable Lyapunov function for the generator of the semigroup.

Let $\mathcal{T}(t)$ be a symmetric Markov semigroup generated by a self-adjoint operator $\mathcal{A}$ associated to an accretive, closed, symmetric form a defined on a domain $\mathcal{V}$ in L^2_μ. Let k_μ be its associated heat kernel. We define a Lyapunov function in the following way (see also [**12**], [**13**])

Definition 2.1. A Lyapunov function is a positive function $\varphi \in L^2_\mu$ such that

$$\mathcal{T}(t)\varphi(x) = \int_{\mathbb{R}^N} k_\mu(t,x,y)\varphi(y)d\mu(y) \le e^{\kappa t}\varphi(x)$$

for any $x \in \mathbb{R}^N$, $t > 0$, and for some real constant κ, called Lyapunov constant.

Now, we define the weighted Nash inequality.

Definition 2.2. Let φ be a positive function on $\mathbb{R}^N$ and ψ be a positive function defined on $(0,\infty)$ with $\frac{\psi(x)}{x}$ non decreasing. The form a on L^2_μ satisfies a weighted Nash inequality with weight φ and rate function ψ if

$$\psi\left(\frac{\|u\|^2_{L^2_\mu}}{\|u\varphi\|^2_{L^1_\mu}}\right) \le \frac{a(u,u)}{\|u\varphi\|^2_{L^1_\mu}}$$

for any functions $u \in \mathcal{V}$ such that $\|u\|^2_{L^2_\mu} > 0$ and $\|u\varphi\|^2_{L^1_\mu} < \infty$.

Theorem 2.1 ([**1**, Theorem 2.5]). *Let $\mathcal{T}(t)$ be a symmetric Markov semigroup on L^2_μ with generator $\mathcal{A}$. Let us assume that there exists a Lyapunov function φ with Lyapunov constant $\kappa \ge 0$ and that the associated form a satisfies a weighted Nash inequality with weight φ and rate function ψ such that $\frac{1}{\psi(x)}$ is integrable in $(a,+\infty)$ for every $a > 0$. Then*

$$\|\mathcal{T}(t)f\|_{L^2_\mu} \le K(2t)e^{\kappa t}\|f\varphi\|_{L^1_\mu}$$

for any functions $f \in L^2_\mu$ such that $f\varphi \in L^1_\mu$. The function K is defined by

$$K(t) = \sqrt{U^{-1}(t)},$$

where

$$U(t) = \int_t^\infty \frac{1}{\psi(u)}du.$$

Finally from [**1**, Corollary 2.8] we get the estimate of k_μ.

Corollary 2.1. *If the Markov semigroup* $\mathcal{T}(t)$ *satisfies the assumptions of Theorem* 2.1 *then the kernel* k_μ *satisfies*

$$k_\mu(2t, x, y) \le K(2t)^2 e^{2\kappa t}\varphi(x)\varphi(y)$$

for any $t > 0$, $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

3 - Heat kernel estimates

In this section we will prove upper bound estimates for the kernel k. First we prove that the function $\varphi(x) = (1 + |x|^\alpha)^{\frac{\gamma}{\alpha}}$ is a Lyapunov function if $\gamma < -\dfrac{N}{2} + \dfrac{\alpha}{2}$ and $\gamma < 0$.

Lemma 3.1. *Let* $\gamma < -\dfrac{N}{2} + \dfrac{\alpha}{2}$ *be a real constant. Then the function* $\varphi(x) = (1 + |x|^\alpha)^{\frac{\gamma}{\alpha}} \in L^2_\mu(\mathbb{R}^N)$ *and satisfies the inequality* $A\varphi \le \kappa\varphi$ *for some* $\kappa > 0$.

Proof. Let us consider $\varphi(x) = (1 + |x|^\alpha)^{\frac{\gamma}{\alpha}}$. It is easy to see that $\varphi \in L^2_\mu$ if $2\gamma - \alpha < -N$. Furthermore we get

$$\begin{aligned}
A\varphi &= \gamma(\gamma - \alpha)(1 + |x|^\alpha)^{\frac{\gamma}{\alpha}-1}|x|^{2\alpha-2} + \gamma(\alpha - 2 + N)(1 + |x|^\alpha)^{\frac{\gamma}{\alpha}}|x|^{\alpha-2} - |x|^\beta(1 + |x|^\alpha)^{\frac{\gamma}{\alpha}} \\
&= \gamma(\gamma - \alpha)\frac{|x|^{2\alpha-2}}{1 + |x|^\alpha}\varphi(x) + \gamma(\alpha - 2 + N)|x|^{\alpha-2}\varphi(x) - |x|^\beta\varphi(x) \\
&= \left[|x|^{\alpha-2}\left(\gamma(\gamma - \alpha)\frac{|x|^\alpha}{1 + |x|^\alpha} + \gamma(\alpha - 2 + N)\right) - |x|^\beta\right]\varphi(x).
\end{aligned}$$

Then, since $\beta > \alpha - 2$, one can see that there exists a positive constant κ such that

$$A\varphi(x) \le \kappa\varphi(x)$$

for all $x \in \mathbb{R}^N$. □

Arguing as in [**11**, Section 2] we have that φ is actually a Lyapunov function.

Theorem 3.1. *Let* $\gamma < -\dfrac{N}{2} + \dfrac{\alpha}{2}$ *be a negative constant. Then the function* φ *is a Lyapunov function with constant* κ_0 *for any* $\kappa_0 > \kappa$.

Proof. Let us observe that $\varphi \in C_0(\mathbb{R}^N)$ if and only if $\gamma < 0$. Then we can consider $u = R(\lambda, A_{\min})\varphi = (\lambda - A_{\min})^{-1}\varphi \in C_0(\mathbb{R}^N)$ (see [**4**, Section 2]). Let $\kappa_0 > \kappa$, $\lambda \geq \frac{\kappa\kappa_0}{\kappa_0 - \kappa}$ and $w = \left(1 + \frac{\kappa_0}{\lambda}\right)\varphi - \lambda u$. Since $Au = \lambda u - \varphi$, we have

$$\begin{aligned} Aw - \lambda w &= \frac{\kappa_0 + \lambda}{\lambda} A\varphi - \lambda(\lambda u - \varphi) - \lambda\left(\frac{\kappa_0 + \lambda}{\lambda}\varphi - \lambda u\right) \\ &\leq \frac{\kappa_0 + \lambda}{\lambda}\kappa\varphi - \kappa_0\varphi = \frac{1}{\lambda}(\kappa_0\kappa + \lambda\kappa - \lambda\kappa_0)\varphi \leq 0. \end{aligned}$$

By the maximum principle we have $w > 0$ in $\mathbb{R}^N$. Then

$$\left(1 + \frac{\kappa_0}{\lambda}\right)\varphi \geq \lambda R(\lambda, A)\varphi.$$

Iterating the last inequality we get

$$\left(1 + \frac{\kappa_0}{\lambda}\right)^n \varphi \geq \lambda^n R^n(\lambda, A)\varphi.$$

So, we obtain

$$T(t)\varphi = \lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t}, A\right)\right]^n \varphi \leq \lim_{n\to\infty}\left(1 + \frac{\kappa_0 t}{n}\right)^n \varphi = e^{\kappa_0 t}\varphi.$$

□

Finally, in order to get kernel estimates, we will prove the weighted Nash inequality (see Definition 2.2) with Lyapunov function

$$\varphi = (1 + |x|^\alpha)^{\frac{2-\theta}{4} + \frac{1}{\alpha}\frac{\theta - N}{2}}$$

and rate function

$$\psi(t) = t^{1+\frac{2}{\theta}}$$

for $\theta \geq N$. We observe that φ satisfies the hypothesis of Theorem 3.1 if $\alpha > 2$.

In order to prove the weighted Nash inequality we use a weighted Sobolev inequality which we recall for reader's convenience (see [**12**, Proposition 3.5]).

Proposition 3.1. *Let β', γ', v, p, q real values such that*

$$1 < p \leq q < \infty \qquad \gamma' - 1 \leq \beta' \leq \gamma',$$

$$0 \leq \frac{1}{p} - \frac{1}{q} = \frac{1 - \gamma' + \beta'}{N}, \qquad N + p(\gamma' - 1) \neq 0, \qquad p \leq q \leq p^*, \qquad p < N.$$

Then there exists a positive constant C such that for any $u \in C_c^\infty(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} (1+|x|)^{q\beta'}|u(x)|^q dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} (1+|x|)^{\gamma' p}|\nabla u(x)|^p dx\right)^{\frac{1}{p}}$$

$$+ C\left(\int_{\mathbb{R}^N} (1+|x|)^{\gamma}|u(x)|^p dx\right)^{\frac{1}{p}}.$$

Theorem 3.2. *If $\alpha > 2$ and $\beta > \alpha - 2$, then the kernel k_μ of the semigroup generated by A satisfies the inequality*

$$k_\mu(t,x,y) \le \frac{C}{t^{\frac{\theta}{2}}}\varphi(x)\varphi(y)$$

for every $0 < t \le 1$, $x,\ y \in \mathbb{R}^N$.

Proof. Let $u \in \mathcal{V}$ be such that $\|u\varphi\|_{L^1_\mu} < \infty$.

Applying Hölder's inequality with $p = \dfrac{\theta+2}{\theta-2}$ we get

$$\Bigg(\int_{\mathbb{R}^N} |u|^2 d\mu = \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta+2}+\frac{4}{\theta+2}}\frac{\varphi^{\frac{4}{\theta+2}}}{\varphi^{\frac{4}{\theta+2}}}\,d\mu$$

$$\le \left(\int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}}\varphi^{\frac{4}{2-\theta}}d\mu\right)^{\frac{\theta-2}{\theta+2}}\left(\int_{\mathbb{R}^N} |u|\varphi d\mu\right)^{\frac{4}{\theta+2}}$$

$$\le \left(\int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}}(1+|x|^\alpha)^{\frac{4}{2-\theta}\left(\frac{2-\theta}{4}+\frac{1}{\alpha}\frac{\theta-N}{2}\right)-1}dx\right)^{\frac{\theta-2}{\theta+2}}\left(\int_{\mathbb{R}^N} |u|\varphi d\mu\right)^{\frac{4}{\theta+2}}$$

$$\le \left(\int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}}(1+|x|)^{2\frac{\theta-N}{2-\theta}}dx\right)^{\frac{\theta-2}{\theta+2}}\left(\int_{\mathbb{R}^N} |u|\varphi d\mu\right)^{\frac{4}{\theta+2}}.$$

Then

$$\left(\|u\|^2_{L^2_\mu}\right)^{1+\frac{2}{\theta}} \le \left(\int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}}(1+|x|)^{2\frac{\theta-N}{2-\theta}}dx\right)^{\frac{\theta-2}{\theta}}\|u\varphi\|^{\frac{4}{\theta}}_{L^1_\mu},$$

from which

$$\psi\left(\frac{\|u\|_{L^2_\mu}^2}{\|u\varphi\|_{L^1_\mu}^2}\right) \leq \frac{\left(\int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}}(1+|x|)^{2\frac{\theta-N}{2-\theta}}dx\right)^{\frac{\theta-2}{\theta}}}{\|u\varphi\|_{L^1_\mu}^2}.$$

Applying the weighted Sobolev inequality with $p=2$, $q=\dfrac{2\theta}{\theta-2}$, $\gamma'=0$, $q\beta'=2\dfrac{\theta-N}{2-\theta}$ and $\nu=-\alpha$ we obtain

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta-2}}(1+|x|)^{2\frac{\theta-N}{2-\theta}}dx\right)^{\frac{\theta-2}{\theta}} \leq C\left(\|\nabla u\|_2^2 + \int_{\mathbb{R}^N}(1+V)u^2 d\mu\right) = C\tilde{a}(u,u),$$

where $\tilde{a}(u,u)=a(u,u)+\int_{\mathbb{R}^N} u^2 d\mu$ is the quadratic form associated with the operator $A+I$. Since φ is a Lyapunov function with constant $\kappa+1$ for the operator $A+I$, applying Corollary 2.1, we get

$$\tilde{k}_\mu(t,x,y) \leq \frac{Ce^{(\kappa+1)t}}{t^{\frac{\theta}{2}}}\varphi(x)\varphi(y),$$

where $\tilde{k}_\mu = e^t k_\mu$ is the kernel associated with $A+I$. This gives the result. □

Taking into account the relation between k and k_μ, by Theorem 3.2 we state the following result:

Theorem 3.3. *Le us assume $\alpha > 2$, $\beta > \alpha - 2$. Then the kernel k of the semigroup generated by A for every $0 < t \leq 1$ satisfies the bound*

$$k(t,x,y) \leq C\frac{e^{\kappa t}}{t^{\frac{\theta}{2}}}\frac{\varphi(x)\varphi(y)}{1+|x|^\alpha}. \tag{8}$$

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Anna Canale
University of Salerno
Dept. of Information Eng., Electrical Eng. and Applied Mathematics
Via Giovanni Paolo II, 132
Fisciano (SA), 84084, Italy
e-mail: acanale@unisa.it

Cristian Tacelli
University of Salerno
Dept. of Information Eng., Electrical Eng. and Applied Mathematics
Via Giovanni Paolo II, 132
Fisciano (SA), 84084, Italy
e-mail: ctacelli@unisa.it