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Ground state solutions for a system of weakly coupled nonlinear fractional equations in the entire space

Abstract. We show the existence of a nontrivial ground state solution for a class of nonlinear pseudo–relativistic systems in the entire space.

Keywords. Fractional operators, nonlinear system, entire solutions, ground state solutions.

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In this paper we consider a system of two massive particles in presence of a pseudorelativistic kinetic energy moving in the whole $N-$dimensional space. More precisely, we consider generalizations of the prototype system

$$(1)\qquad \begin{cases} i\psi_t = \sqrt{-\Delta + m_1^2}\,\psi - |\psi|^{2p-2}\psi - |\phi|^p|\psi|^{p-2}\psi & \text{in } \mathbb{R}^N, \\ i\phi_t = \sqrt{-\Delta + m_2^2}\,\phi - |\phi|^{2p-2}\phi - |\psi|^p|\phi|^{p-2}\phi & \text{in } \mathbb{R}^{\mathrm{N}}, \end{cases}$$

where $\psi, \phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ are the wave functions, $m_i > 0$ denote the masses of the two particles and p is a positive superlinear and subcritical exponent (see below for the precise condition). The nonlinear terms appearing in the equations describe the interaction between the two particles.

The Hamiltonian operator $\mathcal{H}_m = \sqrt{-\Delta + m^2}$, which coincides with the so–called *half Laplacian* $(-\Delta)^{1/2}$ when $m = 0$, describes the kinetic and rest energy of a

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relativistic particle of mass $m \geq 0$ (more precisely, the kinetic energy of a fermion with mass $m > 0$ is described by the pseudo–differential operator $\sqrt{-\Delta + m^2} - m$, after some constant–renormalization). Such an operator can be defined, for example, by associating to $\sqrt{-\Delta + m^2}$ its symbol $\sqrt{k^2 + m^2}$ in Fourier spaces in the following way: for any $f \in H^1(\mathbb{R}^N)$ with Fourier transform $\mathcal{F}f$, set

$$\mathcal{F}\Big(\sqrt{-\Delta + m^2} f\Big)(k) = \sqrt{|k|^2 + m^2}\,\mathcal{F}f(k),$$

which is actually well defined in $H^{1/2}(\mathbb{R}^N)$, see [**11**] for a complete description of this method. The fact that, from this point of view, the natural space where defining the governing operator is $H^{1/2}(\mathbb{R}^N)$, forces to decrease the range of natural exponents in the nonlinearities according to the Sobolev Embedding Theorem, see below.

This type of Hamiltonian has been used in several context, and it turns out that it can be quite useful in Celestial Mechanics or general Physics, see, for instance, [**5**], [**6**], [**7**], [**9**], [**10**], [**12**], [**13**]. Let us also note that, in case of non–massive particles, i.e. when $m = 0$, the system could be be treated directly through the integral definition of $\sqrt{-\Delta}$, namely

$$\sqrt{-\Delta}u(x) := -C_N \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+1}}\,dy,$$

for some $C_N > 0$, as done, for instance, in [**1**], see also [**18**].

However, in place of using the Fourier approach, this Hamiltonian can be treated through an extension to a higher dimensional Euclidean space by the “Dirichlet to Neumann” procedure (see, for example, [**3**] and also [**2**] and [**21**] for related problems in a bounded domain), which consists in realizing the nonlocal operator in $\mathbb{R}^N$ into a local one settled in $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty)$. Let us briefly recall this method: for any function $U \in \mathscr{S}(\mathbb{R}^N)$ there exists a unique function $u \in \mathscr{S}(\mathbb{R}^{N+1}_+)$ such that

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ u(x,0) = U(x) & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\}, \end{cases} \tag{2}$$

i.e. u is the generalized harmonic extension of U in $\mathbb{R}^{N+1}_+$. Now, consider the operator T defined as

$$Tu(x) = -\frac{\partial u}{\partial y}(x,0).$$

Then, the system

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(x,0) = -\dfrac{\partial w}{\partial y}(x,0) = TU(x) & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\}, \end{cases}$$

admits the unique solution $w(x,y) = -\dfrac{\partial u}{\partial y}(x,y)$, and thus, by (2),

$$T(TU)(x) = -\frac{\partial w}{\partial y}(x,0) = \frac{\partial^2 u}{\partial y^2}(x,0) = (-\Delta_x u + m^2 u)(x,0),$$

where we have denoted by Δ_x the Laplace operator in the x–variables. In conclusion, $T^2 = -\Delta_x + m^2$, i.e the operator T mapping the Dirichlet datum U to the Neumann datum $-\dfrac{\partial u}{\partial y}(\cdot, 0)$ is a square root of the operator $-\Delta + m^2$ in $\mathbb{R}^N$, see also [**4**] and [**20**].

We are interested in solitary wave solutions of (1), i.e. solutions of the form $\psi(x,t) = e^{-i\omega_1 t}U(x)$, $\phi = e^{-i\omega_2 t}V(x)$ where $\omega_i \in \mathbb{R}$ and $U, V : \mathbb{R}^N \to \mathbb{R}$; thus, since the Fourier transform acts only on the x–variables, it is readily seen that the couple (U, V) solves the system

$$\begin{cases} \sqrt{-\Delta + m_1^2}\, U = \omega_1 U + |U|^{2p-2}U + |V|^p |U|^{p-2} U & \text{in } \mathbb{R}^N, \\ \sqrt{-\Delta + m_2^2}\, V = \omega_2 V + |V|^{2p-2}V + |U|^p |V|^{p-2} V & \text{in } \mathbb{R}^N. \end{cases} \tag{3}$$

Actually, there is no reason to discard an x–dependence in the potential, or to neglect more general nonlinearities; for this reason we shall consider the following nonlinear system:

$$\begin{cases} \sqrt{-\Delta + m_1^2}\, U = \omega_1 U + G_U(x, U, V) & \text{in } \mathbb{R}^N, \\ \sqrt{-\Delta + m_2^2}\, V = \omega_2 V + G_V(x, U, V) & \text{in } \mathbb{R}^N, \end{cases} \tag{4}$$

where $G : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying suitable assumptions described below. Of course, coherently with system (3), our prototype will be

$$G(x, U, V) = \frac{1}{2p}(|U|^{2p} + |V|^{2p}) + \frac{1}{p}|UV|^p. \tag{5}$$

Using the approach with the operator T introduced above, we rewrite system (4) as

$$\begin{cases} -\Delta u + m_1^2 u = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\Delta v + m_2^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial u}{\partial y} = \omega_1 u + G_u(x, u, v) & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+, \\ -\dfrac{\partial v}{\partial y} = \omega_2 v + G_v(x, u, v) & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+, \end{cases} \tag{6}$$

where, with abuse of notation, we have set $G_U = G_u$ and $G_V = G_v$. In this way, $U = \mathrm{tr}\, u$ and $V = \mathrm{tr}\, v$ will solve system (4), where "tr" stands for the usual boundary trace operator in $\mathbb{R}^N$.

Hence, we shall look for solutions of system (6) in the Hilbert space $\mathcal{H} = H_{m_1}(\mathbb{R}^{N+1}_+) \times H_{m_2}(\mathbb{R}^{N+1}_+)$, where, for any $m \in \mathbb{R}$, $m \neq 0$, we have written $H_m(\mathbb{R}^{N+1}_+) = H^1(\mathbb{R}^{N+1}_+)$, endowed with the inner product

$$\langle u, v\rangle_m = \int_{\mathbb{R}^{N+1}_+} Du \cdot Dv\, dx + m^2 \int_{\mathbb{R}^{N+1}_+} uv\, dx.$$

Definition 0.1. A couple $(u, v) \in \mathcal{H}$ is said to be a (weak) *solution* of problem (6) iff for any $w \in H^1(\mathbb{R}^{N+1}_+)$

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} (Du \cdot Dw + m_1^2 uw)\, dxdy = \int_{\mathbb{R}^N} \big[\omega_1 u + G_u(x, u, v)\big] w\, dx \\ &\int_{\mathbb{R}^{N+1}_+} (Dv \cdot Dw + m_2^2 vw)\, dxdy = \int_{\mathbb{R}^N} \big[\omega_2 v + G_v(x, u, v)\big] w\, dx. \end{aligned} \tag{7}$$

Before going on, we introduce some notations:

(x, y)	a point of $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty)$,
$\lVert u\rVert$	the norm of u in $H^1(\mathbb{R}^{N+1}_+)$,
$\lVert u\rVert_q$	the norm of u in $L^q(\mathbb{R}^{N+1}_+)$,
$\lvert u\rvert_q$	the norm of the trace of u in $L^q(\mathbb{R}^N)$,
$\lVert (u, v)\rVert$	the norm of (u, v) in $\mathcal{H}$,
$\lVert (u, v)\rVert_q$	the norm of (u, v) in $L^q(\mathbb{R}^{N+1}_+) \times L^q(\mathbb{R}^{N+1}_+)$.

Indeed, we recall that any $u \in H^1(\mathbb{R}^{N+1}_+)$ admits trace (still denoted by u for simplicity) on $\partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N$, so that we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x, 0)|^q dx &= -\int_{\mathbb{R}^N} dx \int_0^\infty \frac{\partial}{\partial y} |u(x, y)|^q dy \\ &= -q \int_{\mathbb{R}^{N+1}_+} |u(x, y)|^{q-2} u(x, y)\, \frac{\partial u}{\partial y}(x, y)\, dxdy, \end{aligned}$$

and by the Hölder inequality

$$|u(x, 0)|_q \leq q^{1/q} \lVert u\rVert_{2q-2}^{1-1/q} \lVert Du\rVert_2^{1/q}. \tag{8}$$

By interpolation and the Sobolev Embedding Theorem we get that

$$|u(x,0)|_q \le c_q \|u\| \quad \text{for any } u \in H^1(\mathbb{R}^{N+1}_+) \tag{9}$$

provided that $2 \le 2q - 2 \le \dfrac{2(N+1)}{N-1}$, that is $2 \le q \le \dfrac{2N}{N-1}$.

Applying the Cauchy inequality to (8) we get that

$$|u(x,0)|_q^q \le \frac{\varepsilon q^2}{4} \int_{\mathbb{R}^{N+1}_+} |u|^{2q-2} dxdy + \frac{1}{\varepsilon} \int_{\mathbb{R}^{N+1}_+} \left|\frac{\partial u}{\partial y}\right|^2 dxdy \quad \text{for any } \varepsilon > 0, \tag{10}$$

and in particular, when $q = 2$,

$$|u(x,0)|_2^2 \le \varepsilon \int_{\mathbb{R}^{N+1}_+} |u|^2 dxdy + \frac{1}{\varepsilon} \int_{\mathbb{R}^{N+1}_+} \left|\frac{\partial u}{\partial y}\right|^2 dxdy \quad \text{for any } \varepsilon > 0. \tag{11}$$

As usual, rigorous derivation of the inequalities above are obtained by starting from smooth functions vanishing at infinity, and then by applying a density argument.

From now on, we will assume that the exponent p appearing in the problem satisfies the superlinear and subcritical relation $1 < p < \dfrac{N}{N-1}$. Indeed, if $p = 1$ the term $|u|^{2p-2}u$ is absorbed by the term $\omega_1 u$, while if $p = \dfrac{N}{N-1}$, the fractional Sobolev critical exponent, we loose compactness in the embedding $H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^{\frac{N}{N-1}}(\mathbb{R}^N)$ also in the radial case and locally. Concerning the nonlinearity G, we assume that it satisfies the following set of hypotheses

G:

- **(i)** $G : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1–Carathéodory function, i.e. for a.e. $x \in \mathbb{R}^N$ the map $(u,v) \mapsto G(x,u,v)$ is of class C^1 and for every $(u,v) \in \mathbb{R}^2$ the maps $x \mapsto G(x,u,v)$, $G_u(x,u,v)$ and $G_v(x,u,v)$ are measurable;
- **(ii)** $G(x,0,0) = G_u(x,0,0) = G_v(x,0,0) = 0$ for a.e. $x \in \mathbb{R}^N$ and all $(u,v) \in \mathbb{R}^2$;
- **(iii)** there exist $c > 0$ and $p \in \left(1, \dfrac{N}{N-1}\right)$ such that for a.e. $x \in \mathbb{R}^{N+1}_+$ and all $(u,v) \in \mathbb{R}^2$ we have
$$|G_u(x,u,v)| \le c(|u|^{2p-1} + |v|^p |u|^{p-1})$$
and
$$|G_v(x,u,v)| \le c(|v|^{2p-1} + |u|^p |v|^{p-1});$$
- **(iv)** there exists $\mu > 2$ such that for all $(u,v) \in \mathbb{R}^2$, $u, v \ne 0$, and for a.e. $x \in \mathbb{R}^N$
$$0 < \mu G(x,u,v) \le G_u(x,u,v)u + G_v(x,u,v)v;$$

(v) there exist $\nu \in \left(2, \dfrac{2N}{N-1}\right)$ and $a > 0$ such that for all $(u,v) \in \mathbb{R}^2$ we have

$$G(x,u,v) \geq a(|u|^\nu + |v|^\nu + |uv|^{\nu/2}).$$

Remark 0.1. Of course, **(G)(iv)** is an obvious adaptation of the Ambrosetti–Rabinowitz condition, and if G satisfied additional assumptions or were independent of x, condition **(G)(v)** would be a consequence of **(G)(iv)**, see [**19**]. Moreover, in condition **(G)(iv)** we have $\mu = 2p$ if G is as in (5).

Remark 0.2. By direct integration, from **G(iii)** we get that

$$0 \leq G(x,u,v) \leq \tilde{c}(|u|^{2p} + |v|^{2p} + |uv|^p) \qquad \forall\, (u,v) \in \mathbb{R}^2, \tag{12}$$

for some $\tilde{c} > 0$.

Remark 0.3. With usual standard technicalities, one can ask that **G(iv)** holds only for large values of (u,v), but in order to maintain a light presentation, we prefer to present a global condition.

In view of the previous considerations, the proof of the following result is straightforward.

Proposition 0.1. *Assume* **G(i)-(iii)**. *Then, a couple* $(u,v) \in \mathcal{H}$ *is a solution of system* (6) *iff it is a critical point of the* C^1 *functional* $J : \mathcal{H} \to \mathbb{R}$ *defined as*

$$J(u,v) = \frac{1}{2}\|(u,v)\|^2 - \int_{\mathbb{R}^N} \left[\frac{\omega_1}{2}u^2 + \frac{\omega_2}{2}v^2 + G(x,u,v)\right] dx.$$

We are now ready to state our main result:

Theorem 0.1. *Assume* **G(i)–(v)** *and suppose that* $G(x,u,v) = G(|x|,u,v)$ *for a.e.* $x \in \mathbb{R}^N$ *and all* $(u,v) \in \mathbb{R}^2$. *Then, for every* $\omega_1 < m_1$ *and* $\omega_2 < m_2$ *there exists a couple of nontrivial functions* $(u,v) \in H^1(\mathbb{R}^{N+1}_+) \times H^1(\mathbb{R}^{N+1}_+)$ *which solve problem* (6) *and whose energy level is minimal in the set of energy levels for radial solutions.*

Moreover, if G is independent of x, problem (6) admits a nontrivial solution whose energy level is minimal in the set of energy levels.

In accordance with the usual terminology, the last situation ensures the existence of a nontrivial *ground state solution* for system (6) when G is independent of x.

As it will be clear from the proofs, the first result is obtained by working *a priori* in the set of radial Sobolev functions, while the second one is obtained *directly* by working in the whole Sobolev space.

1 - Proof of Theorem 0.1

In this section we will prove Theorem 0.1, showing that we can apply the Mountain Pass in the space $\mathcal{H} = H^1(\mathbb{R}^{N+1}_+) \times H^1(\mathbb{R}^{N+1}_+)$. We start with the proof of the second case, i.e. for G independent of x, the former one being easier and easily deducible from the latter.

First, let us remark that we can assume $\omega_i > 0$, $i = 1, 2$, otherwise the calculations are simpler, since the term $-\omega_i \int (\cdot)^2$ could be included in the norm of $H_{m_i}(\mathbb{R}^{N+1}_+)$. Next, we prove that J has a strict minimum point in $(u, v) = (0, 0)$. Indeed, taking alternatively $\varepsilon = m_1$ and $\varepsilon = m_2$ in (11), by (12) and the Hölder inequality we have

$$\begin{aligned} J(u,v) \geq & \left(\frac{1}{2} - \frac{\omega_1}{2m_1}\right) \int_{\mathbb{R}^{N+1}_+} |Du|^2 dxdy + \left(\frac{1}{2} - \frac{\omega_2}{2m_2}\right) \int_{\mathbb{R}^{N+1}_+} |Dv|^2 dxdy \\ & + \left(\frac{m_1^2}{2} - \frac{m_1\omega_1}{2}\right) \int_{\mathbb{R}^{N+1}_+} u^2 dxdy + \left(\frac{m_2^2}{2} - \frac{m_2\omega_2}{2}\right) \int_{\mathbb{R}^{N+1}_+} v^2 dxdy \\ & - C\Big(|u|_{2p}^{2p} + |v|_{2p}^{2p} + |u|_{2p}^{p} |v|_{2p}^{p}\Big) \end{aligned}$$

for any $(u, v) \in \mathcal{H}$ and some constant $C > 0$. Recalling that $\omega_1 < m_1$ and $\omega_2 < m_2$, by using (9) and the Cauchy–Schwartz inequality, we finally get the existence of positive universal constants $A, B > 0$ such that

$$J(u,v) \geq A\|(u,v)\|^2 - B(\|(u,v)\|_{2p}^{2p}$$

for all $(u, v) \in \mathcal{H}$. Being $p > 1$, we can find $\rho > 0$ such that $\inf J(S_\rho) > 0$, where S_ρ denotes the sphere of radius ρ and center at the origin in $\mathcal{H}$.

Moreover, if $(u, v) \in \mathcal{H}$ are such that $u(x, 0) \neq 0$ and $v(x, 0) \neq 0$, taken $t > 0$, by **G(v)** we have

$$\begin{aligned} J(tu,tv) = & \frac{t^2}{2}\|(u_n, v_n)\|^2 - \frac{t^2}{2} \int_{\mathbb{R}^N} \left[\omega_1 u_n^2 + \omega_2 v_n^2\right] dx - \int_{\mathbb{R}^N} G(tu,tv)dx \\ \leq & At^2 - Bt^\nu \to -\infty \end{aligned}$$

as $t \to \infty$, A, B being positive constants.

We have thus proved that the functional J has a geometrical structure of mountain pass type. Now, we show that the mountain pass level is indeed a critical level. For this, as usual, we set

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} f(\gamma(t)) > 0,$$

where $\Gamma := \{\gamma \in C([0,1],\mathcal{H}) : \gamma(0) = 0,\ \gamma(1) = (\bar{U},\bar{V})\}$, and $(\bar{U},\bar{V}) \in \mathcal{H}$ is such that $\|(\bar{U},\bar{V})\| > \rho$ and $J(\bar{U},\bar{V}) < 0$.

In order to conclude, we follow a usual strategy, for instance see [**17**]. Let $(u_n,v_n)_n$ in $\mathcal{H}$ be such that $J(u_n,v_n) \to c$ and $J'(u_n,v_n) \to 0$ in $\mathcal{H}'$ as $n \to \infty$, i.e. $(u_n,v_n)_n$ is a Palais–Smale sequence for J at level c. Note that such a sequence exists by Ekeland's Variational Principle.

First, we prove that $(u_n,v_n)_n$ is bounded. Indeed, by assumption there exist $A, B > 0$ such that

$$\mu J(u_n,v_n) - J'(u_n,v_n)(u_n,v_n) \le A + B\|(u_n,v_n)\|$$

for every $n \in \mathbb{N}$. On the other hand, by **G(iv)** and by (11) applied again with $\varepsilon = m_1$ and $\varepsilon = m_2$, we get

$$\begin{aligned}
\mu J(u_n,v_n) - J'(u_n,v_n)(u_n,v_n) &= \Big(\frac{\mu}{2}-1\Big)\|(u_n,v_n)\|^2 \\
&\quad - \Big(\frac{\mu}{2}-1\Big)\int_{\mathbb{R}^N} [\omega_1 u_n^2 + \omega_2 v_n^2]\,dx \\
&\quad + \int_{\mathbb{R}^N} \big[G_u(u_n,v_n)u_n + G_v(u_n,v_n)v_n - \mu G(u_n,v_n)\big]\,dx \\
&\ge \Big(\frac{\mu}{2}-1\Big)\Bigg[\Big(1-\frac{\omega_1}{m_1}\Big)\int_{\mathbb{R}^{N+1}_+} |Du_n|^2 + \Big(1-\frac{\omega_2}{m_2}\Big)\int_{\mathbb{R}^{N+1}_+} |Dv_n|^2 \\
&\quad + m_1(m_1-\omega_1)\int_{\mathbb{R}^{N+1}_+} u_n^2 + m_2(m_2-\omega_2)\int_{\mathbb{R}^{N+1}_+} v_n^2\Bigg]\,dxdy.
\end{aligned}$$

In conclusion, there exists $C > 0$ such that

$$C\|(u_n,v_n)\|^2 \le A + B\|(u_n,v_n)\|$$

for every $n \in \mathbb{N}$, and thus $(u_n,v_n)_n$ is bounded, as claimed, and so we can assume that $(u_n,v_n) \rightharpoonup (u,v)$ in $\mathcal{H}$, i.e. $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^{N+1}_+)$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^{N+1}_+)$.

By (9), (denoting as usual the trace of a function by the function itself) we get that $(u_n)_n$ and $(v_n)_n$ are bounded in $L^q(\mathbb{R}^N)$ for any $q \in \Big[2, \dfrac{2N}{N-1}\Big]$, so that we may assume without loss of generality that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $L^q(\mathbb{R}^N)$. We now show that $u_n \to u$ and $v_n \to v$ in $L^q(\mathbb{R}^N)$ for any $q \in \Big(2, \dfrac{2N}{N-1}\Big)$.

First we show that there exist $(Y_n)_n$ in $\mathbb{R}^{N+1}_+$ and $R,\ \varepsilon > 0$ and such that

$$\int\limits_{\mathcal{B}_R(Y_n)} (u_n^2 + v_n^2)dxdy \geq \varepsilon, \tag{13}$$

where we have set

$$\mathcal{B}_R(Y_n) = \Big\{X \in \mathbb{R}^{N+1}_+ \ : \ |X - Y_n| < R\Big\}.$$

If not, both $(u_n)_n$ and $(v_n)_n$ make vanishing (see [**15**]), that is

$$\lim_{n\to\infty} \sup_{Y\in\mathbb{R}^{N+1}_+} \int\limits_{\mathcal{B}_R(Y)} u_n^2 dxdy = 0 \text{ and } \lim_{n\to\infty} \sup_{Y\in\mathbb{R}^{N+1}_+} \int\limits_{\mathcal{B}_R(Y)} v_n^2 dxdy = 0,$$

and by [**16**, Lemma I.1] we get that $u_n, v_n \to 0$ in $L^q(\mathbb{R}^{N+1}_+)$ for all $q \in \Big(2, \dfrac{2N}{N-1}\Big)$. From the limit $J'(u, v_n)(u_n, v_n) \to 0$, using (11) in the usual way, we easily get

$$\begin{aligned} 0 \leftarrow & \int\limits_{\mathbb{R}^{N+1}_+} \Big[Du_n^2 + m_1^2 u_n^2 + |Dv_n|^2 + m_2^2 v_n^2\Big]\, dxdy \\ & - \int\limits_{\mathbb{R}^N} \big(\omega_1 u_n^2 + \omega_2 v_n^2 + G_u(u_n, v_n)u_n + G_v(u_n, v_n)v_n\big)\, dx \\ & \geq C\|(u_n, v_n)\|^2 - \int\limits_{\mathbb{R}^N} \big(G_u(u_n, v_n)u_n + G_v(u_n, v_n)v_n\big)\, dxdy \end{aligned} \tag{14}$$

for some constant $C = C(m_1, m_2, \omega_1, \omega_2) > 0$. From **G(iii)** we also get

$$\begin{aligned} & \Bigg| \int\limits_{\mathbb{R}^N} \big(G_u(u_n, v_n)u_n + G_v(u_n, v_n)v_n\big)\, dxdy \Bigg| \\ & \leq c\Bigg(\int\limits_{\mathbb{R}^N} \big[|u_n|^{2p} + 2|u_n v_n|^p + |v_n|^{2p}\big]\, dxdy). \end{aligned}$$

But, since $2p \in \Big(2, \dfrac{2N}{N-1}\Big)$, by (8), recalling that $(u_n, v_n)_n$ is bounded in $\mathcal{H}$ and that $u_n, v_n \to 0$ in $L^p(\mathbb{R}^N)$, from the previous inequality we get that

$$\int\limits_{\mathbb{R}^N} \big(G_u(u_n, v_n)u_n + G_v(u_n, v_n)v_n\big)\, dxdy \to 0,$$

and so from (14), we get that $(u_n, v_n) \to (0,0)$ in $\mathcal{H}$. But $2c = 2J(u_n, v_n) - J'(u_n, v_n)(u_n, v_n) + o(1)$, where $o(1) \to 0$ as $n \to \infty$, and then

$$2c = 2J(u_n, v_n) - J'(u_n, v_n)(u_n, v_n) + o(1) \to 0,$$

while $c > 0$, and a contradiction arises. This means that (13) holds.

Now, let us set $\tilde{u}_n(x) = u_n(x + Y_n)$ and $\tilde{v}_n(x) = v_n(x + Y_n)$ for every $x \in \mathbb{R}^{N+1}_+$. J being invariant under translations (recall that we are assuming G independent of x), we have that $(\tilde{u}_n, \tilde{v}_n)_n$ is still a Palais–Smale sequence for J at level c. Of course, also $(\tilde{u}_n, \tilde{v}_n)_n$ is bounded in $\mathcal{H}$, so that there exists $(\tilde{u}, \tilde{v}) \in \mathcal{H}$ such that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ in $\mathcal{H}$. From (13) we get that there exist $R,\ \varepsilon > 0$ such that

$$\int\limits_{\mathcal{B}_R(0_{\mathbb{R}^{N+1}_+})} (u_n^2 + v_n^2) dxdy \geq \varepsilon.$$

Since $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})$ in $L^2(\mathcal{B}_R(0_{\mathbb{R}^{N+1}_+}))$, we find that $(\tilde{u}, \tilde{v}) \neq (0,0)$. Moreover, if $w, z \in C^\infty_c(\mathbb{R}^{N+1}_+)$, from the fact that $J'(\tilde{u}_n, \tilde{v}_n)(w, z) \to 0$, by the weak convergence in $\mathcal{H}$ and the strong convergence in the spaces L^q over bounded domains, we get that

$$\begin{aligned} &\int\limits_{\mathbb{R}^{N+1}_+} (D\tilde{u} \cdot Dw + m_1^2 \tilde{u} w)\, dxdy = \int\limits_{\mathbb{R}^N} \big[\omega_1 \tilde{u} + G_u(x, \tilde{u}, \tilde{v})\big] w\, dx \\ &\int\limits_{\mathbb{R}^{N+1}_+} (D\tilde{v} \cdot Dz + m_2^2 \tilde{v} z)\, dxdy = \int\limits_{\mathbb{R}^N} \big[\omega_2 v + G_v(x, \tilde{u}, \tilde{v})\big] z\, dx. \end{aligned} \tag{15}$$

Since $C^\infty_c(\mathbb{R}^{N+1}_+)$ is dense in $H^1(\mathbb{R}^{N+1}_+)$, (15) also holds for every $w, z \in H^1(\mathbb{R}^{N+1}_+)$, which means that $(\tilde{u}, \tilde{v})$ is a nontrivial solution for problem (6).

Now, let us show that $J(\tilde{u}, \tilde{v}) = c$. Indeed, since $(\tilde{u}_n, \tilde{v}_n)$ is a critical point for J, of course we have that $(\tilde{u}_n, \tilde{v}_n)$ belongs to the Nehari manifold $\mathcal{N}$, defined as usual by

$$\mathcal{N} := \big\{(u, v) \in \mathcal{H} \setminus \{(0,0)\}\ :\ J'(u, v)(u, v) = 0\big\}.$$

As it is well known, for instance see [**23**] (or [**17**]), we have

$$\inf_{\mathcal{N}} J = \inf \Big\{ J(u, v)\ :\ (u, v) \neq (0,0)\ \text{ and }\ J'(u, v) = 0 \Big\} = c.$$

Hence, since $(\tilde{u}, \tilde{v})$ is a critical point for J, we have $J(\tilde{u}, \tilde{v}) \geq c$.

On the other hand, we have

$$\begin{aligned} c &= \lim_{n\to\infty} J(\tilde{u}_n, \tilde{v}_n) + o(1) = \lim_{n\to\infty} \Big(J(\tilde{u}_n, \tilde{v}_n) - \frac{1}{2} J'(\tilde{u}_n, \tilde{v}_n)(\tilde{u}_n, \tilde{v}_n) + o(1) \Big) \\ &= \int\limits_{\mathbb{R}^N} \Big(\frac{1}{2} G_u(\tilde{u}_n, \tilde{v}_n)\tilde{u}_n + \frac{1}{2} G_v(\tilde{u}_n, \tilde{v}_n)\tilde{v}_n - G(\tilde{u}_n, \tilde{v}_n) \Big)\, dx. \end{aligned}$$

Now, condition **G(iv)** ensures that we can apply the Fatou Lemma, and so the previous identity implies that

$$\begin{aligned} c &\geq \int_{\mathbb{R}^N} \Big(\frac{1}{2}G_u(\tilde{u},\tilde{v})\tilde{u} + \frac{1}{2}G_v(\tilde{u},\tilde{v})\tilde{v} - G(\tilde{u},\tilde{v})\Big)\,dx \\ &= J(\tilde{u},\tilde{v}) - \frac{1}{2}J'(\tilde{u},\tilde{v})(\tilde{u},\tilde{v}) = J(\tilde{u},\tilde{v}). \end{aligned}$$

As a consequence, $J(\tilde{u},\tilde{v}) = c$ and $(\tilde{u},\tilde{v})$ is a ground state solution for problem (6), as claimed.

The proof of the first part of the theorem is now easier. Indeed, it is enough to consider J constrained on the space of functions which are radially symmetric in the first N variables

$$\mathcal{H}_r := \Big\{(u,v) \in \mathcal{H} \,:\, u,v \in H^1_r(\mathbb{R}^{N+1}_+)\Big\},$$

where

$$H^1_r(\mathbb{R}^{N+1}_+) = \Big\{v \in H^1(\mathbb{R}^{N+1}_+) \,:\, v(Mx,y) = v(x,y) \text{ for any } M \in O(N)\Big\}$$

and $O(N)$ denotes the orthogonal group in $\mathbb{R}^N$. Now, since the problem under consideration is invariant by rotation around the y–axis, if $(u,v) \in \mathcal{H}_r$ is a critical point of J constrained on $\mathcal{H}_r$, then (u,v) is also a critical point of J on the whole of $\mathcal{H}$ by the Principle of Symmetric Criticality of Palais, see [**22**]. Hence, we are reduced to look for critical points of J constrained on $\mathcal{H}_r$. In particular, we can reproduce the proof above with a small technical change. In particular, we only need to change the proof of the Palais–Smale condition: once showed that any Palais–Smale sequence in $\mathcal{H}_r$ at level c is bounded, we use that functions in $H^1_r(\mathbb{R}^{N+1}_+)$ admit traces in

$$H^{1/2}_r(\mathbb{R}^N) := \Big\{u \in H^1(\mathbb{R}^N) \,:\, u(x) = u(|x|)\Big\}$$

and that $H^{1/2}_r(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ for any $q \in \left(2, \dfrac{2N}{N-1}\right)$, see [**14**], and the proof proceeds as before.

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