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Lipschitz stability for an inverse problem for the 2D-Sellers model on a manifold

Abstract. In this paper, we are interested in some inverse problem that consists in recovering the so-called insolation function in the 2-D Sellers model on a Riemannian manifold that materializes the Earth's surface. For this nonlinear problem, we obtain a Lipschitz stability result in the spirit of the result by Imanuvilov-Yamamoto in the case of the determination of the source term in the linear heat equation. The paper complements an analogous study by Tort-Vancostenoble in the case of the 1-D Sellers model.

Keywords. PDEs on manifolds, nonlinear parabolic equations, climate models, inverse problems, Carleman estimates.

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Contents

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1 - Introduction

1.1 - *The Sellers model*

In this paper, we are interested in some inverse problem that consists in recovering the so-called insolation function q in the nonlinear Sellers climate model. The case of the 1-D Sellers model has been considered in [**39**]. Here we focus on the 2-D

Sellers model on the Earth's surface:

$$\begin{cases} u_t - \Delta_{\mathcal{M}} u = \overbrace{r(t)q(x)\beta(u)}^{\mathcal{R}_a(t,x,u)} - \overbrace{\varepsilon(u)u|u|^3}^{\mathcal{R}_e(u)} & x \in \mathcal{M},\ t > 0, \\ u(0,x) = u^0(x) & x \in \mathcal{M}. \end{cases} \tag{1}$$

The Earth's surface is materialized by a sub-manifold $\mathcal{M}$ of $\mathbb{R}^3$ which is assumed to be of dimension 2, compact, connected, oriented, and without boundary. The function u represents the mean annual or seasonal temperature, and $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$. The right hand side of the equation corresponds to

- the mean radiation flux depending on the solar radiation $\mathcal{R}_a$,
- and the radiation $\mathcal{R}_e$ emitted by the Earth.

For more details on the model, we refer the reader to [**14, 15**] and the references therein.

1.2 - *Assumptions and main results*

1.2.1 - Geometrical and regularity assumptions

Consider a sub-manifold $\mathcal{M}$ of $\mathbb{R}^3$ which is assumed to be of dimension 2, compact, connected, oriented, and without boundary.

Throughout this paper, we make the following assumptions (that are compatible with the applications, see [**39**]):

Assumption 1.1.

$$\beta \in \mathcal{C}^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ \beta' \in L^\infty(\mathbb{R}),\ \beta' \text{ is } k\text{-Lipschitz } (k > 0), \tag{2}$$

$$\exists \beta_{min} > 0, \forall u \in \mathbb{R},\ \beta(u) \geq \beta_{min}, \tag{3}$$

$$q \in L^\infty(\mathcal{M}),\ q \geq 0, \tag{4}$$

$$r \in \mathcal{C}^1(\mathbb{R}) \textit{ is } \tau\textit{-periodic } (\tau > 0), \tag{5}$$

$$\exists r_{min} > 0,\ \forall t \in \mathbb{R},\ r(t) \geq r_{min}, \tag{6}$$

$$\varepsilon \in \mathcal{C}^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ \varepsilon' \textit{ is } K\textit{-Lipschitz } (K > 0), \tag{7}$$

$$\exists \varepsilon_{min} > 0, \forall u \in \mathbb{R},\ \varepsilon(u) > \varepsilon_{min}\,. \tag{8}$$

We also make the following geometrical assumption:

Assumption 1.2. *Let ω be a non empty open subset of $\mathcal{M}$. We assume that there exists a weight function $\psi \in C^\infty(\mathcal{M})$ that satisfies:*

$$\nabla\psi(m) = 0 \quad \Longrightarrow \quad m \in \omega. \tag{9}$$

(Here ∇ stands for the usual gradient associated to the Riemannian structure, see Section 2.)

1.2.2 - Main results

As in [**39**], our aim is to prove some Lipschitz stability result for the inverse problem that consists in recovering the insolation function q in (1) from partial measurements. We introduce

- the set of admissible initial conditions: given $A > 0$, we consider $\mathcal{U}_A$:

$$\mathcal{U}_A := \{u^0 \in D(\Delta_\mathcal{M}) \cap L^\infty(\mathcal{M}) : \Delta_\mathcal{M} u^0 \in L^\infty(\mathcal{M}), \\ \|u_0\|_{L^\infty(\mathcal{M})} + \|\Delta_\mathcal{M} u_0\|_{L^\infty(\mathcal{M})} \leq A\}, \tag{10}$$

  where $D(\Delta_\mathcal{M})$ is the domain of the Laplace-Beltrami operator in $L^2(\mathcal{M})$ (we will recall the definition of $\Delta_\mathcal{M}$ and $D(\Delta_\mathcal{M})$ in Section 2),
- and the set of admissible coefficients: given $B > 0$, we consider

$$\mathcal{Q}_B := \{q \in L^\infty(\mathcal{M}) : \|q\|_{L^\infty(\mathcal{M})} \leq B\}. \tag{11}$$

The main result of this paper is the following one:

Theorem 1.1. *Consider*

- $t_0 \in [0, T)$ *and* $T' \in (t_0, T)$,
- $A > 0$ *and* $u_1^0, u_2^0 \in \mathcal{U}_A$ *(defined in* (10)),
- $B > 0$ *and* $q_1, q_2 \in \mathcal{Q}_B$ *(defined in* (11)),
- u_1 *the solution of* (1) *associated to* q_1 *and the initial condition* u_1^0, *and* u_2 *the solution of* (1) *associated to* q_2 *and the initial condition* u_2^0,
- $\omega \subset \mathcal{M}$ *such that Assumption* 1.2 *holds.*

Then there exists $C(t_0, T', T, A, B) > 0$ *such that, for all* $u_1^0, u_2^0 \in \mathcal{U}_A$, *for all* $q_1, q_2 \in \mathcal{Q}_B$, *the corresponding solutions* u_1, u_2 *of problem* (1) *satisfy*

$$\|q_1 - q_2\|^2_{L^2(\mathcal{M})} \leq C\big(\|u_1(T') - u_2(T')\|^2_{D(\Delta_\mathcal{M})} + \|u_{1,t} - u_{2,t}\|^2_{L^2((t_0,T)\times\omega)}\big). \tag{12}$$

We complete Theorem 1.1 by the following remarks:

- the geometrical assumption 1.2 is satisfied when $\mathcal{M}$ is simply connected (hence in particular for the sphere $\mathbb{S}^2$):

Proposition 1.1. *Additionnally, assume that* $\mathcal{M}$ *is simply connected. Consider any* ω *non empty open set of* $\mathcal{M}$. *Then Assumption* 1.2 *is fullfilled: there exists some smooth function* ψ *that satisfies* (9).

- as a consequence of the stability estimate (12) and of the Carleman estimate that we will prove in Theorem 3.1, we obtain a weighted stability estimate for the difference $u_1 - u_2$: there exists $C'(t_0, T', T, A, B) > 0$ such that,

$$(13)\quad \|e^{-R\sigma}(u_1 - u_2)\|^2_{L^2((0,T)\times\mathcal{M})} \leq C'\big(\|u_1(T') - u_2(T')\|^2_{D(\Delta_\mathcal{M})} + \|u_{1,t} - u_{2,t}\|^2_{L^2((t_0,T)\times\omega)}\big),$$

where σ is the weight function defined in (35).

The proof of Theorem 1.1 is based on

- global Carleman estimates for the heat equation (see Theorem 3.1),
- maximum principles, useful to study this nonlinear problem (see Theorem 5.2 and Corollary 5.1),
- and Riemannian geometry tools, since we are in the manifold setting.

The proof of Proposition 1.1 is based on

- a direct construction when $\mathcal{M}$ is the sphere $\mathbb{S}^2$, using the stereographic projection,
- the celebrated uniformisation theorem ([**1, 40**]) when $\mathcal{M}$ is simply connected.

(Remark: we no not know if the result remains true if $T' = T$.)

1.3 - *Relation to literature*

A similar problem is considered in [**39**], where stability estimates for the insolation function are obtained combining Carleman estimates with maximum principles, the main difference with the present paper being that the problem in [**39**] is stated and studied in the interval $(-1, 1)$ and with a degenerate diffusion coefficient.

Global Carleman estimates have proved their usefulness in the context of null controllability, unique continuation properties, we refer in particular to [**25**] for the seminal paper on the null controllability of the heat equation on compact manifolds, to [**18, 21**] for Carleman estimates in a general setting, to [**29**] for unique continuation properties for the heat equation on non compact manifolds, to [**31, 32**] for uniqueness results for manifolds with poles, to [**6**] for stabilization results of the wave equation on manifolds.

Concerning inverse problems, Isakov [**23**] provided many results for elliptic, hyperbolic and parabolic problems. Imanuvilov-Yamamoto [**22**] developped a general method to solve some standard inverse source problem for the linear heat equation, using global Carleman estimates. In the context of semilinear parabolic equations in bounded domains of $\mathbb{R}^n$, we can also mention in particular [**33, 34**], where uniqueness results are obtained under analyticity assumptions, [**11**], that combines also Carleman estimates with maximum principles to obtain stability estimates (for two coefficients but under rather strong assumptions on the time interval of observation).

1.4 - *Contents of the paper*

Let us now precise the organization of the paper.

- First of all, since the equation is stated on a surface, the operators needed for the definitions and the computations (Laplacian, divergence, gradient) are defined through a Riemaniann metric associated to the surface. So, in order to fix the ideas, we begin in Section 2 by introducing all the notations and recalling all the definitions and the properties useful for computations on manifolds.
- Next, in Section 3, we state and prove some global Carleman estimate for the heat operator on a compact manifold without boundary. This will be a crucial tool in order to study our inverse problem.
- In Section 4, we prove Proposition 1.1, studying first the case of the sphere $\mathbb{S}^2$, and then the general case of a simply connected manifold.
- In Section 5, we make some preliminary studies concerning the 2-D Sellers model on the manifold $\mathcal{M}$ (well-posedness of course but also regularity results and maximum principles that will also be essential in the proof of the stability result for the inverse problem).
- Finally, in Section 6, we prove Theorem 1.1.

2 - Notations, computations and heat operator on manifolds

In this section, we fix the notations and recall some classical definitions and results on manifolds. We refer in particular to [**9, 19**].

2.1 - *Notions on topological and Riemannian manifolds*

Charts, atlas, smooth manifolds. A topological manifold $\mathcal{M}$ of dimension n is a separated topological space such that every point $m \in \mathcal{M}$ has a neighbourhood U which is homeomorphic to some connected open subset of $\mathbb{R}^n$. For any neighbourhood U and any homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$, we say that (U, ϕ) is a coordinate chart on U. A set $(U_i, \phi_i)_{i\in I}$ such that the set of neighbourhoods U_i covers $\mathcal{M}$ is called an atlas on $\mathcal{M}$.

When two coordinate charts (U_1, ϕ_1) and (U_2, ϕ_2) have overlapping domains U_1 and U_2, there is a transition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ which is a homeomorphism between two open subsets of $\mathbb{R}^n$. A smooth manifold (or a $\mathcal{C}^\infty$-manifold) is a manifold for which all the transition maps are $\mathcal{C}^\infty$-diffeomorphims. *In the following, $\mathcal{M}$ always denotes a smooth manifold.*

Tangent vectors, tangent spaces, basis. A tangent vector at $m \in \mathcal{M}$ is an equivalence class $[c]$ of differentiable curves $c : I \to \mathcal{M}$ with I sub-interval of $\mathbb{R}$ such that $0 \in I$ and $c(0) = m$, modulo the equivalence relation of first order contact between curves i.e.

$$c_1 \equiv c_2 \ \Leftrightarrow c_1(0) = c_2(0) = m \ \text{ and } \ (\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$$

for every coordinate chart (U, ϕ) such that $m \in U$.

The tangent space to $\mathcal{M}$ at m, denoted by $T_m\mathcal{M}$, is the collection of all tangent vectors at m. Let (U, ϕ) be a chart such that $m \in U$ and define the map θ_ϕ:

$$\begin{array}{rccc} \theta_\phi : & T_m\mathcal{M} & \longrightarrow & \mathbb{R}^n \\ & [c] & \longmapsto & (\phi \circ c)'(0). \end{array}$$

Then $\theta_\phi : T_m\mathcal{M} \to \mathbb{R}^n$ is a bijection (see [**24**, p. 64]). Therefore $T_m\mathcal{M}$ can be endowed with a structure of a vector space. It is possible to exhibit a basis $(\partial_i(m))_{1\le i\le n}$ of $T_m\mathcal{M}$ in the following way. Let $m \in \mathcal{M}$ and (U, ϕ) be a chart of $\mathcal{M}$ such that $m \in U$. In $\phi(U) \subset \mathbb{R}^n$, we have n coordinate fields:

$$\forall 1 \le i \le n, \qquad \frac{\partial}{\partial x_i} : \begin{cases} \phi(U) & \to \mathbb{R}^n \\ x & \mapsto (0, 0, \dots, 1, 0, \dots, 0) \end{cases}$$

where 1 is at position i. Then we set

$$\forall 1 \le i \le n, \qquad \partial_i(m) = \theta_\phi^{-1}\left(\frac{\partial}{\partial x_i}(\phi(m))\right).$$

Regularity, derivatives. A continuous function $f : \mathcal{M} \longrightarrow \mathbb{R}$ is of class C^k if, for any $m \in \mathcal{M}$ and for any chart (U, ϕ) with $m \in U$, $f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is of class C^k.

Assume $f : \mathcal{M} \longrightarrow \mathbb{R}$ is of class C^1 and $m \in \mathcal{M}$. For any vector $\xi \in T_m\mathcal{M}$, the directional derivative of f at m along ξ, denoted by $\xi.f_m$ or $(\xi.f)(m)$, is:

$$\xi.f_m := (f \circ \omega)'(0),$$

where $\omega : I \longrightarrow \mathcal{M}$ satisfies $\omega(0) = m$ and $\omega'(0) = \xi$. For all $m \in \mathcal{M}$, the map $\alpha_m : \xi \longmapsto \xi.f_m$ is a linear form on $T_m\mathcal{M}$.

Let us explicit now the derivatives of f along each vector of the basis of the tangent space. Let $f : \mathcal{M} \longrightarrow \mathbb{R}$ be regular, $m \in \mathcal{M}$ and (U, ϕ) be a chart of $\mathcal{M}$ containing m. Then $\partial_i(m).f_m = (f \circ \omega_i)'(0)$ where $\omega_i : t \longmapsto \phi^{-1}(\phi(m) + t(0, ..., 1, 0, ...0))$. Moreover $(f \circ \omega_i)(t) = (f \circ \phi^{-1})(\phi(m) + t(0, ..., 1, 0, ...0))$. Hence $\partial_i(m).f_m = \dfrac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(m))$.

Tangent bundle, vector fields. The tangent bundle of a differentiable manifold $\mathcal{M}$ is a manifold $T\mathcal{M}$, which assembles all the tangent vectors at $\mathcal{M}$, that is $T\mathcal{M} = \cup_{m\in\mathcal{M}} T_m\mathcal{M} = \cup_{m\in\mathcal{M}}\{m\} \times T_m\mathcal{M}$. We denote by $\Pi : (m, \xi) \in T\mathcal{M} \to m \in \mathcal{M}$ the canonical projection.

Vector fields, derivative along a vector field. A vector field X on a manifold $\mathcal{M}$ is a regular map $X : \mathcal{M} \longrightarrow T\mathcal{M}$ such that $\Pi \circ X = Id_{\mathcal{M}}$ (i.e. $X(m) \in T_m\mathcal{M}$ for any $m \in \mathcal{M}$).

Let $X : \mathcal{M} \to T\mathcal{M}$ be a vector field on $\mathcal{M}$ and $f : \mathcal{M} \longrightarrow \mathbb{R}$ regular. We define $X.f : \mathcal{M} \longrightarrow \mathbb{R}$ the derivative of f along X in the following way: for all $m \in \mathcal{M}$, for any chart (U, ϕ) with $m \in U$,

$$(X.f)(m) = (f \circ \omega)'(0),$$

where $\omega : I \longrightarrow \mathcal{M}$ satisfies $\omega(0) = m$ and $\omega'(0) = X(m)$.

Lie bracket of two vector fields. The Lie bracket of two vector fields X and Y is a third vector field $[X, Y]$ defined by

$$\forall f : \mathcal{M} \longrightarrow \mathbb{R}, \qquad [X, Y].f := X.(Y.f) - Y.(X.f).$$

For the computations of Carleman inequalities, we will need the following result (see e.g. the proof in [38]): for all $1 \leq i, j \leq n$, then $[\partial_i, \partial_j] = 0$.

Riemannian manifolds. Let $\mathcal{M}$ be a smooth manifold. A Riemannian metric on $\mathcal{M}$ is a family $g = (g_m)_{m\in\mathcal{M}}$ of (positive definite) inner products $g_m := \langle , \rangle_m$ on $T_m\mathcal{M}$ for all $m \in \mathcal{M}$. Moreover the map $m \longmapsto g_m$ is assumed to be regular. Then we say that $(\mathcal{M}, g)$ is a Riemannian manifold.

Let $m \in \mathcal{M}$ and (U, ϕ) be a chart containing m, the matrix $G = (g_{j,k}) \in \mathcal{M}(n, \mathbb{R})$ of the scalar product $g_m := \langle , \rangle_m$ in the basis of $T_m\mathcal{M}$ is given by:

$$g_{j,k} := \langle \partial_j, \partial_k \rangle_m. \tag{14}$$

As $\langle , \rangle_m$ is a scalar product, G is invertible. We also denote

$$g := \det(G) \neq 0 \quad \text{and} \quad G^{-1} := (g^{i,l}). \tag{15}$$

Connexion on a manifold. A connexion on a manifold $\mathcal{M}$ is an operator D which associates to any vectors fields X and Y a third vector field $D_X Y$ on $\mathcal{M}$ such that, for all X, Y, Z vector fields and for all regular function $f : \mathcal{M} \longrightarrow \mathbb{R}$,

$$D_X(Y+Z) = D_X Y + D_X Z, \tag{16}$$

$$D_X(fY) = f D_X Y + (X.f)Y, \tag{17}$$

$$\xi \longmapsto D_\xi Y \text{ is linear on } T_m\mathcal{M} \text{ for all } m \in \mathcal{M}. \tag{18}$$

Levi-Civita connexion. From the fundamental theorem of Riemannian geometry, there is a unique connection Γ, called Levi-Civita connection, on the tangent bundle of a Riemannian manifold $(\mathcal{M}, g)$ such that:

- Γ is torsion-free, i.e. for all vectors fields X and Y on $\mathcal{M}$, then

$$\Gamma_X Y - \Gamma_Y X = [X, Y]; \tag{19}$$

- and Γ preserves the Riemannian metric g, i.e., for all vector fields X, Y, Z,

$$X.g(Y,Z) = g(\Gamma_X Y, Z) + g(Y, \Gamma_X Z). \tag{20}$$

Gradient. Let $f : \mathcal{M} \longrightarrow \mathbb{R}$ be a regular function. The gradient of f, denoted by grad (f) or ∇f, is the vector field on $\mathcal{M}$ defined for any $m \in \mathcal{M}$ as the unique vector grad $(f)_m$ such that

$$\forall \xi \in T_m\mathcal{M}, \quad \langle \mathrm{grad}(f)_m, \xi\rangle_m = (\xi.f)(m),$$

where $(\xi.f)(m)$ is the derivative of f at m in the direction ξ.

Divergence. For X vector field on $\mathcal{M}$, we define the function div (X) on $\mathcal{M}$ by

$$\forall m \in \mathcal{M}, \quad \mathrm{div}(X)(m) := Tr(\xi \longmapsto \Gamma_\xi X), \text{ where } \xi \text{ belongs to } T_m\mathcal{M}.$$

Laplacian. Let $f : \mathcal{M} \longrightarrow \mathbb{R}$ be a regular function. The Laplacian of f is the function Δf defined by:

$$\forall m \in \mathcal{M}, \qquad \Delta f_m := \mathrm{div}(\mathrm{grad}(f)_m)(m). \tag{21}$$

Hessian. Let f be a regular function on $\mathcal{M}$. Then, for all $m \in \mathcal{M}$, the Hessian of f at m is the bilinear form defined by:

$$\forall (\xi_1, \xi_2) \in (T_m\mathcal{M})^2, \qquad (\mathrm{Hess}(f)_m)(\xi_1, \xi_2) := \langle \Gamma_{\xi_1} \nabla f_m, \xi_2\rangle. \tag{22}$$

Rules for computations.

$$\mathrm{grad}(fh) = f\ \mathrm{grad}(h) + h\ \mathrm{grad}(f), \tag{23}$$

$$\mathrm{div}(X+Y) = \mathrm{div}(X) + \mathrm{div}(Y), \tag{24}$$

$$\mathrm{div}(fX) = f\,\mathrm{div}(X) + \langle \mathrm{grad}(f), X\rangle. \tag{25}$$

Expressions in local coordinates. It can be proved (see [**9**] p. 4-5), that for $f : \mathcal{M} \longrightarrow \mathbb{R}$ regular, X regular vector field on $\mathcal{M}$ and for all $m \in \mathcal{M}$, then

$$\text{grad}(f)_m := \sum_{k=1}^{n} \sum_{l=1}^{n} g^{k,l} \partial_l . f \partial_k . \tag{26}$$

$$\text{div}(X(m))_m = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \partial_i . (\eta^i \sqrt{g}) \quad \text{if } X = \sum_{i=1}^{n} \eta^i \partial_i . \tag{27}$$

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \sum_{l=1}^{n} \partial_i . (g^{i,l} \sqrt{g} \partial_l . f). \tag{28}$$

2.2 - *Integration on a compact manifold and Sobolev spaces*

In the following, $\mathcal{M}$ *is a compact connected oriented Riemannian manifold without boundary.* With the Riemann metric is associated an integration theory, the measure $d\mathcal{M}$ being defined globally on $\mathcal{M}$ with the help of a partition of unity (see [**9**], p. 5-6).

Then we have ([**9**] p. 6):

Proposition 2.1.

$$\forall X : \mathcal{M} \to T\mathcal{M} \textit{ regular}, \quad \int_{\mathcal{M}} \text{div}(X) d\mathcal{M} = 0, \tag{29}$$

and

$$\forall h, f : \mathcal{M} \to \mathbb{R} \textit{ regular}, \quad \int_{\mathcal{M}} h \Delta f + \langle \text{grad}(h), \text{grad}(f) \rangle d\mathcal{M} = 0. \tag{30}$$

***L^2*-spaces.** A function $f : \mathcal{M} \longrightarrow \mathbb{R}$ is measurable if, for any chart (U, Φ), $f \circ \Phi^{-1}$ is measurable. The space $L^2(\mathcal{M})$, constituted of the measurable functions $f : \mathcal{M} \longrightarrow \mathbb{R}$ such that $\int_{\mathcal{M}} |f|^2 d\mathcal{M}$ is finite, is a Hilbert space for the scalar product

$$(f, h)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} fh d\mathcal{M}.$$

Let X and Y be two regular vector fields. We define their scalar product by

$$(X, Y)_{L^2(T\mathcal{M})} := \int_{\mathcal{M}} \langle X, Y \rangle d\mathcal{M}. \tag{31}$$

Then $L^2(T\mathcal{M})$ is defined as the completion for the associated norm of the set of regular vector fields. It is a Hilbert space constituted of the vector fields whose components in the local basis of the tangent space are measurable and such that the integral $\int_{\mathcal{M}} |X|^2 d\mathcal{M}$ is finite.

Sobolev space $H^1(\mathcal{M})$. Let $\mathcal{M}$ be a compact Riemannian manifold of dimension n without boundary. If $f \in C(\mathcal{M})$ then $f \in L^2(\mathcal{M})$. As $\mathcal{M}$ is compact, the set of compactly supported C^∞-functions on $\mathcal{M}$ is simply the set of C^∞-functions on $\mathcal{M}$ and it is dense in $L^2(\mathcal{M})$ ([**2**] p. 79).

We define on $C^\infty(\mathcal{M})$ the scalar product $(.,.)_1$ in the following way:

$$\forall f, \tilde{f} \in C^\infty(\mathcal{M}), \qquad (f, \tilde{f})_1 := (f, \tilde{f})_{L^2(\mathcal{M})} + (\nabla f, \nabla \tilde{f})_{L^2(T\mathcal{M})}.$$

$H^1(\mathcal{M})$ is defined as the completion of $C^\infty(\mathcal{M})$ for the norm associated to $(.,.)_1$.

Weak derivative. Let $f \in L^2(\mathcal{M})$ be given. f admits a weak derivative in $L^2(T\mathcal{M})$ if there exists a vector field $\varsigma \in L^2(T\mathcal{M})$ such that, for any regular vector field X,

$$\int_{\mathcal{M}} f \ \mathrm{div}(X) d\mathcal{M} = - \int_{\mathcal{M}} \langle \varsigma, X \rangle d\mathcal{M}. \tag{32}$$

Then we denote $\varsigma = \nabla f$. Of course, if $f \in C^1(\mathcal{M})$, then it coincides with the classical gradient of f. $H^1(\mathcal{M})$ is also the set of functions in $L^2(\mathcal{M})$ having a weak derivative in $L^2(T\mathcal{M})$. It is endowed with the scalar product $(.,.)_1$.

Let us end this subsection by a general result (see [**35**] for its proof), that will be useful for the proofs of maximum principles:

Proposition 2.2. *Let $(U_i, \Phi_i)_{1 \le i \le N}$ be an atlas of $\mathcal{M}$. Then $f \in H^1(\mathcal{M})$ if and only if, for all $1 \le i \le N$, $f \circ \Phi_i^{-1} \in H^1(\Phi_i(U_i))$.*

2.3 - *The heat equation on a Riemannian manifold*

The Laplace Beltrami operator in $L^2(\mathcal{M})$. $f \in L^2(\mathcal{M})$ admits a weak Laplacian in $L^2(\mathcal{M})$ if there exists $F \in L^2(\mathcal{M})$ such that, for any $\Phi \in C^\infty(\mathcal{M})$,

$$(F, \Phi)_{L^2(\mathcal{M})} = (f, \Delta \Phi)_{L^2(\mathcal{M})}.$$

Then we denote $F = \Delta f$. Of course, if $f \in C^2(\mathcal{M})$, the weak Laplacian of f coincides with the classical one.

Proposition 2.3. *Let $f \in H^1(\mathcal{M})$ admitting a weak Laplacian in $L^2(\mathcal{M})$. Then, for all $\Phi \in H^1(\mathcal{M})$, $(\Delta f, \Phi)_{L^2(\mathcal{M})} = -(\nabla f, \nabla \Phi)_{L^2(T\mathcal{M})}$.*

The Laplace Beltrami operator is the unbounded operator in $L^2(\mathcal{M})$ defined by the domain $D(\Delta) := \{u \in H^1(\mathcal{M})$ having a weak Laplacian in $L^2(\mathcal{M})\}$ and the weak Laplacian. Note that, as $C^\infty(\mathcal{M}) \subset D(\Delta)$, $D(\Delta)$ is dense in $L^2(\mathcal{M})$. For all $u, v \in H^1(\mathcal{M})$, we define $a(u,v) := \int_{\mathcal{M}} \langle \nabla u, \nabla v \rangle d\mathcal{M}$. Then we define an unbounded operator in $L^2(\mathcal{M})$ by:

$$D(A) := \{u \in H^1(\mathcal{M}) : w \in H^1(\mathcal{M}) \longmapsto a(u,w) \text{ is } C^0 \text{ for the norm } \|.\|_{L^2(\mathcal{M})}\}$$

and for all $u \in D(A)$, $v \in H^1(\mathcal{M})$, $(Au, v)_{L^2(\mathcal{M})} = -a(u,v)$. The operator $(A, D(A))$ coincides with the Laplace-Beltrami operator $(\Delta, D(\Delta))$. Moreover, $(\Delta, D(\Delta))$ is the infinitesimal generator of an analytical semigroup.

The heat equation on a compact Riemannian manifold. We consider

$$\begin{cases} u_t - \Delta u = f & (0,T) \times \mathcal{M}, \\ u(0) = u_0 & \mathcal{M}. \end{cases} \tag{33}$$

The interpolation space $[D(\Delta), L^2(\mathcal{M})]_{\frac{1}{2}}$ is $H^1(\mathcal{M})$, (see [**27**, Prop. 21 p. 22]).

Theorem 2.1. *If $u_0 \in D(\Delta)$ et $f \in H^1(0,T; L^2(\mathcal{M}))$,* (33) *has a unique classical solution $u \in C([0,T], D(\Delta)) \cap C^1([0,T]; L^2(\mathcal{M}))$.*

If $u_0 \in H^1(\mathcal{M})$ et $f \in L^2(0,T; L^2(\mathcal{M}))$, (33) *has a unique solution such that $u \in L^2(0,T, D(\Delta)) \cap H^1(0,T; L^2(\mathcal{M}))$.*

If $u_0 \in L^2(\mathcal{M})$ et $f \in L^2(0,T; L^2(\mathcal{M}))$, (33) *has a unique weak solution such that $u \in C([0,T]; L^2(\mathcal{M})) \cap L^2(0,T; H^1(\mathcal{M}))$, i.e. for any $v \in H^1(\mathcal{M})$,*

$$\begin{cases} \dfrac{d}{dt}\Big(u(t), v\Big)_{L^2(\mathcal{M})} + \displaystyle\int_{\mathcal{M}} \langle \nabla u(t), \nabla v \rangle d\mathcal{M} = \Big(f(t), v\Big)_{L^2(\mathcal{M})}, \\ u(0) = u_0. \end{cases} \tag{34}$$

Moreover, for all $\varepsilon > 0$, $u \in L^2(\varepsilon, T; D(\Delta)) \cap H^1(\varepsilon, T; L^2(\mathcal{M}))$.

Proof. Apply Prop. 3.3 p. 68, Thm. 3.1 p. 80 and Prop. 3.8 of [3]. □

In order to treat later the questions of inverse problems, we will need some more regularity results for the time derivative of the solution:

Proposition 2.4. *Let $u_0 \in D(\Delta)$ and $f \in H^1(0,T; L^2(\mathcal{M}))$ be given. Let u be the classical solution of* (33) *associated to u_0 and f. Then $z := u_t \in L^2(0,T; H^1(\mathcal{M}))$ and z is the weak solution of*

$$\begin{cases} z_t - \Delta z = f_t & (0,T) \times \mathcal{M}, \\ z(0) = \Delta u_0 + f(0) & \mathcal{M}. \end{cases}$$

For the proof, we refer for example to [**38**, Proposition 2.5]. Finally, we end this section with a result concerning regular solutions (see [**10**] p. 139):

Theorem 2.2. *Let $u_0 \in C^\infty(\mathcal{M})$ and $f \in C^\infty((0,T) \times \mathcal{M})$ be given. Then* (33) *has a unique regular solution.*

3 - Global Carleman estimates for the heat operator on a compact manifold without boundary

In this section, we state and prove some global Carleman estimate for the heat operator on a compact Riemannian manifold without boundary $\mathcal{M}$ with a locally distributed observation in some non empty open set ω of $\mathcal{M}$.

3.1 - *Global Carleman estimate*

We define the heat operator on $\mathcal{M}$:

$$\forall z \in C([0,T]; D(\Delta_{\mathcal{M}})) \cap C^1([0,T]; L^2(\mathcal{M})), \qquad Pz := z_t - \Delta_{\mathcal{M}} z.$$

We denote $Q_{\mathcal{M}}^{0,T} := (0,T) \times \mathcal{M}$, $Q_{\omega}^{0,T} := (0,T) \times \omega$ and we consider $R > 0$, $S > 0$, ψ satisfying Assumption 1.2. Then we introduce first $0 < T_0 < T_1 < T$ and $\theta : (0,T) \to \mathbb{R}_+^*$ smooth, convex, such that

$$\theta(t) = \begin{cases} \dfrac{1}{t}, & t \in (0, T_0) \\ \dfrac{1}{T-t}, & t \in (T_1, T), \end{cases}$$

next

$$\forall (t,x) \in Q_{\mathcal{M}}^{0,T}, \quad p(x) := e^{2S\|\psi\|_\infty} - e^{S\psi(x)}, \quad \rho(t,x) := RS\theta e^{S\psi},$$

and finally

$$\forall (t,x) \in Q_{\mathcal{M}}^{0,T}, \quad \sigma(t,x) := \theta(t)p(x). \tag{35}$$

And we prove the following

Theorem 3.1. *Let ω be such that Assumption* 1.2 *holds. There exists constants $C = C(T, T_0, T_1, \omega) > 0$, $R_0 = R_0(T, T_0, T_1, \omega) > 0$, $S_0 = S_0(T, T_0, T_1, \omega) > 0$ such that, for all $S \geq S_0$ and all $R \geq R_0 e^{2S\|\psi\|_\infty}$, we have for all $z \in C([0,T]; D(\Delta_{\mathcal{M}})) \cap$*

$C^1([0,T];L^2(\mathcal{M}))$

$$(36)\quad \iint\limits_{Q_{\mathcal{M}}^{0,T}} \rho^3 e^{-2R\sigma} z^2 + \iint\limits_{Q_{\mathcal{M}}^{0,T}} \rho e^{-2R\sigma} |\nabla z|^2 + \iint\limits_{Q_{\mathcal{M}}^{0,T}} \frac{1}{\rho} e^{-2R\sigma} z_t^2 \leq C \left(\left\| e^{-R\sigma} P z \right\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + \iint\limits_{Q_{\omega}^{0,T}} \rho^3 e^{-2R\sigma} z^2 \right).$$

The proof of Theorem 3.1 is classical. It follows combining the proof of the Carleman estimate for the heat operator in a bounded domain of $\mathbb{R}^n$ with the properties of the operators divergence, gradient, laplacian on the manifold $\mathcal{M}$. We refer to [**38**] for detailed proofs, and we mention here the main properties and steps:

3.2 - *The basic properties*

The following property are basic:

Lemma 3.1. *For any regular function h on $\mathcal{M}$, one has:*

$$(37)\quad \nabla(h^2) = 2h\nabla h,$$

$$(38)\quad \nabla e^h = e^h \nabla h,$$

$$(39)\quad \Delta(h^2) = 2h\Delta h + 2|\nabla h|^2.$$

$$(40)\quad \langle \nabla(|\nabla h|^2), \nabla h\rangle = 2\,\mathrm{Hess}(h)(\nabla h, \nabla h).$$

Lemma 3.2. *For any $w \in C^\infty((0,T)\times\mathcal{M})$, one has:*

$$(41)\quad \nabla(w_t) = (\nabla w)_t.$$

$$(42)\quad \mathrm{Hess}(w)(\nabla w, \nabla p) = \mathrm{Hess}(w)(\nabla p, \nabla w).$$

Proof of Lemmas 3.1 and 3.2. The proofs are classical and derive from the basic material of Chavel [**9**], and can be found in [**38**], lemmas 3.3.4-3.3.7, p. 128-132. As an exercise, we prove (38): let $m \in \mathcal{M}$, (U,ϕ) be a chart such that $m \in U$ and $\xi \in T_m\mathcal{M}$. Consider $\omega : I \longrightarrow \mathcal{M}$ a smooth curve with $0 \in I$, $\omega(0) = m$ and $\omega'(0) = \xi$. Then, if we set $f = e^h$, we have (using the definition of the gradient):

$$\langle \nabla f(m), \xi\rangle_m = (\xi.f)(m) = (f\circ\omega)'(0) = (e^{h\circ\omega})'(0) = (e^{h\circ\omega})(0)(h\circ\omega)'(0) = e^{h(m)}(h\circ\omega)'(0)$$

and, on the other side, $\langle \nabla h(m), \xi\rangle_m = (\xi.h)(m) = (h\circ\omega)'(0)$. So, identifying the two expressions, we get $\nabla e^h = e^h\nabla h$, hence (38). The other proofs are in the same spirit. □

3.3 - *The main steps to prove Theorem* 3.1

First we note that it is sufficient to prove (36) for regular functions. Indeed we have the following result (see the proof in [38]):

Lemma 3.3. *Let* $u \in C([0,T]; D(\Delta)) \cap C^1([0,T]; L^2(\mathcal{M}))$ *be given. Consider* $(f_n)_n \subset \mathcal{D}((0,T)\times\mathcal{M})$ *converging to* Pu *in* $L^2((0,T)\times\mathcal{M})$ *and* $(u_{0,n})_n \subset C^\infty(\mathcal{M})$ *converging to* $u_0 \in H^1(\mathcal{M})$. *We denote by* u_n *the regular solution* (*given in Theorem* 2.2) *of* (33) *associated to* $u_{0,n}$ *and* f_n. *Then we have*

$$u_n \longrightarrow u \ \text{ in } \ L^2(0,T;L^2(\mathcal{M})), \qquad \nabla u_n \longrightarrow \nabla u \ \text{ in } \ L^2(0,T;L^2(T\mathcal{M})),$$
$$\text{and} \qquad (u_n)_t \longrightarrow u_t \ \text{in } L^2(0,T;L^2(\mathcal{M})).$$

3.3.1 - The decomposition of the weighted heat operator

So let $z \in C^\infty((0,T)\times\mathcal{M}) \cap C([0,T]\times\mathcal{M})$ be given and let us prove that z satisfies (36). We set $w := ze^{-R\sigma}$. Then we have

$$(we^{R\sigma})_t - \Delta(we^{R\sigma}) = P(we^{R\sigma}) = Pz. \tag{43}$$

We have $(we^{R\sigma})_t = w_t e^{R\sigma} + R\theta_t p w e^{R\sigma}$ and

$$\begin{aligned}\Delta(we^{R\sigma}) &= \operatorname{div}(\nabla(we^{R\sigma})) = \operatorname{div}(\nabla w e^{R\sigma}) + \operatorname{div}(w\nabla(e^{R\sigma}))\\ &= e^{R\sigma}\Delta w + 2\langle\nabla(w), Re^{R\sigma}\nabla\sigma\rangle + \Delta(e^{R\sigma})w.\end{aligned}$$

Of course $\nabla\sigma = \theta(t)\nabla p$. And $\Delta(e^{R\sigma}) = \operatorname{div}(\nabla(e^{R\sigma})) = \operatorname{div}(R\theta\nabla p e^{R\sigma})$. Hence

$$\Delta(e^{R\sigma}) = R\theta(e^{R\sigma}\Delta p + \langle\nabla p, \nabla(e^{R\sigma})\rangle) = R\theta\Delta p e^{R\sigma} + R^2\theta^2|\nabla p|^2 e^{R\sigma}.$$

This allows us to consider P_R^+ and P_R^- as follows:

$$P_R^+ w = R\theta_t p w - R^2\theta^2|\nabla p|^2 w - \Delta w, \tag{44}$$

$$P_R^- w = w_t - R\theta\Delta p w - 2R\theta\langle\nabla w, \nabla p\rangle, \tag{45}$$

so that

$$P_R^+ w + P_R^- w = e^{-R\sigma} Pz.$$

This implies that

$$\|P_R^+ w\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + \|P_R^- w\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + 2\langle P_R^+ w, P_R^- w\rangle_{L^2(Q^{0,T}_{\mathcal{M}})} = \|e^{-R\sigma}Pz\|^2_{L^2(Q^{0,T}_{\mathcal{M}})}. \tag{46}$$

3.3.2 - The expression of the scalar product

With some integrations by parts (see [**38**]), using Proposition 2.1 and the properties stated in Lemmas 3.1 and 3.2, we obtain

$$(47)\quad 2\langle P_R^+ w, P_R^- w\rangle_{L^2(Q_{\mathcal{M}}^{0,T})} = \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2\theta\theta_t|\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt})w^2 - 4\iint_{Q_{\mathcal{M}}^{0,T}} R^3\theta^3\,\mathrm{Hess}\,(p)(\nabla p,\nabla p)w^2 - 4\iint_{Q_{\mathcal{M}}^{0,T}} R\theta\,\mathrm{Hess}\,(p)(\nabla w,\nabla w).$$

The proof of Theorem 3.1 follows from suitable lower bounds of the terms appearing in (47).

3.3.3 - A bound from below of the zero order term of the scalar product

The main property is the following:

Lemma 3.4. *There exists $C>0$ independent of R and S such that*

$$(48)\qquad -4R^3\theta^3\,\mathrm{Hess}\,(p)(\nabla p,\nabla p) \ge -CR^3S^3\theta^3e^{3S\psi} + R^3S^4\theta^3e^{3S\psi}|\nabla\psi|^4.$$

Proof of Lemma 3.4. Since $\nabla p = -Se^{S\psi}\nabla\psi$, we have

$$\begin{aligned} -\mathrm{Hess}\,(p)(\nabla p,\nabla p) &= -\langle \Gamma_{\nabla p}\nabla p,\nabla p\rangle \\ &= -\langle \Gamma_{-Se^{S\psi}\nabla\psi}(-Se^{S\psi}\nabla\psi), -Se^{S\psi}\nabla\psi\rangle \\ &= -\langle -Se^{S\psi}\Gamma_{\nabla\psi}(-Se^{S\psi}\nabla\psi), -Se^{S\psi}\nabla\psi\rangle \\ &= -S^2e^{2S\psi}\langle \Gamma_{\nabla\psi}(-Se^{S\psi}\nabla\psi),\nabla\psi\rangle \\ &= -S^2e^{2S\psi}\langle -Se^{S\psi}\Gamma_{\nabla\psi}(\nabla\psi) + \nabla\psi.(-Se^{S\psi})\nabla\psi,\nabla\psi\rangle \\ &= -S^2e^{2S\psi}\Big(-Se^{S\psi}\langle\Gamma_{\nabla\psi}(\nabla\psi),\nabla\psi\rangle + \nabla\psi.(-Se^{S\psi})\langle\nabla\psi,\nabla\psi\rangle\Big). \end{aligned}$$

Now choose $m\in\mathcal{M}$, ω a smooth curve such that $\omega(0)=m$, $\omega'(0)=\nabla\psi$. Then

$$\begin{aligned} \nabla\psi.(-Se^{S\psi}) &= \frac{d}{dt_{/t=0}}(-Se^{S\psi(\omega(t))}) = -S^2e^{S\psi(m)}\frac{d}{dt_{/t=0}}(\psi(\omega(t))) \\ &= -S^2e^{S\psi(m)}\nabla\psi.\psi = -S^2e^{S\psi(m)}\langle\nabla\psi,\nabla\psi\rangle. \end{aligned}$$

Hence

$$-\operatorname{Hess}(p)(\nabla p,\nabla p)=S^2e^{2S\psi}\Big(Se^{S\psi}\langle\Gamma_{\nabla\psi}(\nabla\psi),\nabla\psi\rangle+S^2e^{S\psi}|\nabla\psi|^4\Big).$$

Hence

$$-R^3\theta^3\operatorname{Hess}(p)(\nabla p,\nabla p)=R^3S^3\theta^3e^{3S\psi}\big(\langle\Gamma_{\nabla\psi}(\nabla\psi),\nabla\psi\rangle+S|\nabla\psi|^4\big).$$

Therefore, there exists $C>0$ independent of R and S such that

$$-4R^3\theta^3\operatorname{Hess}(p)(\nabla p,\nabla p)\geq -CR^3S^3\theta^3e^{3S\psi}+R^3S^4\theta^3e^{3S\psi}|\nabla\psi|^4.$$

Hence (48) is proved. □

3.3.4 - A bound from below of the first order term of the scalar product

Now we turn to the last term of (47), and we prove the following

Lemma 3.5. *There exists $C>0$ independent of R and S such that*

$$-4\iint_{Q_{\mathcal M}^{0,T}}R\theta\operatorname{Hess}(p)(\nabla w,\nabla w)\geq\iint_{Q_{\mathcal M}^{0,T}}RS\theta e^{S\psi}|\nabla w|^2 -\frac{C}{S}\|P_R^+w\|^2_{L^2(Q_{\mathcal M}^{0,T})}-C\iint_{Q_{\mathcal M}^{0,T}}R^3S^3\theta^3e^{3S\psi}w^2. \tag{49}$$

Proof of Lemma 3.5. We have

$$\begin{aligned}\operatorname{Hess}(p)(\xi,\xi)&=\langle\Gamma_\xi\nabla p,\xi\rangle\\&=\langle\Gamma_\xi(-Se^{S\psi}\nabla\psi),\xi\rangle=\langle -Se^{S\psi}\Gamma_\xi(\nabla\psi)+\xi.(-Se^{S\psi})\nabla\psi,\xi\rangle\\&=-Se^{S\psi}\langle\Gamma_\xi(\nabla\psi),\xi\rangle+\langle -S^2e^{S\psi}\langle\nabla\psi,\xi\rangle\nabla\psi,\xi\rangle\\&=-Se^{S\psi}\langle\Gamma_\xi(\nabla\psi),\xi\rangle-S^2e^{S\psi}\langle\nabla\psi,\xi\rangle^2.\end{aligned}$$

Hence, there exists c_1 such that

$$\begin{aligned}-R\theta\operatorname{Hess}(p)(\nabla w,\nabla w)&=RS\theta e^{S\psi}\langle\Gamma_{\nabla w}(\nabla\psi),\nabla w\rangle+RS^2\theta e^{S\psi}\langle\nabla\psi,\nabla w\rangle^2\\&\geq -c_1RS\theta e^{S\psi}|\nabla w|^2+RS^2\theta e^{S\psi}\langle\nabla\psi,\nabla w\rangle^2,\end{aligned}$$

hence

$$-R\theta\operatorname{Hess}(p)(\nabla w,\nabla w)\geq -c_1RS\theta e^{S\psi}|\nabla w|^2. \tag{50}$$

On the other hand,

$$\langle RS\theta e^{S\psi} w, P_R^+ w\rangle = \langle RS\theta e^{S\psi} w, R\theta_t p w - R^2\theta^2 |\nabla p|^2 w - \Delta w\rangle$$
$$= \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}(R\theta_t p - R^2 S^2\theta^2 e^{2S\psi}|\nabla\psi|^2)w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} \langle \nabla(RS\theta e^{S\psi} w), \nabla w\rangle$$
$$= \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}(R\theta_t p - R^2 S^2\theta^2 e^{2S\psi}|\nabla\psi|^2)w^2$$
$$+ \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}|\nabla w|^2 + RS\theta e^{S\psi} w\langle \nabla\psi, \nabla w\rangle,$$

hence

$$\iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}|\nabla w|^2 = \langle RS\theta e^{S\psi} w, P_R^+ w\rangle$$
$$- \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}(R\theta_t p - R^2 S^2\theta^2 e^{2S\psi}|\nabla\psi|^2)w^2$$
$$- \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} w\langle \nabla\psi, \nabla w\rangle$$
$$\leq \frac{1}{2S}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + C\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3\theta^3 e^{3S\psi} w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} \frac{1}{2} RS\theta e^{S\psi}|\nabla w|^2.$$

Hence

$$\iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}|\nabla w|^2 \leq \frac{1}{S}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + 2C\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3\theta^3 e^{3S\psi} w^2. \tag{51}$$

From (50) and (51), we deduce that

$$- \iint_{Q_{\mathcal{M}}^{0,T}} R\theta \operatorname{Hess}(p)(\nabla w, \nabla w) \geq \frac{-c_1}{S}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} - 2Cc_1\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3\theta^3 e^{3S\psi} w^2$$
$$\geq \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi}|\nabla w|^2 - \frac{1+c_1}{S}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})}$$
$$-2C(1+c_1)\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3\theta^3 e^{3S\psi} w^2,$$

hence (49) is proved. □

3.3.5 - A first Carleman estimate

Now we are in position to obtain a first Carleman estimate: using (46), (47), (48), (49), and classical estimates of the type $|\theta_t| \leq C\theta^2$, $|\theta_{tt}| \leq C\theta^3$, we obtain that

$$\begin{aligned}
&\left\|e^{-R\sigma}Pz\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})}\\
&\quad = \left\|P^+_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + \left\|P^-_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + 2\langle P^+_R w, P^-_R w\rangle_{L^2(Q^{0,T}_{\mathcal{M}})}\\
&\quad \geq \left\|P^+_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + \left\|P^-_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})}\\
&\qquad + \iint_{Q^{0,T}_{\mathcal{M}}} (4R^2\theta\theta_t|\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt})w^2\\
&\qquad - 4\iint_{Q^{0,T}_{\mathcal{M}}} R^3\theta^3\,\mathrm{Hess}\,(p)(\nabla p, \nabla p)w^2\\
&\qquad - 4\iint_{Q^{0,T}_{\mathcal{M}}} R\theta\,\mathrm{Hess}\,(p)(\nabla w, \nabla w)\\
&\quad \geq \left\|P^+_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + \left\|P^-_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})}\\
&\qquad + \iint_{Q^{0,T}_{\mathcal{M}}} (4R^2\theta\theta_t|\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt})w^2\\
&\qquad + \iint_{Q^{0,T}_{\mathcal{M}}} \left(-CR^3S^3\theta^3e^{3S\psi} + R^3S^4\theta^3e^{3S\psi}|\nabla\psi|^4\right)w^2\\
&\qquad + \iint_{Q^{0,T}_{\mathcal{M}}} RS\theta e^{S\psi}|\nabla w|^2 - \frac{C}{S}\|P^+_R w\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} - C\iint_{Q^{0,T}_{\mathcal{M}}} R^3S^3\theta^3e^{3S\psi}w^2.
\end{aligned}$$

Hence, for S large enough,

$$\begin{aligned}
&\iint_{Q^{0,T}_{\mathcal{M}}} R^3S^4\theta^3e^{3S\psi}|\nabla\psi|^4w^2 + \iint_{Q^{0,T}_{\mathcal{M}}} RS\theta e^{S\psi}|\nabla w|^2\\
&\qquad + \frac{1}{2}\left\|P^+_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + \left\|P^-_R w\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})}\\
&+ \iint_{Q^{0,T}_{\mathcal{M}}} (4R^2\theta\theta_t|\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt})w^2 - C\int_0^T\int_{\mathcal{M}\setminus\omega} R^3S^3\theta^3e^{3S\psi}w^2\\
&\qquad \leq \left\|e^{-R\sigma}Pz\right\|^2_{L^2(Q^{0,T}_{\mathcal{M}})} + C\iint_{Q^{0,T}_{\omega}} R^3S^3\theta^3e^{3S\psi}w^2.
\end{aligned}$$

Moreover, assuming that Assumption 1.2 is satisfied, there exists $C_0 > 0$ such that $|\nabla(m)\psi| > C_0$ for all $m \in \mathcal{M} \setminus \omega$. Thus

$$\int_0^T \int_{\mathcal{M}\setminus\omega} R^3 S^3 \theta^3 e^{3S\psi} w^2 \leq \frac{C}{S} \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^4 \theta^3 e^{3S\psi} |\nabla\psi|^4 w^2.$$

We deduce, for S large enough,

$$\begin{aligned}
&\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} \Big(1 + \frac{S}{2}|\nabla\psi|^4\Big) w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 \\
&\qquad + \frac{1}{2}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + \|P_R^- w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} \\
&\qquad + \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2\theta\theta_t|\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt}) w^2 \\
&\qquad \leq \|e^{-R\sigma} P z\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + C \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.
\end{aligned}$$

Finally, using the properties of the function θ and $R \geq R_0 e^{2S\|\psi\|_\infty}$, we get

$$\begin{aligned}
(52)\quad &\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} \Big(1 + \frac{S}{4}|\nabla\psi|^4\Big) w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 \\
&\qquad + \frac{1}{2}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + \|P_R^- w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} \\
&\qquad \leq \|e^{-R\sigma} P z\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + C \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.
\end{aligned}$$

Going back to $z = e^{R\sigma} w$, we have

$$\begin{aligned}
(53)\quad &\iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} \Big(1 + \frac{S}{4}|\nabla\psi|^4\Big) e^{-2R\sigma} z^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} e^{-2R\sigma} |\nabla z|^2 \\
&\qquad + \frac{1}{2}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + \|P_R^- w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} \\
&\qquad \leq C'\|e^{-R\sigma} P z\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + C' \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} e^{-2R\sigma} z^2.
\end{aligned}$$

3.3.6 - End of the proof of Theorem 3.1

To complete the proof of Theorem 3.1, we only need to estimate z_t. First we estimate w_t, using $P_R^- w$: we have

$$w_t = P_R^- w + R\theta \Delta p w + 2R\theta \langle \nabla w, \nabla p\rangle = P_R^- w - \rho(\Delta\psi + S|\nabla\psi|^2)w - 2\rho\langle\nabla w, \nabla\psi\rangle,$$

Hence

$$\|\frac{w_t}{\sqrt{\rho}}\| \le C\|\frac{P_R^- w}{\sqrt{\rho}}\| + CS\|\sqrt{\rho}\, w\| + C\|\sqrt{\rho}\, \nabla w\|.$$

Using (52), we obtain that

$$\begin{aligned}(54)\quad & \iint\limits_{Q_{\mathcal{M}}^{0,T}} \Big(1+\frac{S}{4}|\nabla\psi|^4\Big)\rho^3 w^2 + \iint\limits_{Q_{\mathcal{M}}^{0,T}} \rho|\nabla w|^2 + \iint\limits_{Q_{\mathcal{M}}^{0,T}} \frac{1}{\rho} w_t^2 \\ &\qquad + \frac{1}{2}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + \frac{1}{2}\|P_R^- w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} \\ &\qquad\qquad \le C\|e^{-R\sigma} P z\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + C\iint\limits_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.\end{aligned}$$

Finally, going back to $z = e^{R\sigma} w$, we have

$$\begin{aligned}(55)\quad & \iint\limits_{Q_{\mathcal{M}}^{0,T}} \Big(1+\frac{S}{4}|\nabla\psi|^4\Big)e^{-2R\sigma}\rho^3 z^2 + \iint\limits_{Q_{\mathcal{M}}^{0,T}} e^{-2R\sigma}\rho|\nabla z|^2 + \iint\limits_{Q_{\mathcal{M}}^{0,T}} e^{-2R\sigma}\frac{1}{\rho} z_t^2 \\ &\qquad + \frac{1}{2}\|P_R^+ w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + \frac{1}{2}\|P_R^- w\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} \\ &\qquad \le C\|e^{-R\sigma} P z\|^2_{L^2(Q_{\mathcal{M}}^{0,T})} + C\iint\limits_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} e^{-2R\sigma} z^2.\end{aligned}$$

This gives (36) and completes the proof of Theorem 3.1. □

4 - Proof of Proposition 1.1

In this section, we study the validity of the geometrical Assumption 1.2.

4.1 - *The case of the sphere* $\mathbb{S}^2$

Let us prove that Assumption 1.2 is satisfied in the case of the sphere $\mathbb{S}^2$. Consider $\omega_{\mathbb{S}^2}$ a non-empty open domain of the sphere. Choose $N \in \omega_{\mathbb{S}^2}$, that will play the role of the North pole. Choose $S \in \omega_{\mathbb{S}^2}, S \neq N$. Consider a small neighborhood ω_N of N included in $\omega_{\mathbb{S}^2}$, and a small neighborhood ω_S of S included in $\omega_{\mathbb{S}^2}$ such that $\omega_N \cap \omega_S = \emptyset$.

Now consider π the stereographic projection of pole N:

$$\pi : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2.$$

Then $\Omega_\pi := \pi(\mathbb{S}^2 \setminus \omega_N)$ is a bounded domain of $\mathbb{R}^2$, $\pi(\omega_S)$ is an open subdomain of Ω_π. The classical geometrical lemma of Fursikov-Imanuvilov [**18**] (see also [**7**]) ensures that there exists

$$\psi_\pi : \Omega_T \to \mathbb{R}, \quad y \mapsto \psi_\pi(y)$$

smooth such that

$$\nabla \psi_\pi(y) = 0 \quad \Longrightarrow \quad y \in \pi(\omega_S).$$

Then consider

$$\psi_{\mathbb{S}^2} : \mathbb{S}^2 \setminus \omega_N \to \mathbb{R}, \quad \psi_{\mathbb{S}^2}(x) := \psi_\pi(\pi(x)).$$

Let us prove that

$$\nabla \psi_{\mathbb{S}^2}(x) = 0 \quad \Longrightarrow \quad x \in \omega_S.$$

Indeed, fix $x \in \mathbb{S}^2 \setminus \omega_N$ and consider any $\xi \in T_x\mathbb{S}^2$, and take a smooth curve $\gamma : I \to \mathbb{S}^2$, $\gamma(0) = x$, $\gamma'(0) = \xi$. Then

$$\langle \nabla \psi_{\mathbb{S}^2}(x), \xi \rangle = (\xi.\psi_{\mathbb{S}^2})(x) = \frac{d}{dt}_{/t=0} (\psi_{\mathbb{S}^2}(\gamma(t))) = \frac{d}{dt}_{/t=0} (\psi_\pi(\pi(\gamma(t)))).$$

Denote

$$\gamma_\pi : I \to \mathbb{R}^2, \quad \gamma_\pi(t) := \pi(\gamma(t)).$$

Then

$$\langle \nabla \psi_{\mathbb{S}^2}(x), \xi \rangle = \frac{d}{dt}_{/t=0} (\psi_\pi(\gamma_\pi(t)) = \nabla \psi_\pi(\pi(x)) \cdot \gamma_\pi'(0).$$

Since $\gamma_\pi'(0)$ can be taken arbitrary in $\mathbb{R}^2$, we obtain that

$$\nabla \psi_{\mathbb{S}^2}(x) = 0 \quad \Longrightarrow \quad \nabla \psi_\pi(\pi(x)) = 0,$$

which implies $\pi(x) \in \pi(\omega_S)$, hence $x \in \omega_S$. Then it is sufficient to extend $\psi_{\mathbb{S}^2}$ to $\mathbb{S}^2$. This can be done, it can bring new zeros of $\nabla \psi_{\mathbb{S}^2}$, but inside ω_N, hence inside $\omega_{\mathbb{S}^2}$. This proves that Assumption 1.2 is satisfied in the case of the sphere $\mathbb{S}^2$. □

4.2 - *The case of a simply connected oriented manifold of dimension* 2

Assume that $\mathcal{M}$ is simply connected, and still compact, oriented, of dimension 2 and without boundary. Then the celebrated theorem of uniformisation of

Riemann [1, 40] implies that there exists a C^1-diffeomorphism between $\mathcal{M}$ and the sphere $\mathbb{S}^2$. We denote it

$$\Phi : \mathcal{M} \to \mathbb{S}^2, \quad m \mapsto \Phi(m).$$

Consider also a (small) non-empty open subdomain $\omega_{\mathcal{M}}$ of $\mathcal{M}$, and denote

$$\omega_{\mathbb{S}^2} := \Phi(\omega_{\mathcal{M}}).$$

Then consider $\psi_{\mathbb{S}^2}$ constructed in the previous section, that satisfies

$$\nabla\psi_{\mathbb{S}^2}(x) = 0 \implies x \in \omega_{\mathbb{S}^2},$$

and

$$\psi_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}, \quad \psi_{\mathcal{M}}(m) := \psi_{\mathbb{S}^2}(\Phi(m)).$$

Then let us prove that

$$\nabla\psi_{\mathcal{M}}(m) = 0 \implies m \in \omega_{\mathcal{M}}.$$

Indeed, fix $m \in \mathcal{M}$ and consider any $\xi \in T_m\mathcal{M}$, $\gamma : I \to \mathcal{M}$ such that $\gamma(0) = m$, $\gamma'(0) = \xi$. Then

$$\langle \nabla\psi_{\mathcal{M}}(m), \xi\rangle_{\mathcal{M}} = (\xi.\psi_{\mathcal{M}})(m) = \frac{d}{dt}_{/t=0}(\psi_{\mathcal{M}}(\gamma(t))) = \frac{d}{dt}_{/t=0}(\psi_{\mathbb{S}^2}(\Phi(\gamma(t)))).$$

Denote

$$\gamma_{\mathbb{S}^2} : I \to \mathbb{S}^2, \quad \gamma_{\mathbb{S}^2}(t) := \Phi(\gamma(t)).$$

Then

$$\langle \nabla\psi_{\mathcal{M}}(m), \xi\rangle_{\mathcal{M}} = \frac{d}{dt}_{/t=0}(\psi_{\mathbb{S}^2}(\gamma_{\mathbb{S}^2}(t))) = \langle \nabla\psi_{\mathbb{S}^2}(\Phi(m)), \gamma'_{\mathbb{S}^2}(0)\rangle_{\mathbb{S}^2}.$$

Since $\gamma'_{\mathbb{S}^2}(0)$ may describe all the tangent directions at $\Phi(m)$, we obtain that

$$\nabla\psi_{\mathcal{M}}(m) = 0 \implies \nabla\psi_{\mathbb{S}^2}(\Phi(m)) = 0,$$

which implies $\Phi(m) \in \omega_{\mathbb{S}^2} = \Phi(\omega_{\mathcal{M}})$, hence $m \in \omega_{\mathcal{M}}$. This completes the proof of Proposition 1.1. □

5 - Preliminary study of the Sellers model on a manifold

5.1 - *Local existence of classical solutions*

In order to apply the theory in [28], we need to rewrite (1) as an evolution equation in $L^2(\mathcal{M})$. We recall that $(\Delta, D(\Delta)) = (A, D(A))$ defined in subsection 2.3. The natural energy space is $H^1(\mathcal{M})$ and the bilinear form a is $H^1(\mathcal{M})$-$L^2(\mathcal{M})$ coercive, i.e.

$$\exists\alpha > 0, \exists\beta \in \mathbb{R}, \forall v \in H^1(\mathcal{M}), \quad a(v, v) + \beta\|v\|^2_{L^2(\mathcal{M})} \geq \alpha\|v\|^2_{H^1(\mathcal{M})}.$$

To rewrite (1) as an evolution equation in $L^2(\mathcal{M})$, it remains to check that the second member of the equation takes its values in $L^2(\mathcal{M})$. So we define G by

$$G : \begin{cases} [0,T] \times H^1(\mathcal{M}) & \longrightarrow L^2(\mathcal{M}) \\ \qquad (t,u) & \longmapsto r(t)q\beta(u) - \varepsilon(u)u|u|^3. \end{cases}$$

If G is well-defined, then problem (1) on $[0,T]$ is equivalent to the evolution equation in $L^2(\mathcal{M})$

$$\begin{cases} u_t(t) + Au(t) = G(t,u(t)), \quad t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{56}$$

We prove

Lemma 5.1. *G is well defined on $[0,T] \times H^1(\mathcal{M})$ with values in $L^2(\mathcal{M})$. Moreover, G satisfies*

- $\forall t \in [0,T], \forall R > 0, \exists C > 0, \forall u_1, u_2 \in B_{H^1(\mathcal{M})}(0,R),$

$$\|G(t,u_1) - G(t,u_2)\|_{L^2(\mathcal{M})} \le C\|u_1 - u_2\|_{H^1(\mathcal{M})}. \tag{57}$$

- $\forall R > 0, \exists\, \theta \in (0,1), \exists C > 0, \forall u \in B_{H^1(\mathcal{M})}(0,R), \forall s,t \in [0,T],$

$$\|G(t,u) - G(s,u)\|_{L^2(\mathcal{M})} \le C|t-s|^\theta. \tag{58}$$

Proof. For the proof, we will need the following result (see [**37**, p. 14]):

Lemma 5.2. *For all $q \in [1,+\infty)$, $H^1(\mathcal{M}) \subset L^q(\mathcal{M})$ with continuous embedding .*

Let us first prove that G is well defined on $[0,T] \times H^1(\mathcal{M})$, with values in $L^2(\mathcal{M})$. We set $Q = rq$ and $Q_1 = \|Q\|_{L^\infty(\mathbb{R}\times I)}$. For $t \in [0,T]$, $u \in H^1(\mathcal{M})$, we write

$$\begin{aligned} \|G(t,u)\|^2_{L^2(\mathcal{M})} &= \int_I |\mathcal{R}_a(t,u) - \mathcal{R}_e(u)|^2 \le 2\int_{\mathcal{M}} Q(t,x)^2\beta(u)^2 + 2\int_{\mathcal{M}} \varepsilon(u)^2u^8 \\ &\le 2Q_1^2\|\beta\|^2_{L^\infty(\mathbb{R})} + 2\|\varepsilon\|^2_{L^\infty(\mathbb{R})}\int_{\mathcal{M}} u^8 \le 2Q_1^2\bar{C}\|\beta\|^2_{L^\infty(\mathbb{R})} + C\|u\|^8_{H^1(\mathcal{M})}, \end{aligned}$$

where we used Lemma 5.2 (with $\bar{C} = \int_{\mathcal{M}} 1 d\mathcal{M} < +\infty$).

Next, we prove that (57) is satisfied. Let $t \in [0,T]$, $R > 0$ and u_1, u_2 in $B_{H^1(\mathcal{M})}(0,R)$. Then

$$\|G(t,u_1)-G(t,u_2)\|^2_{L^2(\mathcal{M})} = \int_{\mathcal{M}} \Big|Q(t,x)(\beta(u_1)-\beta(u_2)) + \mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)\Big|^2$$
$$\leq 2Q_1^2\|\beta'\|^2_{L^\infty(\mathbb{R})} \int_{\mathcal{M}} |u_1-u_2|^2 + 2\int_{\mathcal{M}} |\mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)|^2$$
$$\leq 2Q_1^2\|\beta'\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_V + 2\int_{\mathcal{M}} |\mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)|^2.$$

To conclude the proof of (57), it remains to show

$$\int_{\mathcal{M}} |\mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)|^2 d\mathcal{M} \leq C\|u_1-u_2\|^2_{H^1(\mathcal{M})}, \tag{59}$$

for some $C>0$. We compute

$$\int_{\mathcal{M}} |\mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)|^2 \leq 3\int_{\mathcal{M}} |\varepsilon(u_1)-\varepsilon(u_2)|^2|u_1|^8 + 3\int_{\mathcal{M}} \varepsilon(u_2)^2|u_1-u_2|^2|u_1|^6 + 3\int_{\mathcal{M}} \varepsilon(u_2)^2|u_2|^2\Big(|u_1|^3-|u_2|^3\Big)^2. \tag{60}$$

So it remains to estimate the three terms in the right hand side of the above inequality. From the assumptions on ε (Assumption 1.1), we have:

$$\int_{\mathcal{M}} |\varepsilon(u_1)-\varepsilon(u_2)|^2|u_1|^8 \leq \|\varepsilon'\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_{L^4(\mathcal{M})}\|u_1\|^8_{L^{16}(\mathcal{M})},$$
$$\int_{\mathcal{M}} \varepsilon(u_2)^2|u_1-u_2|^2|u_1|^6 \leq \|\varepsilon\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_{L^4(\mathcal{M})}\|u_1\|^6_{L^{12}(\mathcal{M})},$$

and

$$\int_{\mathcal{M}} \varepsilon(u_2)^2|u_2|^2\big(|u_1|^3-|u_2|^3\big)^2$$
$$\leq \|\varepsilon\|^2_{L^\infty(\mathbb{R})}\int_{\mathcal{M}} |u_2|^2\big(|u_1|-|u_2|\big)^2\big(|u_1|^2+|u_1||u_2|+|u_2|^2\big)^2$$
$$\leq \|\varepsilon\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_{L^4(\mathcal{M})}\bigg(\int_{\mathcal{M}} |u_2|^4\big(|u_1|^2+|u_1||u_2|+|u_2|^2\big)^4\bigg)^{1/2}.$$

Using Lemma 5.2 and $u_1,u_2\in B_{H^1(\mathcal{M})}(0,R)$, we end the proof of (57).

Finally, we prove condition (58): for all $t, s \in [0, T]$,

$$\|G(t,u) - G(s,u)\|^2_{L^2(\mathcal{M})} = \int_{\mathcal{M}} |r(t) - r(s)|^2 q(x)^2 \beta(u(x))^2$$
$$\leq \bar{C}\|r'\|^2_{L^\infty(\mathbb{R})}\|q\|^2_{L^\infty(\mathcal{M})}\|\beta\|^2_{L^\infty(\mathbb{R})}|t-s|^2,$$

where $\bar{C} = \int_{\mathcal{M}} 1 d\mathcal{M} < +\infty$. This implies (58). □

We are now ready to deduce a result of local existence:

Theorem 5.1. *For all $u^0 \in D(\Delta)$, there exists $T^\star(u^0) \in (0, +\infty]$ such that, for all $0 < T < T^\star(u^0)$, problem (56) has a unique solution $u \in \mathcal{C}([0,T], D(\Delta)) \cap \mathcal{C}^1([0,T], L^2(\mathcal{M}))$. Moreover, if $T^\star(u^0) < +\infty$, then $\|u(t)\|_{H^1(\mathcal{M})} \to +\infty$ as $t \to T^\star(u^0)$.*

Proof. Since $(\Delta, D(\Delta))$ generates an analytical semigroup and since the interpolation space $[D(\Delta), L^2(\mathcal{M})]_{1/2}$ is $H^1(\mathcal{M})$, Lemma 5.1 allows to apply [**28**, Theorem 7.1.2] to (56). So there exists a unique weak solution defined until a maximal time $T^\star(u^0)$. Then [**28**, Proposition 7.1.8] implies that, if $T^\star(u^0) < +\infty$ then $\|u(t)\|_{H^1(\mathcal{M})} \to +\infty$ as $t \to T^\star(u^0)$. Moreover, since $Au^0 + G(0,u^0) \in L^2(\mathcal{M})$, [**28**, Proposition 7.1.10] ensures that, for all $T < T^\star(u^0)$, $u \in \mathcal{C}([0,T], D(\Delta)) \cap \mathcal{C}^1([0,T], L^2(\mathcal{M}))$. □

5.2 - *Weak maximum principle*

First we prove

Lemma 5.3. *Let $v \in H^1(\mathcal{M})$ and $M \geq 0$. Then $(u - M)^+ := \sup(u - M, 0) \in H^1(\mathcal{M})$ and $(u + M)^- := \sup(-(u+M), 0) \in H^1(\mathcal{M})$. Moreover*

$$\operatorname{grad}(u - M)^+(m) = \begin{cases} \operatorname{grad}(u)(m) & \textit{if } u(m) \geq M \\ 0 & \textit{otherwise} \end{cases} \tag{61}$$

$$\operatorname{grad}(u + M)^-(m) = \begin{cases} -\operatorname{grad}(u)(m) & \textit{if } u(m) \leq -M \\ 0 & \textit{otherwise} \end{cases} \tag{62}$$

Proof. In the context of a manifold, Lemma 5.3 replaces [**39**, Lemma 6.1] that is the classical result when working in an open subset of $\mathbb{R}^n$. Consider $(U_i, \Phi_i)_{1 \leq i \leq N}$ an atlas of $\mathcal{M}$. Let us first prove that: $\forall 1 \leq i \leq N$, $\forall f \in L^2(\mathcal{M})$,

$$\sup(f, 0) \circ \Phi_i^{-1} = \sup(f \circ \Phi_i^{-1}, 0) \text{ on } \Phi_i(U_i). \tag{63}$$

Indeed, let $y \in \Phi_i(U_i)$ be such that $(f \circ \Phi_i^{-1})(y) \geq 0$. Then $f(x) \geq 0$ with $x = \Phi_i^{-1}(y) \in U_i$. Consequently,

$$\sup(f \circ \Phi_i^{-1}, 0)(y) = (f \circ \Phi_i^{-1})(y) = f(x) = (\sup(f, 0) \circ \Phi_i^{-1})(y).$$

The reasoning is similar when $(f \circ \Phi_i^{-1})(y) \leq 0$. This proves (63).

Let us now prove Lemma 5.3. From Proposition 2.2, it suffices to show that, for all $1 \leq i \leq N$, $(u - M)^+ \circ \Phi_i^{-1} \in H^1(\Phi_i(U_i))$. But $u - M \in H^1(\mathcal{M})$, so, for all $1 \leq i \leq N$, $(u - M) \circ \Phi_i^{-1} \in H^1(\Phi_i(U_i))$. Using [**12**, Proposition 6 p. 934], $((u - M) \circ \Phi_i^{-1})^+ \in H^1(\Phi_i(U_i))$. But, from (63),

$$(u - M)^+ \circ \Phi_i^{-1} = ((u - M) \circ \Phi_i^{-1})^+.$$

So we proved that $(u - M)^+ := \sup(u - M, 0) \in H^1(\mathcal{M})$. Moreover, from [**12**, Prop. 6, p. 934], we know that

$$\nabla((u - M)^+ \circ \Phi_i^{-1})(y) = \begin{cases} \nabla(u \circ \Phi_i^{-1})(y) & \text{if } u(\Phi_i^{-1}(y)) \geq M, \\ 0 & \text{otherwise.} \end{cases}$$

From the local definition of the weak gradient (see the proof of Proposition 2.2),

$$\mathrm{grad}(u - M)^+(m) = \begin{cases} \displaystyle\sum_{l=1}^{n} \sum_{j=1}^{n} g^{lj} \frac{\partial}{\partial x_j}(((u - M) \circ \Phi_i^{-1}) \circ \Phi_i)\partial_l & \text{if } u(m) \geq M, \\ 0 & \text{otherwise.} \end{cases}$$

We immediately deduce (61). The proof of (62) is similar. □

Then we prove the following maximum principle:

Theorem 5.2. *Let $u^0 \in D(\Delta) \cap L^\infty(\mathcal{M})$ and $T^\star(u^0)$ defined by Theorem* 5.1. *We denote*

$$M := \max\left\{ \|u^0\|_{L^\infty(\mathcal{M})}, \left(\frac{\|q\|_{L^\infty(\mathcal{M})} \|r\|_{L^\infty(\mathbb{R})} \|\beta\|_{L^\infty(\mathbb{R})}}{\varepsilon_{min}} \right)^{1/4} \right\}. \quad (64)$$

Then the solution u of problem (1) *satisfies* $\|u\|_{L^\infty((0,T^\star(u^0))\times\mathcal{M})} \leq M$.

Proof. Theorem 5.2 replaces [**39**, theorem 3.3] obtained in case of the 1-dimensional Sellers model. The proof (based on Lemma 5.3) is similar so we omit it here. It can also be found in [**38**]. □

From Theorem 5.2, we deduce that, for all $u^0 \in D(\Delta) \cap L^\infty(\mathcal{M})$, $\|u\|_{L^2(\mathcal{M})}$ does not blow up as $t \to T^\star(u^0)$. However, this is not sufficient to ensure the existence of a *global* classical solution since we did not prove that $\|u\|_{H^1(\mathcal{M})}$ does not blow up. Before showing this, we begin by proving some regularity result on the time derivative of the solution.

5.3 - *Regularity of the time derivative of the solution of* (1)

We work with initial conditions defined in

$$\mathcal{U} := \{u^0 \in D(\Delta_{\mathcal{M}}) \cap L^\infty(\mathcal{M}) : Au^0 \in L^\infty(\mathcal{M})\}. \tag{65}$$

We denote:

$$W(0,T;H^1(\mathcal{M}),(H^1(\mathcal{M}))') := \{v \in L^2(0,T;H^1(\mathcal{M})) : v_t \in L^2(0,T;(H^1(\mathcal{M}))')\}.$$

Then we prove

Theorem 5.3. *Let $u^0 \in \mathcal{U}$ and u the corresponding solution of* (1). *Let T be such that $0 < T < T^\star(u^0)$ (where $T^\star(u^0)$ is defined in Theorem* 5.1). *Then $z := u_t$ belongs to $L^2(0,T;H^1(\mathcal{M}))$ and is solution of the following variational problem:*

$$\begin{cases} z \in W(0,T;H^1(\mathcal{M}),(H^1(\mathcal{M}))'), \\ \forall w \in H^1(\mathcal{M}), \quad \langle z_t(t), w\rangle + b(t,z(t),w) = \Big(r'(t)q\beta(u(t)), w\Big)_{L^2(\mathcal{M})}, \\ z(0) = -Au^0 + G(0,u^0), \end{cases} \tag{66}$$

where $b : [0,T] \times H^1(\mathcal{M}) \times H^1(\mathcal{M}) \longrightarrow \mathbb{R}$ is the time-dependent bilinear form:

$$b(t,v,w) = \int_{\mathcal{M}} \langle \mathrm{grad}(v), \mathrm{grad}(w)\rangle d\mathcal{M} + \int_{\mathcal{M}} \tilde{\pi}(t,x) vw d\mathcal{M},$$

with $\tilde{\pi}(t,x) := \mathcal{R}_e'(u(t,x)) - r(t)q(x)\beta'(u(t,x))$.

Proof. Consider $u^0 \in \mathcal{U}$. Multiplying the equation satisfied by u by $w \in H^1(\mathcal{M})$, we obtain, thanks to Proposition 2.3 : $\forall t \in [0,T]$,

$$\Big(z(t), w\Big)_{L^2(\mathcal{M})} + \Big(\nabla u(t), \nabla w\Big)_{L^2(T\mathcal{M})} = \Big(r(t)q\beta(u(t)) - \varepsilon(u(t))u(t)|u(t)|^3, w\Big)_{L^2(\mathcal{M})}.$$

In order to prove that $z \in L^2(0,T;H^1(\mathcal{M}))$, we use the method of differential quotients (see e.g. [**26**]). Let $0 < \delta < \dfrac{T}{2}$, $t \in (\delta, T-\delta)$ and $-\delta < s < \delta$. We observe that

$$\begin{cases} u_t(t+s) - \Delta u(t+s) &= Q(t+s)\beta(u(t+s)) - \mathcal{R}_e(u(t+s)), \\ u_t(t) - \Delta u(t) &= Q(t)\beta(u(t)) - \mathcal{R}_e(u(t)). \end{cases} \tag{67}$$

Then we define, for all $t \in (\delta, T-\delta)$,

$$u^{(s)}(t) := \frac{u(t+s)-u(t)}{s}.$$

For all $t \in (\delta, T-\delta)$, $u^s(t) \in H^1(\mathcal{M})$ and (67) implies

$$\frac{\partial u^{(s)}}{\partial t}(t) - \Delta u^{(s)}(t) = \frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s}. \tag{68}$$

Multiplying (68) by $u^{(s)}(t)$, using Proposition 2.3 and integrating over $(\delta, T-\delta)$, we get

$$\frac{1}{2}\|u^{(s)}(T-\delta)\|^2_{L^2(\mathcal{M})} + \int_\delta^{T-\delta} \left(\nabla u^{(s)}(t), \nabla u^{(s)}(t)\right)_{L^2(T\mathcal{M})} dt = \frac{1}{2}\|u^{(s)}(\delta)\|^2_{L^2(\mathcal{M})}$$
$$+ \int_\delta^{T-\delta}\int_{\mathcal{M}} \left[\frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s}\right] u^s(t). \tag{69}$$

With computations identical to [**39**, equations (6.11) and (6.12), p. 697], we have

$$\int_\delta^{T-\delta}\int_{\mathcal{M}} \frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} u^s(t) \tag{70}$$
$$\leq \bar{C}T\|\beta\|_{L^\infty(\mathbb{R})}\|q\|^2_{L^\infty(\mathcal{M})}\|r'\|^2_{L^\infty(\mathbb{R})}$$
$$+ \left(\frac{1}{2}\|\beta\|_{L^\infty(\mathbb{R})} + \|Q\|_{L^\infty(\mathbb{R}\times\mathcal{M})}\|\beta'\|_{L^\infty(\mathbb{R})}\right)\int_\delta^{T-\delta}\int_{\mathcal{M}} |u^{(s)}(t)|^2$$

where $\bar{C} = \int_{\mathcal{M}} 1 d\mathcal{M}$ and

$$\int_\delta^{T-\delta}\int_{\mathcal{M}} \frac{\mathcal{R}_e(u(t,x)) - \mathcal{R}_e(u(t+s,x))}{s} u^s(t) \leq C \int_\delta^{T-\delta}\int_{\mathcal{M}} |u^{(s)}(t,x)|^2. \tag{71}$$

Thanks to (70) and (71), (69) becomes

$$\int_\delta^{T-\delta} \|\nabla u^{(s)}(t)\|^2_{L^2(T\mathcal{M})} dt \leq \frac{1}{2}\|u^{(s)}(\delta)\|^2_{L^2(\mathcal{M})} + C + C\int_\delta^{T-\delta}\int_{\mathcal{M}} |u^{(s)}(t,x)|^2 d\mathcal{M}dt.$$

As $u \in \mathcal{C}^1([0,T]; L^2(\mathcal{M}))$, we obtain

$$\int_\delta^{T-\delta} \|\nabla u^{(s)}(t)\|^2_{L^2(T\mathcal{M})} dt \leq \frac{1}{2}\sup_{t\in[0,T]}\|u_t\|^2_{L^2(\mathcal{M})} + C + CT\sup_{t\in[0,T]}\|u_t\|^2_{L^2(\mathcal{M})}.$$

Consequently, the quantity $\int_{\delta}^{T-\delta} \|\nabla u^{(s)}(t)\|^2_{L^2(T\mathcal{M})} dt$ is bounded by a constant independent of s. So, there exists a sub-sequence, still denoted by $(u^{(s)})_s$, that weakly converges to some $v \in L^2(\delta, T-\delta; H^1(\mathcal{M}))$ as $s \to 0$. But $L^2(\delta, T-\delta; H^1(\mathcal{M}))$ is continuously embedded in $L^2(\delta, T-\delta; L^2(\mathcal{M}))$. So the sub-sequence $(u^{(s)})_s$ weakly converges to v in $L^2(\delta, T-\delta; L^2(\mathcal{M}))$, (see e.g. [**5**, Theorem III.9, p. 39]). But, from [8, Corollary 1.4.39, p. 15], $(u^{(s)})_s$ strongly converges to u_t in $L^2(\delta, T-\delta; L^2(\mathcal{M}))$. Hence $u_t = v \in L^2(\delta, T-\delta; H^1(\mathcal{M}))$. Moreover,

$$\begin{aligned}\|u_t\|_{L^2(\delta,T-\delta;H^1(\mathcal{M}))} &\le \limsup_{s\to 0} \|u^{(s)}\|_{L^2(\delta,T-\delta;H^1(\mathcal{M}))}\\ &\le \frac{1}{2}\sup_{t\in[0,T]}\|u_t\|^2_{L^2(\mathcal{M})} + C + CT\sup_{t\in[0,T]}\|u_t\|^2_{L^2(\mathcal{M})}.\end{aligned}$$

As the right hand side above does not depend on δ, we may let δ tends to 0 and we obtain that $z \in L^2(0,T;H^1(\mathcal{M}))$. □

Corollary 5.1. *Let* $u^0 \in \mathcal{U}$ *and* $0 < T < T^\star(u^0)$ *with* $T^\star(u^0)$ *defined by Theorem* 5.1. *Then the solution* z *of* (66) *satisfies*

$$\|z\|_{L^\infty((0,T)\times\mathcal{M})} \le e^{(\|\tilde{\pi}\|_{L^\infty((0,T)\times\mathcal{M})}+1)T} N,$$

with $N := max\left\{\|-Au^0 + G(0,u^0)\|_{L^\infty(\mathcal{M})}, \|r'\|_{L^\infty(\mathbb{R})}\|q\|_{L^\infty(\mathcal{M})}\|\beta\|_{L^\infty(\mathbb{R})}\right\}$.

Proof. This result replaces [**39**, Corollary 3.1] obtained in the case of the 1-dimensional Sellers model. The proof (that uses Lemma 5.3) is similar to the proof of [**39**, Corollary 3.1] for dimension 1. The main difficulty in the proof relies on the lack of coercivity of the bilinear form b so one has to introduce some auxiliary variational problem associated to some coercive bilinear form b_1. We omit the proof here. It can also be found in [38]. □

5.4 - *Global existence of the solutions of* (1)

Theorem 5.4. *Let* $u^0 \in \mathcal{U}$. *Then the solution* u *of* (1) *is defined on* $[0,+\infty)$, *i.e.* $T^\star(u^0) = +\infty$. *Consequently, Theorem* 5.2 *and Corollary* 5.1 *hold true with* $T^\star(u^0) = +\infty$.

Proof. Theorem 5.4 replaces [**39**, Theorem 3.5] obtained in the 1-dimensional case and it can be proved in a similar way (except the fact that computations are now on a manifold). So the proof (that can be found in [38]) is omitted. □

6 - Proof of Theorem 1.1

STEP 1: Reduction to some non standard linear inverse source problem. Let $T > 0$, $u_1, u_2 \in \mathcal{C}([0,T]; D(\Delta)) \cap \mathcal{C}^1\big([0,T]; L^2(\mathcal{M})\big)$ be the solutions of (1) corresponding respectively to q_1 with the initial condition u_1^0, and to q_2 with the initial condition u_2^0. We introduce $w := u_1 - u_2$. Then one can prove that $w \in \mathcal{C}([0,T]; D(\Delta)) \cap \mathcal{C}^1\big([0,T]; L^2(\mathcal{M})\big)$ solves

$$\begin{cases} w_t - \Delta w = H^* + H + \tilde{H} & (t,x) \in (0,T) \times \mathcal{M}, \\ w(0,x) = u_1^0 - u_2^0 & x \in \mathcal{M}, \end{cases} \tag{72}$$

with

$$H^* := r(q_1 - q_2)\beta(u_1), \tag{73}$$

$$H := rq_2(\beta(u_1) - \beta(u_2)), \tag{74}$$

$$\tilde{H} := \varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3. \tag{75}$$

As r and β are bounded from below (see Assumption 1.1), it suffices to estimate H^* to deduce an estimate of $q_1 - q_2$ in $L^2(\mathcal{M})$. So we reduced the problem to the determination of H^* in the above *linear* problem (72).

STEP 2: Condition satisfied by h_1. Let us recall that in inverse source problems, the source term has to satisfy some condition otherwise uniqueness may be false. Motivated by [22], we introduce the following condition: given $C_0 > 0$, we consider the condition

$$\left|\frac{\partial h}{\partial t}(t,x)\right| \le C_0 |h(T', x)| \text{ for almost all } (t,x) \in (0,T) \times \mathcal{M}, \tag{76}$$

and we define the set of C_0-admissible source terms:

$$\mathcal{G}(C_0) := \left\{h \in H^1(0,T; L^2(\mathcal{M})) | h \text{ satisfies (76)}\right\}.$$

Coming back to (72), we prove that the part H^* defined in (73) (and which is the part we wish to identify) is admissible (with some explicit C_0):

Lemma 6.1. *The function* $H^* = r(q_1 - q_2)\beta(u_1)$ *belongs to* $\mathcal{G}(C_0)$ *with* $C_0 > 0$ *defined by*

$$C_0 := \frac{\|r'\|_{L^\infty(\mathbb{R})}\|\beta\|_{L^\infty(\mathbb{R})} + \|r\|_{L^\infty(\mathbb{R})}\|\beta'\|_{L^\infty(\mathbb{R})} e^{(\|\tilde{\pi}_1\|_{L^\infty((0,T)\times\mathcal{M})}+1)T} N_1}{\beta_{min} r(T')},$$

where $\tilde{\pi}_1$ *is given in Theorem* 5.3 *with* $u^0 = u_1^0$ *and* N_1 *is given in Corollary* 5.1 *with* $u^0 = u_1^0$.

Proof. The proof is based on Corollary 5.1. As it is identical to the similar result established in [39, Lemma 7.1], we omit it. □

STEP 3: Application of global Carleman estimates and link with some more standard inverse source problem. In the following computations, C stands for generic constant depending on T, t_0, T', B, ω and the parameters in Assumption 1.1. Let us introduce $Z := w_t = u_{1,t} - u_{2,t}$ where w solves (72). Using Proposition 2.4, $Z \in L^2(t_0, T; D(\Delta)) \cap H^1\big(t_0, T; L^2(\mathcal{M})\big)$ and satisfies

$$Z_t - \Delta Z = H_t^* + H_t + \tilde{H}_t \quad (t,x) \in (t_0, T) \times \mathcal{M}. \tag{77}$$

Then we apply the Carleman estimate (36) to Z on the time interval (t_0, T), with $\theta : (t_0, T) \to \mathbb{R}_+^*$ smooth, convex, such that

$$\theta(t) = \begin{cases} \dfrac{1}{t - t_0} & t \in \left(t_0, \dfrac{t_0 + T'}{2}\right), \\[2ex] \dfrac{1}{T - t} & t \in \left(\dfrac{T' + T}{2}, T\right), \end{cases}$$

and θ attains its global minimum at T'. And we obtain

$$\begin{aligned} I_0 := \int_{t_0}^{T}\int_{\mathcal{M}} \rho^3 Z^2 e^{-2R\sigma} + \int_{t_0}^{T}\int_{\mathcal{M}} \rho |\nabla Z|^2 e^{-2R\sigma} + \int_{t_0}^{T}\int_{\mathcal{M}} \frac{1}{\rho} Z_t^2 e^{-2R\sigma} \\ \le C\left(\left\| e^{-R\sigma} P Z \right\|^2_{L^2((t_0,T)\times\mathcal{M})} + \int_{t_0}^{T}\int_{\omega} \rho^3 Z^2 e^{-2R\sigma} \right). \end{aligned} \tag{78}$$

Inequality (78) is the first step when dealing with standard inverse source problem, see [22]. Here the problem consists is retrieving only the part H^* in the source term $H^* + H + \tilde{H}$. First we estimate $\int_{t_0}^{T}\int_{\mathcal{M}} (H_t^2 + \tilde{H}_t^2) e^{-2R\sigma} d\mathcal{M} dt$ in the left hand side of (78):

Lemma 6.2. *There exists a constant $C > 0$ such that*

$$\int_{t_0}^{T}\int_{\mathcal{M}} (H_t^2 + \tilde{H}_t^2) e^{-2R\sigma} \le C\left(\int_{t_0}^{T}\int_{\mathcal{M}} Z^2 e^{-2R\sigma} + \int_{\mathcal{M}} w(T')^2 \right). \tag{79}$$

Proof. The proof is similar to the proof of [39, lemma 5.2] (using Theorem 5.2 and Corollary 5.1 instead of their analogous 1-dimensional forms) and can also be found in [38]. □

Coming back to (78), we deduce:

$$I_0 \leq C\left(\int_{t_0}^{T}\int_{\mathcal{M}} (H_t^*)^2 e^{-2R\sigma} + \int_{t_0}^{T}\int_{\mathcal{M}} Z^2 e^{-2R\sigma} + \int_{\mathcal{M}} w(T')^2 + \int_{t_0}^{T}\int_{\omega} \rho^3 Z^2 e^{-2R\sigma} \right). \tag{80}$$

For all $t \in (t_0, T)$, $1 \leq C\theta(t)$, so that, for R large,

$$C\int_{t_0}^{T}\int_{\mathcal{M}} Z^2 e^{-2R\sigma} \leq \frac{1}{2}\int_{t_0}^{T}\int_{\mathcal{M}} \rho^3 Z^2 e^{-2R\sigma}.$$

Hence, there exists $R_1 > 0$ and $C > 0$ such that: $\forall R \geq R_1$,

$$I_0 \leq C\left(\underbrace{\int_{t_0}^{T}\int_{\mathcal{M}} (H_t^*)^2 e^{-2R\sigma} + \int_{\mathcal{M}} w(T')^2 + \int_{t_0}^{T}\int_{\omega} \rho^3 Z^2 e^{-2R\sigma}}_{:=I_1} \right). \tag{81}$$

Let us note that, without the term $\int_{\mathcal{M}} w(T')^2 d\mathcal{M}$, inequality (81) would be the kind of inequality that one would obtain when dealing with the standard inverse source problem that consists in retrieving H^* in the equation $w_t - \Delta w = H^*$. Let us observe that this extra term satisfies

$$\int_{\mathcal{M}} w(T')^2 = \|(u_1 - u_2)(T', \cdot)\|^2_{L^2(\mathcal{M})} \leq \|(u_1 - u_2)(T', \cdot)\|^2_{D(\Delta)}.$$

Consequently, it can easily be estimated by the right hand side of (12).

STEP 4: Estimate from above of I_1. Let us prove that there exists $C > 0$ such that

$$I_1 \leq C\left[\frac{1}{\sqrt{R}} \int_{\mathcal{M}} (H^*(T'))^2 e^{-2R\sigma(T')} + \|w(T')\|^2_{L^2(\mathcal{M})} + \|w_t\|^2_{L^2((t_0,T)\times\omega)} \right]. \tag{82}$$

Indeed, there exists $p_{min} > 0$ such that $p(x) \geq p_{min}$ for all $x \in \mathcal{M}$, hence $\rho^3 e^{-2R\sigma(t,x)} \leq R^3 S^3 e^{3S\|\psi\|_\infty} \theta(t)^3 e^{-2Rp_{min}\theta(t)}$, and since $\theta(t)^3 e^{-2Rp_{min}\theta(t)} \to 0$ as $t \to t_0$ and as $t \to T$, there exists C such that

$$\int_{t_0}^{T}\int_{\omega} \rho^3 Z^2 e^{-2R\sigma} \leq C\|Z\|^2_{L^2((t_0,T)\times\omega)} = C\|w_t\|^2_{L^2((t_0,T)\times\omega)}.$$

Finally, the proof of (82) follows from

Lemma 6.3. *There exists $C > 0$ such that*

$$\int_{t_0}^{T} \int_{\mathcal{M}} (H_t^*)^2 e^{-2R\sigma} d\mathcal{M} dt \le C \frac{1}{\sqrt{R}} \int_{\mathcal{M}} (H^*(T'))^2 e^{-2R\sigma(T')} d\mathcal{M}. \tag{83}$$

Proof. Lemma 6.3 is classical in inverse source problems. We refer to [**22**] for its proof. Indeed, the fact that one works on a manifold does not change the reasoning. The key point is the form of the weight function θ. □

STEP 5: Estimate from below of I_0. Let us show that there exists $C = C(t_0, T) > 0$ such that

$$\int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} \le C I_0. \tag{84}$$

Indeed, since $Z(t,x)^2 e^{-2R\sigma(t,x)} \to 0$ as $t \to t_0$ for a.a. $x \in \mathcal{M}$, we can write

$$\int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} = \int_{t_0}^{T'} \frac{\partial}{\partial t} \left(\int_{\mathcal{M}} Z(t,x)^2 e^{-2R\sigma(t,x)} \right) = \int_{t_0}^{T'} \int_{\mathcal{M}} \left[2ZZ_t - 2R\sigma_t Z^2 \right] e^{-2R\sigma}. \tag{85}$$

First, we estimate

$$\int_{t_0}^{T'} \int_{\mathcal{M}} 2ZZ_t e^{-2R\sigma} = \int_{t_0}^{T'} \int_{\mathcal{M}} 2\sqrt{\rho} Z e^{-R\sigma} \frac{Z_t e^{-R\sigma}}{\sqrt{\rho}} \le \int_{t_0}^{T'} \int_{\mathcal{M}} \left(\rho Z^2 e^{-2R\sigma} + \frac{Z_t^2 e^{-2R\sigma}}{\rho} \right) \le C I_0. \tag{86}$$

Next we estimate the other term of (85): since $|\theta_t(t)| \le C\theta(t)^3$, we have

$$\int_{t_0}^{T'} \int_{\mathcal{M}} 2R|\sigma_t| Z^2 e^{-2R\sigma} \le C \int_{t_0}^{T'} \int_{\mathcal{M}} \rho^3 Z^2 e^{-2R\sigma} \le C I_0. \tag{87}$$

Finally, (85), (86) and (87) imply (84).

STEP 6: Conclusion. Using (84), (81) and next (82), there exists $C > 0$ such that

$$\int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} \leq \frac{C}{\sqrt{R}} \int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} + C\|w(T')\|^2_{L^2(\mathcal{M})} + C\|w_t\|^2_{L^2((t_0,T)\times\omega)}. \tag{88}$$

Let us recall that

$$Z(T') = w_t(T') = \Delta w(T') + H^*(T') + H(T') + \tilde{H}(T'),$$

hence

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \leq C\Bigg(\int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} |\Delta w(T')|^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')}\Bigg).$$

Applying (88), we get

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \leq C\Bigg(\frac{1}{\sqrt{R}}\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} + \|w_t\|^2_{L^2((t_0,T)\times\omega)} + \|w(T')\|^2_{D(\Delta)} + \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')}\Bigg).$$

Choosing R large enough so that $C/\sqrt{R} = 1/2$, we get

$$\frac{1}{2}\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \leq C\Big(\|w_t\|^2_{L^2((t_0,T)\times\omega)} + \|w(T')\|^2_{D(\Delta)} + \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')}\Big). \tag{89}$$

Let us now estimate the two last terms of the right hand side of (89). First, we recall that $|H| = |rq_2(\beta(u_1) - \beta(u_2))| \leq \|r\|_{L^\infty(\mathbb{R})} B \|\beta'\|_{L^\infty(\mathbb{R})} |u_1 - u_2|$. Therefore

$$\int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} \leq C \int_{\mathcal{M}} w(T')^2 e^{-2R\sigma(T')} \leq C\|w(T')\|^2_{L^2(\mathcal{M})}. \tag{90}$$

Next, we write

$$\begin{aligned}
|\tilde{H}| &= \big|\big(\varepsilon(u_2) - \varepsilon(u_1)\big)u_2|u_2|^3 + \varepsilon(u_1)(u_2 - u_1)|u_2|^3 + \varepsilon(u_1)u_1\big(|u_2|^3 - |u_1|^3\big)\big| \\
&\leq \|\varepsilon'\|_{L^\infty(\mathbb{R})}|u_2 - u_1||u_2|^4 + \|\varepsilon\|_{L^\infty(\mathbb{R})}|u_2 - u_1||u_2|^3 \\
&\quad + \|\varepsilon\|_{L^\infty(\mathbb{R})}|u_1|\big||u_2| - |u_1|\big|\big(|u_2|^2 + |u_2 u_1| + |u_1|^2\big).
\end{aligned}$$

By Theorem 5.2, for $i = 1, 2$, $\|u_i\|_{L^\infty((0,T)\times\mathcal{M})} \le C$. Hence,

$$|\tilde{H}| \le C|u_2 - u_1| + C\big||u_2| - |u_1|\big| \le C|u_2 - u_1|.$$

We deduce

$$\int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')} \le C\|w(T')\|^2_{L^2(\mathcal{M})}. \tag{91}$$

Finally, putting (90) and (91) into (89), we get

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \le C\Big[\|w_t\|^2_{L^2((t_0,T)\times\omega)} + \|w(T')\|^2_{D(\varDelta)}\Big].$$

On the other hand, R being now fixed, there exists some $C_{min} > 0$ such that $e^{-2R\sigma(T')} \ge C_{min} > 0$. Hence

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} d\mathcal{M} = \int_{\mathcal{M}} r(t)^2 |q_1(x) - q_2(x)|^2 \beta(u_1(T'))^2 e^{-2R\sigma(T')} d\mathcal{M}$$
$$\ge C_{min} r^2_{min} \beta^2_{min} \|q_1 - q_2\|^2_{L^2(\mathcal{M})}.$$

It follows

$$\|q_1 - q_2\|^2_{L^2(\mathcal{M})} \le C\Big[\|w_t\|^2_{L^2((t_0,T)\times\omega)} + \|w(T')\|^2_{D(\varDelta)}\Big].$$

This concludes the proof of Theorem 1.1. □

And (13) follows then immediately from the Carleman estimate of Theorem 3.1 and the stability estimate (12).

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