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Hidden regularity for wave equations with memory

Abstract. Our goal is to show a "hidden regularity" result for integro-differential equations, when the integral term is of convolution type. Under general assumptions on the integral kernel we are able to define the trace of the normal derivative of a weak solution. In such a way we extend to integro-differential equations well-known results available in the literature for wave equations without memory.

Keywords. Hidden regularity, Integro-differential equations.

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1 - Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) a bounded open domain of class C^2. Let us denote by ν the outward unit normal vector to the boundary Γ. By trace theory in Sobolev spaces it is well known that for any function $u \in H^2(\Omega)$ one can define the normal derivative (with respect to ν).

More precisely, we have that (see e.g. [**2**])

$$u \in H^2(\Omega) \Rightarrow \partial_\nu u \in L^2(\Gamma).$$

Giving a meaning to the normal derivative of functions belonging to $H^2(\Omega)$ is a basic step for extending Green's formula to Sobolev spaces.

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On the other hand, several authors investigated the regularity of solutions of PDEs, looking for properties that do not necessarily follow from the classical theory of Sobolev spaces. In the huge literature about this subject, it is worth to mention [**6, 7**], where the authors found out a trace theory interpretation for solutions of hyperbolic systems. In [**6, 9**] the authors studied the problem by means of pseudo-differential operator techniques.

Now we briefly recall some well-known results for the Cauchy problem for the wave equation with Dirichlet boundary conditions.

The hidden regularity property of the weak solutions means

$$\partial_\nu u \in L^2_{loc}(\mathbb{R}; L^2(\Gamma)).$$

The expression "hidden regularity" has been introduced in [**11**] for the case of a semilinear wave equation. Moreover, the hidden regularity of $\partial_\nu u$ has been established in [**10**], by using a direct PDEs method, see also [**12**] where that property has been applied to solve exact controllability problems for distributed systems.

For another approach based on Fourier series see [**13, 14**], although it works for special domains.

In this paper we will prove a hidden regularity result for weak solutions of integro-differential equations.

Indeed, we will consider the Cauchy problem for wave equations with a general integral term and Dirichlet boundary conditions:

$$\begin{cases} u_{tt}(t,x) - \triangle u(t,x) + \int_0^t k(t-s)\triangle u(s,x)ds = 0, \quad t \geq 0,\ x \in \Omega, \\ u(t,x) = 0, \qquad t \geq 0,\ x \in \Gamma, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \Omega. \end{cases} \tag{1}$$

We will assume that the integral kernel $k : [0,\infty) \to [0,\infty)$ is a locally absolutely continuous function such that $k(0) > 0$, $k'(t) \leq 0$ for a.e. $t \geq 0$ and $\int_0^\infty k(t)dt < 1$.

The result we will establish is the following.

Theorem 1.1. *Let $T > 0$. There is a constant $c_0 > 0$ depending on T and $\|k\|_{L^1(0,T)}$ such that for every $u_0 \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$, denoted by u the weak solution of* (1)*, one can define $\partial_\nu u$ in such a way that*

$$\int_0^T \int_\Gamma |\partial_\nu u|^2 d\Gamma dt \leq c_0(\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)}). \tag{2}$$

Inequality (2) extends to integro-differential equations the inequality obtained in [**6, 7**] for solutions of wave equations.

We will prove our result by showing first that inequality (2) holds true for strong solutions by means of the multipliers method. In particular, we note that we have to use a multiplier which depends on the integral kernel k, see below the proof of Lemma 3.1.

In a forthcoming paper we will extend our hidden regularity result to weak solutions of semilinear equations of the type

$$u_{tt}(t,x) - \triangle u(t,x) + \int_0^t k(t-s)\triangle u(s,x)ds + F(u) = 0, \tag{3}$$

where the nonlinearity F satisfies suitable assumptions, see [**1**] for an abstract version of equation (3). It is noteworthy to mention that such problem for equation (3) without memory has been investigated in [**11, 15**].

2 - Preliminaries

Throughout the paper, we will assume that the integral kernel satisfies the following conditions:

$$\begin{aligned} &k : [0,\infty) \to [0,\infty) \quad \text{is a locally absolutely continuous function,}\\ &k(0) > 0 \qquad\qquad k'(t) \le 0 \quad \text{for a.e. } t \ge 0,\\ &\int_0^\infty k(t)\, dt < 1. \end{aligned} \tag{4}$$

For reader's convenience we begin with recalling some known notions and results.

Definition 2.1. For *strong solution* of equation

$$u_{tt}(t,x) - \triangle u(t,x) + \int_0^t k(t-s)\triangle u(s,x)ds = 0 \qquad t \ge 0,\ x \in \Omega, \tag{5}$$

with Dirichlet boundary conditions

$$u(t,x) = 0 \qquad t \ge 0,\ x \in \Gamma, \tag{6}$$

we mean any function u belonging to $C^2([0,\infty);L^2(\Omega)) \cap C([0,\infty);H^2(\Omega)\cap H_0^1(\Omega))$ and satisfying the equation for all $t \ge 0$.

We call *weak solution* of (5)-(6) a function $u \in C^1([0,\infty);\ L^2(\Omega)) \cap C([0,\infty);$ $H_0^1(\Omega))$ such that for any $v \in H_0^1(\Omega)$, $t \to \int_\Omega u_t v\, dx \in C^1([0,\infty))$ and

$$(7)\quad \frac{d}{dt}\int_\Omega u_t v\, dx - \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega \int_0^t k(t-s)\nabla u(s)\, ds \cdot \nabla v\, dx = 0, \qquad \forall t \ge 0.$$

Clearly, a strong solution is a weak solution.

The following result states the existence of solutions and the dissipation of energy, see e.g. [**1, 16**].

Theorem 2.2. *Let us assume* (4)*. For any $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ there exists a unique weak solution u of the Cauchy problem*

$$(8)\quad \begin{cases} u_{tt}(t,x) - \triangle u(t,x) + \displaystyle\int_0^t k(t-s)\triangle u(s,x)ds = 0, & t \ge 0,\ x \in \Omega, \\ u(t,x) = 0, \qquad t \ge 0,\ x \in \Gamma, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \Omega. \end{cases}$$

In addition, if the initial data are more regular, that is $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ the weak solution of (8) *is a strong one.*

Moreover, the energy of a weak solution u, defined by means of formula

$$(9)\quad \begin{aligned} E(t) :=& \frac{1}{2}\int_\Omega |u_t(t,x)|^2\, dx + \frac{1}{2}\Big(1 - \int_0^t k(s)\, ds\Big)\int_\Omega |\nabla u(t,x)|^2\, dx \\ &+ \frac{1}{2}\int_\Omega \int_0^t k(t-s)|\nabla u(s,x) - \nabla u(t,x)|^2\, ds\, dx \qquad t \ge 0, \end{aligned}$$

is a decreasing function. In particular, we have

$$(10)\quad E'(t) = \frac{1}{2}\int_\Omega\int_0^t k'(t-s)|\nabla u(s,x) - \nabla u(t,x)|^2\, ds\, dx - \frac{1}{2}k(t)\int_\Omega |\nabla u(t,x)|^2\, dx \quad t \ge 0.$$

A well-known result concerning integral equations (see e.g. [**3**, Theorem 2.3.5]), that we will use later to establish our hidden regularity finding, is the following.

Lemma 2.3. *Let $k \in L^1(0,T)$ $(T > 0)$ and X a Banach space with norm $\|\cdot\|_X$. Then, for any $\varphi \in L^2(0,T;X)$ the function $\varphi(t) - \int_0^t k(t-s)\varphi(s)ds$ belongs to $L^2(0,T;X)$ and vice versa. Moreover, there exist two positive constants $c_i = c_i(\|k\|_{L^1(0,T)})$, $i = 1, 2$, depending on the norm $\|k\|_{L^1(0,T)}$, such that*

$$c_1 \int_0^T \|\varphi(t)\|_X^2 \, dt \le \int_0^T \left\|\varphi(t) - \int_0^t k(t-s)\varphi(s)ds\right\|_X^2 dt \le c_2 \int_0^T \|\varphi(t)\|_X^2 \, dt. \tag{11}$$

Throughout the paper we will also use a standard notation for the integral convolution between two functions, that is

$$k * u(t) := \int_0^t k(t-s)u(s)\, ds. \tag{12}$$

We denote with the symbol $\cdot$ the Euclidean scalar product in $\mathbb{R}^N$.

3 - The hidden regularity result

For sake of completeness we prefer to give a complete proof of the results of this section. We refer to [8] for the proof of the results for wave equations without memory. We will follow the approach pursued in [**4, 5**]. First, we need to introduce a technical lemma, that we will use in the proof of our main result.

Lemma 3.1. *Let $u \in H^2_{loc}((0,\infty); H^2(\Omega))$ be a function satisfying the following equation*

$$u_{tt}(t,x) - \triangle u(t,x) + \int_0^t k(t-s)\triangle u(s,x)ds = 0, \quad \text{in } (0,\infty)\times\Omega. \tag{13}$$

If $h : \overline{\Omega} \to \mathbb{R}^N$ is a vector field of class C^1, then for any fixed $S, T \in [0,\infty)$, $S < T$, the following identity holds true

$$(14)\quad \int_S^T\int_\Gamma \left[2\partial_\nu(u-k*u)\,h\cdot\nabla(u-k*u)-h\cdot\nu|\nabla(u-k*u)|^2+h\cdot\nu\,(u_t)^2\right]d\Gamma\,dt$$

$$=2\Big[\int_\Omega u_t\;h\cdot\nabla(u-k*u)\;dx\Big]_S^T+\int_S^T\int_\Omega\sum_{j=1}^N\partial_j h_j\;(u_t)^2\;dx\;dt$$

$$+2\int_S^T\int_\Omega u_t\;h\cdot\int_0^t k'(t-s)\big(\nabla u(s)-\nabla u(t)\big)ds\;dx\;dt$$

$$+2\int_S^T k(t)\int_\Omega u_t\;h\cdot\nabla u\;dx\;dt$$

$$+2\int_S^T\sum_{i,j=1}^N\int_\Omega \partial_i h_j\partial_i(u-k*u)\partial_j(u-k*u)\;dx\;dt$$

$$-\int_S^T\int_\Omega\sum_{j=1}^N\partial_j h_j\;|\nabla(u-k*u)|^2\;dx\;dt.$$

Proof. To begin with, we multiply the equation (13) by

$$2h\cdot\nabla\Big(u(t)-\int_0^t\;k(t-s)u(s)\;ds\Big)$$

and integrate over $[S,T]\times\Omega$. For simplicity, here and in the following we often drop the dependence on the variables.

First, we will handle the term with u_{tt}. Indeed, integrating by parts in the variable t gives

$$(15)\quad 2\int_S^T\int_\Omega u_{tt}\;h\cdot\nabla\Big(u(t)-\int_0^t\;k(t-s)u(s)\;ds\Big)\;dx\;dt$$

$$=2\Big[\int_\Omega u_t\;h\cdot\nabla\Big(u(t)-\int_0^t\;k(t-s)u(s)\;ds\Big)\;dx\Big]_S^T$$

$$-2\int_S^T\int_\Omega u_t\;h\cdot\nabla u_t\;dx\;dt+2\int_S^T\int_\Omega u_t\;h\cdot\nabla\Big(\int_0^t\;k'(t-s)u(s)\;ds+k(0)u(t)\Big)\;dx\;dt.$$

Now, we note that, if we integrate by parts in the variable x then we obtain

$$2\int_\Omega u_t\, h\cdot\nabla u_t\, dx = \int_\Omega h\cdot\nabla(u_t)^2\, dx = \int_\Gamma h\cdot\nu\, (u_t)^2\, d\Gamma - \int_\Omega \sum_{j=1}^N \partial_j h_j\, (u_t)^2\, dx. \tag{16}$$

In addition, we can write

$$\begin{aligned}\int_0^t k'(t-s)u(s)\, ds &= \int_0^t k'(t-s)(u(s)-u(t))\, ds + \int_0^t k'(s)u(t)\, ds\\ &= \int_0^t k'(t-s)(u(s)-u(t))\, ds + k(t)u(t) - k(0)u(t).\end{aligned}\tag{17}$$

Therefore, plugging (16) and (17) into (15) yields

$$\begin{aligned}&2\int_S\int_\Omega u_{tt}\, h\cdot\nabla\Big(u(t) - \int_0^t k(t-s)\, u(s)\, ds\Big)\, dx\, dt \\ &= 2\Big[\int_\Omega u_t\, h\cdot\nabla\Big(u(t) - \int_0^t k(t-s)\, u(s)\, ds\Big)\, dx\Big]_S^T - \int_S\int_\Gamma h\cdot\nu\, (u_t)^2\, d\Gamma\, dt \\ &+ \int_S\int_\Omega \sum_{j=1}^N \partial_j h_j\, (u_t)^2\, dx\, dt + 2\int_S\int_\Omega u_t\, h\cdot\int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big)\, ds\, dx\, dt \\ &+ 2\int_S k(t)\int_\Omega u_t\, h\cdot\nabla u\, dx\, dt.\end{aligned}\tag{18}$$

Now, to manage the terms with $\triangle u$, we set

$$w(t) = u(t) - \int_0^t k(t-s)u(s)\, ds, \tag{19}$$

so, we have to evaluate the term

$$2\int_S\int_\Omega \triangle w\, h\cdot\nabla w\, dx\, dt.$$

Integrating by parts in the variable x we get

$$(20)\quad \begin{aligned} &2\int_S^T\int_\Omega \triangle w\; h\cdot\nabla w\; dx\; dt \\ &= 2\int_S^T\int_\Gamma \partial_\nu w\; h\cdot\nabla w\; d\Gamma\, dt - 2\int_S^T\int_\Omega \nabla w\cdot\nabla(h\cdot\nabla w)\, dx\, dt. \end{aligned}$$

We observe that

$$(21)\quad \begin{aligned} 2\int_\Omega \nabla w\cdot\nabla(h\cdot\nabla w)\; dx &= 2\sum_{i,j=1}^N\int_\Omega \partial_i w\; \partial_i(h_j\partial_j w)\, dx \\ &= 2\sum_{i,j=1}^N\int_\Omega \partial_i h_j\partial_i w\partial_j w\, dx + 2\sum_{i,j=1}^N\int_\Omega h_j\partial_i w\; \partial_j(\partial_i w)\, dx, \end{aligned}$$

and

$$(22)\quad \begin{aligned} 2\sum_{i,j=1}^N\int_\Omega h_j\partial_i w\; \partial_j(\partial_i w)\, dx &= \sum_{j=1}^N\int_\Omega h_j\; \partial_j\Big(\sum_{i=1}^N(\partial_i w)^2\Big)dx \\ &= \int_\Gamma h\cdot\nu|\nabla w|^2\; d\Gamma - \int_\Omega\sum_{j=1}^N\partial_j h_j\; |\nabla w|^2\, dx. \end{aligned}$$

Therefore, by putting (21) and (22) into (20) we obtain

$$(23)\quad \begin{aligned} &2\int_S^T\int_\Omega \triangle w\; h\cdot\nabla w\; dx\, dt \\ &= 2\int_S^T\int_\Gamma \partial_\nu w\; h\cdot\nabla w\; d\Gamma\, dt - \int_S^T\int_\Gamma h\cdot\nu|\nabla w|^2\; d\Gamma\, dt \\ &\quad - 2\int_S^T\sum_{i,j=1}^N\int_\Omega \partial_i h_j\partial_i w\partial_j w\, dx\, dt + \int_S^T\int_\Omega\sum_{j=1}^N\partial_j h_j\; |\nabla w|^2\, dx\, dt. \end{aligned}$$

Finally, by (18) and (23), taking into account (19) we have the identity (14). □

Theorem 3.2. *Let $T > 0$. There is a constant $c_0 = c_0(T) > 0$ such that for every $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ the strong solution of*

$$\begin{cases} u_{tt}(t,x) - \triangle u(t,x) + \int_0^t k(t-s)\triangle u(s,x)ds = 0, & t \in (0,T),\ x \in \Omega, \\ u(t,x) = 0, \qquad t \in (0,T),\ x \in \Gamma, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{24}$$

satisfies the inequality

$$\int_0^T \int_\Gamma \left|\partial_\nu u - k * \partial_\nu u\right|^2 d\Gamma dt \le c_0(\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)}). \tag{25}$$

Proof. To begin with we consider a vector field $h \in C^1(\overline{\Omega}; \mathbb{R}^N)$ such that

$$h = \nu \qquad \text{on} \quad \Gamma, \tag{26}$$

see e.g. [4] for the construction of such vector field, and we denote with c a positive constant such that

$$|h(x)| \le c \quad \text{and} \quad \sum_{i,j=1}^N |\partial_i h_j(x)| \le c, \quad \forall x \in \overline{\Omega}. \tag{27}$$

We will apply the identity (14) with the vector field h satisfying (26) and with $S = 0$. First, we observe that

$$u_t = 0, \qquad \nabla u = (\partial_\nu u)\nu \quad \text{on} \quad (0,T) \times \Gamma. \tag{28}$$

For a detailed proof of the second identity see e.g. [**15**, Lemma 2.1]. Therefore, thanks to (28) the left-hand side of (14) becomes

$$\int_0^T \int_\Gamma \left|\partial_\nu u - k * \partial_\nu u\right|^2 d\Gamma dt.$$

To prove (25) we will show that every term on the right-hand side of (14) can be majorized by a positive constant depending on T multiplied by the initial energy $E(0)$. Indeed,

$$\begin{aligned} (29)\quad & 2\Big[\int_\Omega u_t\ h \cdot \nabla\big(u - k * u\big)\ dx\Big]_0^T \\ &= 2\int_\Omega u_t(T)\ h \cdot \nabla\big(u - k * u\big)(T)\ dx - 2\int_\Omega u_1\ h \cdot \nabla u_0\, dx \\ &\le c\int_\Omega |u_t(T)|^2\ dx + c\int_\Omega |\nabla\big(u - k * u\big)(T)|^2\, dx + c\int_\Omega |u_1|^2\ dx + c\int_\Omega |\nabla u_0|^2\, dx. \end{aligned}$$

We proceed to evaluate for all $t \in [0,T]$ the term $\int_\Omega |\nabla(u - k * u)(t)|^2\,dx$, because that evaluation will be also useful later. Since for all $t \in [0,T]$

$$\nabla u(t) - k * \nabla u(t) = \Big(1 - \int_0^t k(s)\,ds\Big)\nabla u(t) - \int_0^t k(t-s)\big(\nabla u(s) - \nabla u(t)\big)\,ds,$$

in view also of $k(t) \geq 0$ and $\int_0^\infty k(s)\,ds < 1$ we have

$$\begin{aligned}
|\nabla(u - k * u)(t)|^2 &\leq 2\Big(1 - \int_0^t k(s)\,ds\Big)^2 |\nabla u(t)|^2 \\
&\quad + 2\Big(\int_0^t k(t-s)\big|\nabla u(s) - \nabla u(t)\big|\,ds\Big)^2 \\
&\leq 2\Big(1 - \int_0^t k(s)\,ds\Big)|\nabla u(t)|^2 \\
&\quad + 2\int_0^t k(t-s)\big|\nabla u(s) - \nabla u(t)\big|^2\,ds,
\end{aligned}$$

whence, by means of formula (9) for the energy, we get

$$\begin{aligned}
(30)\qquad & \int_\Omega |\nabla(u - k * u)(t)|^2\,dx \\
&\leq 2\int_\Omega \left(\Big(1 - \int_0^t k(s)\,ds\Big)|\nabla u(t)|^2 + \int_0^t k(t-s)\big|\nabla u(s) - \nabla u(t)\big|^2\,ds\right)dx \\
&\leq 4E(t).
\end{aligned}$$

By putting (30) with $t = T$ into (29) and using again (9), we obtain

$$2\Big[\int_\Omega u_t\, h \cdot \nabla(u - k * u)\,dx\Big]_0^T \leq 6cE(T) + 2cE(0),$$

and hence, since the energy $E(t)$ is decreasing, see Theorem 2.2, we have

$$2\Big[\int_\Omega u_t\, h \cdot \nabla(u - k * u)\,dx\Big]_0^T \leq 8cE(0).$$

Now, we estimate the second term on the right-hand side of (14) by using (27), the expression of energy (9) and $E(t) \leq E(0)$ for all $t \in (0, T)$, that is

$$\int_0^T \int_\Omega \sum_{j=1}^N |\partial_j h_j| \, |u_t|^2 \, dx \, dt \leq 2c \int_0^T E(t) \, dt \leq 2cTE(0).$$

In order to bound the term

$$2\int_0^T \int_\Omega \Big| u_t \, h \cdot \int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big) \, ds \Big| dx \, dt$$

we note that, thanks also to (27), we have

$$2c \int_0^T \int_\Omega |u_t| \, \Big| \int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big) ds \Big| \, dx \, dt$$
$$\leq c \int_0^T \int_\Omega |u_t|^2 \, dx \, dt + c \int_0^T \int_\Omega \Big| \int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big) ds \Big|^2 dx \, dt. \tag{31}$$

To evaluate the second term on the right-hand side of the previous formula, we observe

$$\Big| \int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big) ds \Big|^2$$
$$\leq \Big(\int_0^t |k'(t-s)|^{1/2} |k'(t-s)|^{1/2} \big| \nabla u(s) - \nabla u(t) \big| ds \Big)^2$$
$$\leq \int_0^t |k'(s)| \, ds \int_0^t |k'(t-s)| \big| \nabla u(s) - \nabla u(t) \big|^2 ds$$
$$= -(k(0) - k(t)) \int_0^t k'(t-s) \big| \nabla u(s) - \nabla u(t) \big|^2 ds.$$

Therefore, in view of $k(t) \geq 0$ and formula (10), giving the derivative of the energy, from the above inequality we obtain

$$\int_0^T \int_\Omega \Big| \int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big)ds\Big|^2 dx\,dt$$

$$\leq -k(0)\int_0^T \int_\Omega \int_0^t k'(t-s)\big|\nabla u(s) - \nabla u(t)\big|^2\, ds\,dx\,dt \tag{32}$$

$$\leq -2k(0)\int_0^T E'(t)dt \leq 2k(0)E(0).$$

Plugging (32) into (31) and using Theorem 2.2 yield

$$2\int_0^T \int_\Omega \Big|u_t\, h \cdot \int_0^t k'(t-s)\big(\nabla u(s) - \nabla u(t)\big)\, ds\Big| dx\,dt$$

$$\leq c\int_0^T \int_\Omega |u_t|^2\, dx\; dt + 2ck(0)E(0)$$

$$\leq 2c\int_0^T E(t)\; dt + 2ck(0)E(0) \leq\; 2c(T + k(0))E(0).$$

Keeping in mind that $k(t) \leq k(0)$ and by using again (27), we get

$$\int_0^T k(t)\int_\Omega |u_t\, h \cdot \nabla u|\, dx\,dt$$

$$\leq ck(0)\int_0^T \int_\Omega |u_t||\nabla u|\, dx\,dt \;\leq\; \frac{ck(0)}{2}\int_0^T \int_\Omega |u_t|^2 + |\nabla u|^2\, dx\,dt.$$

In addition, thanks to $k(t) \geq 0$, $\int_0^\infty k(s)ds < 1$ and the definition of energy (9), we have

$$|\nabla u(t)|^2 \;\leq\; \frac{1 - \int_0^t k(s)ds}{1 - \int_0^\infty k(s)ds}\; |\nabla u(t)|^2 \;\leq\; \frac{2}{1 - \int_0^\infty k(s)ds}\; E(t),$$

so, in view also of $E(t) \le E(0)$ for all $t \in (0,T)$, we deduce

$$\int_0^T k(t) \int_\Omega |u_t \, h \cdot \nabla u| \, dx \, dt$$

$$\le ck(0)\Big(1 + \frac{1}{1 - \int_0^\infty k(s)ds}\Big) \int_0^T E(t) \; dt \;\le\; ck(0)\Big(1 + \frac{1}{1 - \int_0^\infty k(s)ds}\Big) TE(0).$$

Finally, to evaluate the last two terms on the right-hand side of (14) we will use the estimate (30). Indeed, as regards the first one, by means of (27) we have that

$$\int_0^T \sum_{i,j=1}^N \int_\Omega |\partial_i h_j \partial_i \big(u - k * u\big) \partial_j \big(u - k * u\big)| \, dx \, dt$$

$$\le c \int_0^T \int_\Omega \Big(\sum_{i=1}^N |\partial_i \big(u - k * u\big)| \Big)^2 dx \, dt \;\le\; 2^{N-1} c \int_0^T \int_\Omega |\nabla \big(u - k * u\big)|^2 \, dx \, dt.$$

Since, from (30) and $E(t) \le E(0)$, we obtain

$$\int_0^T \int_\Omega |\nabla \big(u - k * u\big)|^2 \, dx \, dt \;\le\; 4 \int_0^T E(t) \, dt \;\le\; 4TE(0), \tag{33}$$

thus it follows

$$\int_0^T \sum_{i,j=1}^N \int_\Omega |\partial_i h_j \partial_i \big(u - k * u\big) \partial_j \big(u - k * u\big)| \, dx \, dt \;\le\; c 2^{N+1} TE(0).$$

In a similar way, thanks again to (27) and (33) we have

$$\int_0^T \int_\Omega \sum_{j=1}^N |\partial_j h_j| \; |\nabla \big(u - k * u\big)|^2 \, dx \, dt \;\le\; c \int_0^T \int_\Omega |\nabla \big(u - k * u\big)|^2 \, dx \, dt \;\le\; 4cTE(0).$$

In conclusion, the previous argumentations show that all terms on the right-hand side of (14) can be majorized by a positive constant depending on T multiplied by $E(0)$. So, since

$$E(0) = \frac{1}{2} \|u_0\|^2_{H^1_0(\Omega)} + \frac{1}{2} \|u_1\|^2_{L^2(\Omega)},$$

we have proved estimate (25) for a suitable constant $c_0 > 0$. □

Corollary 3.3. *There exists a unique continuous linear map*

$$\mathcal{L} : H_0^1(\Omega) \times L^2(\Omega) \to L^2_{loc}((0,\infty); L^2(\Gamma))$$

such that for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$*, called* u *the strong solution of* (24)*, we have*

$$\mathcal{L}(u_0, u_1) = \partial_\nu u.$$

Proof. For $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, if we denote by u the strong solution of problem (24) and apply Lemma 2.3 with $X = L^2(\Gamma)$, then for any $T > 0$, thanks to (25) and (11) there exists a constant $c_0 = c_0(T, \|k\|_{L^1(0,T)}) > 0$ such that

$$\int_0^T \int_\Gamma |\partial_\nu u|^2 d\Gamma dt \le c_0(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}).$$

By density our claim follows. □

Remark 3.4. Corollary 3.3 allows us to introduce the notation $\partial_\nu u$ instead of $\mathcal{L}(u_0, u_1)$ for $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and u the weak solution of (24). Indeed, we have the following trace theorem:

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \Rightarrow \partial_\nu u \in L^2_{loc}((0,\infty); L^2(\Gamma)),$$

and for any $T > 0$ there is a positive constant c_0 depending on T and $\|k\|_{L^1(0,T)}$ such that

$$(34) \qquad \int_0^T \int_\Gamma |\partial_\nu u|^2 d\Gamma dt \le c_0(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}) \quad \forall (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega).$$

This result does not follow from the usual trace theorems of the Sobolev spaces. For this reason it is called a hidden regularity result. The corresponding inequality (34) is often called a direct inequality.

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