Riv. Mat. Univ. Parma, Vol. **8** (2017), 1-82

Giovanni Naldi

A journey through multiscale, some episodes from approximation and modelling

Abstract. The present notes contains both a survey of and some novelties about mathematical problems which emerged in multiscale based approach in approximation of evolutionary partial differential equations. Specifically, we present a relaxed systems approximation for nonlinear diffusion problems, which can tackle also the cases of degenerate and strongly degenerate diffusion equations. Relaxation schemes take advantage of the replacement of the original partial differential equation with a semi-linear hyperbolic system of equations, with a stiff source term, tuned by a relaxation parameter ε. When $\varepsilon \to 0^+$, the system relaxes onto the original PDE: in this way, a consistent discretization of the relaxation system for vanishing ε yields a consistent discretization of the original PDE. The advantage of this procedure is that numerical schemes obtained in this fashion do not require to solve implicit nonlinear problems and possess the robustness of upwind discretizations. We also review a unified framework, including BGK-based diffusive relaxation methods and new relaxed numerical schemes. A stability analysis for the new methods is sketched and high order extensions are provided. Finally some numerical tests in one and two dimensions are shown with preliminary results for nonlocal problems and multiscale hyperbolic systems.

Keywords. Multiscale modelling, Relaxation approximation, nonlinear evolutionary differential equations, Relaxed methods.

Mathematics Subject Classification (2010): 65M99, 35Q70, 35Q92, 65M06.

Received: February 16, 2017; accepted in revised form: July 25, 2017.

This research was partially supported by ADAMSS Center of the Università degli Studi di Milano.

Contents

1 - Introduction

The concept of multiscale modelling and of multiscale analysis follows the idea that a comprehensive description of many complex systems will require an understanding over multiple time and length scales [**50**]. The structure of these systems requires suitable mathematical and computational tools. Indeed, physical, chemical, and biological processes for many problems in computational physics, biology, and material science span length and time scales of many orders of magnitude. Traditionally, scientists and research groups have focused on methods that are particularly applicable in only one regime, and knowledge of the system at one scale has been transferred to another scale only

indirectly. Microscopic models, for example, have been often used to find the effective parameters of macroscopic models, but for obvious computational reasons, microscopic and macroscopic scales have been treated separately [**51**]. For the numerical simulation of a multiscale system, though the enormous increase in computational power available (due to the improvement both in computer speed and in efficiency of the numerical methods) allows in some cases the treatment of systems involving scales of different orders of magnitude, the numerical solution of such problems by classical methods often leads to an inefficient use of the computational resources and the problem cannot be solved by direct numerical simulation. The main reasons for this are that the necessary resolution of a fine scale entails an over-resolution of coarser scales, the position of the singularity is not known beforehand, the gap between the scales is too big for a treatment in the same framework. In other cases, the structure of the mathematical models that treat the system at the different scales varies a lot, and therefore new mathematical techniques are required for the systems described by different mathematical models. Finally, in many cases one is interested in the accurate treatment of a small portion of a large system, and it is too expensive to consider the whole system at the required accuracy [**97**].

During the 8^{th} Summer School on "*Methods and Models of Kinetic Theory*" I have selected three topics: the relaxation/relaxed approximation of nonlinear evolutionary differential equations, a description of a suitable collective dynamics, a multiscale model for the muscle contraction. The first topic was the introduction and analysis of the relaxed approximation for nonlinear problems, which can tackle also the cases of degenerate and strongly degenerate diffusion equations. The collective dynamics of self-propelled particles (agents) such as flocking of birds and mobile agents, describes phenomena as schooling of fishes, swarming of bacteria, mobile network, and swarm robotics. More complex systems are those that involve multiple types of groups and interactions, such as intra-species competition for resources or inter-species relationships. During school we focused on the interactions between two (or more) different populations of agents and the interactions with possible obstacles. Starting from the discrete models, we deduced the hydrodynamic/macroscopic descriptions of collective motion via kinetic theory (see e.g. [**1**, **88**]). For the third topic, we discussed a multiscale mathematical model for the simulation of the force response and length change of individual myofibril in order to reproduce the phenomena of the skeletal muscle contraction. The myofibril is modeled as a group of segments placed in series, each segment represent a half-sarcomere with active and elastic properties. The corresponding macro-scale model is a system of nonlinear nonlocal partial differential equations (see e.g. [**93**]).

We point out that there are not explicit connections between the three

subjects but one key point in all the topics is the notion of the bridging which consists in establishing links between the various involved scales. In these notes we consider only the relaxation approximation and the relaxed schemes for the numerical discretization of nonlinear evolutionary partial differential equations including nonlinear degenerate and strongly degenerate diffusion equations and some work in progress about the multiscale hyperbolic systems and nonlocal problems. A warning: I will skip several technical details, but the interested reader may find most of these in the available references on the various models and problems.

2 - Around the relaxation and the relaxed approximation

Many kinetic models of the Boltzmann equation have a diffusive scaling that leads to the Navier-Stokes type equations such as a suitably small parameter goes to zero. In such problems the diffusive relaxation parameter may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic (diffusive) regimes. Numerical approximation of these equations is challenging due to the presence of stiff source, collision, forcing terms, or when different scales coexist. Moreover, in real applications, it is desirable to develop a class of numerical schemes that can work uniformly with respect to this relaxation parameter, from the rarefied kinetic regimes to the hydrodynamic diffusive regimes. We observe, at least at the formal level, that, when the relaxation parameter ε is small, the Boltzmann equation can be approximated by the Euler equations to the leading order and the Navier-Stokes equations to $O(\varepsilon)$ [**34**]. For many kinetic models the Navier-Stokes or the diffusive limit is well established mathematically. For example, in some recent papers [**56**, **83**, **106**], it was shown that the equations of slow and fast diffusion, the porous media equation, and the Burgers equation can be obtained as the diffusive limit of kinetic models of Boltzmann type. Also for the reactive Boltzmann equation, an extension of the classical Boltzmann equation, for a mixture of different species of molecules with chemical reactions, it is possible to show, under suitable scalings, that the solution converges to the solutions of a reaction-diffusion system [**12**, **114**].

Following a similar framework, relaxation approximations to partial differential equations (PDE's) of various type have been recently introduced: from classical kinetic schemes for gas dynamics [**40**, **44**] to the relaxation schemes for conservation laws [**3**, **74**] and Hamilton-Jacobi equations [**75**], and the diffusive relaxation schemes for convection-diffusion problems [**4**, **28**, **30**, **31**, **32**, **72**, **95**]. In these cases, in order for a scheme to be useful in the hydrodynamic regime (where ε is small), allowing the use of Δt, $\Delta x \gg \varepsilon$, where Δt and Δx are the time steps and grid size respectively, the scheme should possess the correct

diffusion limit in the sense that the asymptotic limit that leads from the kinetic equations to the hydrodynamic (Euler, Navier-Stokes, or diffusion) equations should be preserved at the discrete level. We call such schemes asymptotic-preserving (AP) [**53**, **67**, **71**], a robust AP scheme should allow an implicit discretization for better numerical stability.

To motivate the idea we first illustrate the major ingredient that leads to the development of the diffusive relaxation schemes for discrete-velocity kinetic models considering the one-dimensional GoldsteinTaylor model [**58**, **116**]. The GoldsteinTaylor model describes the behavior of a one-dimensional fictitious gas composed of two kinds of particles moving parallel to the $x-$axis with constant speeds, of equal modulus c, one in the positive $x-$direction with density $u(x,t)$, the other in the negative $x-$direction with density $v(x,t)$. The corresponding system of equations is:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = \sigma(x)(v-u) \\ \dfrac{\partial v}{\partial t} - c\dfrac{\partial v}{\partial x} = \sigma(x)(u-v) \end{cases} \tag{1}$$

where $u := u(x,t)$, $v := v(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, and σ is a nonnegative function which characterizes the interactions between gas particles (or the cross section in the vocabulary of transport equations). Let $c = 1$, and $\sigma = 1$, the Goldstein-Taylor model under the diffusive scaling ($t \to \varepsilon^2 t$, $x \to \varepsilon x$, with a positive relaxation parameter ε) read

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u_\varepsilon}{\partial x} = \dfrac{1}{\varepsilon^2}(v_\varepsilon - u_\varepsilon) \\ \dfrac{\partial v_\varepsilon}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v_\varepsilon}{\partial x} = \dfrac{1}{\varepsilon^2}(u_\varepsilon - v_\varepsilon). \end{cases} \tag{2}$$

The macroscopic variables for this model are the mass density $\rho_\varepsilon = \rho_\varepsilon(x,t) = u_\varepsilon(x,t) + v_\varepsilon(x,t)$, and the flux,

$$j_\varepsilon = j_\varepsilon(x,t) = \frac{u_\varepsilon(x,t) - v_\varepsilon(x,t)}{\varepsilon}. \tag{3}$$

Since u_ε and v_ε can be expressed in terms of ρ_ε and j_ε, system (2) is equivalent to the following system for the mass density and the flux,

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \dfrac{\partial j_\varepsilon}{\partial x} = 0 \\ \dfrac{\partial j_\varepsilon}{\partial t} + \dfrac{1}{\varepsilon^2}\dfrac{\partial \rho_\varepsilon}{\partial x} = \dfrac{-2}{\varepsilon^2} j_\varepsilon. \end{cases} \tag{4}$$

Formally, In the zero relaxation (or diffusion) limit ($\varepsilon \to 0^+$), system (4) can be approximated to leading order by

$$j = -\frac{1}{2}\frac{\partial \rho}{\partial x} \tag{5}$$

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial \rho}{\partial x}\right) = 0. \tag{6}$$

The state satisfying (5) will be called the local equilibrium, while (6) is the equation of continuum mechanics generated by the kinetic model, that in this case is the linear heat equation. System (4) is often called the hyperbolic heat equation or Maxwell-Cattaneo model (Maxwell first introduced the concept of a relaxation time). The diffusive limit for (4) was first studied by Kurtz and McKean [**80**, **89**]. In particular, it was shown that starting with initial data $u_\varepsilon(x,0) = u_0(x)$, $v_\varepsilon(x,0) = v_0(x)$, the solution u_ε, and v_ε to system (4) converges strongly in L^1_x for all $t \geq 0$ to limit density ρ which satisfies the heat equation (6), with $\rho_0 = u_0 + v_0$ as initial data. Moreover, $\varepsilon j_\varepsilon$ converges to zero in $L^2_{x,t}$. A similar study for a more general class of 2×2 hyperbolic systems with relaxation can be found e.g. in [**87**]. Note that solving (4) numerically is challenging due to the stiffness of the problem for both the convection and collision terms. If an explicit scheme is used, one needs a CFL condition $\Delta t < \varepsilon \Delta x$, which is very expensive when ε is small. Characteristic based numerical schemes to solve (4) that are able to handle the small relaxation parameter ε were given in [**107**]. If an implicit scheme is used, due to the nonlinear nature of the numerical flux for the convection term, a major ingredient for all modern high order shock capturing schemes for hyperbolic conservation laws, one needs an iterative method, such as the Newtons method, to solve the resulting nonlinear systems. A natural way to compute numerically the solution of the system (4) is to consider a splitting method. For the moment, we consider the semi-discrete time approximation, this means that the variable t is discretized dividing the time interval $[0,T]$ into $M > 0$ equal subintervals $[t_n, t_{n+1}]$ where $t_n = n\Delta_{Mt}$, and $\Delta_{Mt} = T/M$. Now let ρ^n, J^n the discrete solution at time t_n and set $\rho^0 = \rho_\varepsilon(x,0)$, $J^0 = j_\varepsilon(x,0)$. Now suppose that the values ρ^n, J^n have been computed, then approximations ρ^{n+1}, J^{n+1} are obtained in two steps. First we solve the following homogeneous linear system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial J}{\partial x} = 0 \\ \\ \varepsilon^2 \dfrac{\partial J}{\partial t} + \dfrac{\partial \rho}{\partial x} = 0 \end{cases} \tag{7}$$

on the time subinterval $[n\Delta t, (n+1)\Delta t]$ with ρ_n, J^n as the initial data. We denote with $\rho^{n+1/2}$, $J^{n+1/2}$ the solution of (7) at the time t_{n+1}. In the second step we solve the following system of ordinary differential equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = 0 \\ \\ \varepsilon^2 \dfrac{\partial J}{\partial t} = -2J \end{cases} \tag{8}$$

on the same interval $[t_n, t_{n+1}]$ with the values $\rho^{n+1/2}$, $J^{n+1/2}$ taken as the initial data. The solution of (8) at time t_{n+1} was denoted by ρ^{n+1}, J^{n+1}. We note that the system (8) is equivalent to the kinetic system

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{1}{\varepsilon^2}(V-U) \\ \\ \dfrac{\partial V}{\partial t} = \dfrac{1}{\varepsilon^2}(U-V) \end{cases} \tag{9}$$

where $\rho = U + V$, and $J = (U - V)$. Let us take the partial derivative with respect to x, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x}\right) = \frac{1}{\varepsilon^2}\frac{\partial}{\partial x}(V-U)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) = \frac{1}{\varepsilon^2}\frac{\partial}{\partial x}(U-V)$$

and

$$\frac{\partial}{\partial t}\left|\frac{\partial U}{\partial x}\right| = \frac{1}{\varepsilon^2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial x}\right) sign\left(\frac{\partial U}{\partial x}\right)$$

$$\frac{\partial}{\partial t}\left|\frac{\partial V}{\partial x}\right| = \frac{1}{\varepsilon^2}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial x}\right) sign\left(\frac{\partial V}{\partial x}\right).$$

Let us take the sum of both above equations,

$$\frac{\partial}{\partial t}\left(\left|\frac{\partial U}{\partial x}\right| + \left|\frac{\partial V}{\partial x}\right|\right) =$$

$$-\frac{1}{\varepsilon^2}\left[\left(\left|\frac{\partial U}{\partial x}\right| + \left|\frac{\partial V}{\partial x}\right|\right) + \left(\frac{\partial U}{\partial x} sign\left(\frac{\partial V}{\partial x}\right) + \frac{\partial V}{\partial x} sign\left(\frac{\partial U}{\partial x}\right)\right)\right].$$

If

$$sign\left(\frac{\partial U}{\partial x}\right) = sign\left(\frac{\partial V}{\partial x}\right),$$

the right-hand side of the previous equation is equal to zero, while if

$$sign\left(\frac{\partial U}{\partial x}\right) \neq sign\left(\frac{\partial V}{\partial x}\right),$$

the right-hand side is equal to

$$-\frac{2}{\varepsilon^2}\left[\left(\left|\frac{\partial U}{\partial x}\right| + \left|\frac{\partial V}{\partial x}\right|\right)\right].$$

Hence,

$$\frac{\partial}{\partial t}\left(\left|\frac{\partial U}{\partial x}\right| + \left|\frac{\partial V}{\partial x}\right|\right) \leq 0,$$

and if U, V are in the space BV, so it is the solution of (8) at any subsequent time, and the same conclusion can be drawn for the solution to the linear hyperbolic system (7), so we have

$$\|U(t)\|_{BV} + \|V(t)\|_{BV} \leq \|U(0)\|_{BV} + \|V(0)\|_{BV},$$

for any initial data $U(0)$, and $V(0)$ in BV. Then,

Proposition 2.1. *Let $U(0)$, $V(0) \in BV(\mathbb{R})$, then $\forall T > 0$, the solution to the splitting* (7), (8), *ρ^M, J^M, is uniformly bounded in BV, and converge to the unique solution $\rho_\varepsilon(\cdot, T)$, $j_\varepsilon(\cdot, T)$ of the original system as $M \to \infty$.*

From the numerical point of view a special care should be used in the space discretization of a system like (7) since the convection term is stiff. For example a standard upwinding may introduce excessive numerical viscosity [**94**]. For the system (8) an implicit scheme should be considered to avoid stability conditions of the type $\Delta t \sim \varepsilon^2$.

Remark 2.1. In order to write a stable discretization to system (4), we can also avoid the splitting approach and use implicit temporal integrators on the stiff terms for the original system. Now, we have to modify upwind schemes in order to have the correct asymptotic behavior and eliminate the diffusive effect introduced by the numerical viscosity. The main idea is to use a modified upwind numerical fluxes for the variable j_ε and by letting the numerical fluxes for the variable ρ_ε unmodified [**70**,**94**,**95**]. A particular effective way to compute the numerical fluxes has been introduced in [**70**]. The idea is to build into the numerical schemes the asymptotic that leads from the hyperbolic system to the parabolic equation in an implicit way. These techniques guarantee that the numerical schemes possess the correct asymptotic limit. However, it is not clear how to extend these approaches to high order schemes or in the non-stiff regime.

A key idea to attack the problem of the stiffness when $0 < \varepsilon << 1$ is to reformulate (4) as a (nonstiff) linear hyperbolic system with stiff relaxation term, usually called the diffusive relaxation system [**72**, **73**, **94**],

$$(10)\qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial j}{\partial x} = 0 \\ \dfrac{\partial j}{\partial t} + \phi_\varepsilon^2 \dfrac{\partial \rho}{\partial x} = -\dfrac{1}{\varepsilon^2}\left(2j + (1-\varepsilon^2\phi_\varepsilon^2)\dfrac{\partial \rho}{\partial x}\right). \end{cases}$$

The function ϕ_ε permits to remove the stiffness from the characteristic velocities when $\varepsilon << 1$, a simple choice for ϕ_ε is [**72**, **95**]

$$\phi_\varepsilon = \begin{cases} 1, & \text{if } \varepsilon \leq 1 \\ 1/\varepsilon, & \text{if } \varepsilon > 1. \end{cases}$$

The above reformulation is equivalent to rewrite the partial derivative $\partial\rho/\partial x$ as a linear combination [**45**]

$$\frac{\partial \rho}{\partial x} = \underbrace{\varepsilon^2\phi_\varepsilon^2 \frac{\partial \rho}{\partial x}}_{hyperbolic\,term} + \underbrace{(1-\varepsilon^2\phi_\varepsilon^2)\frac{\partial \rho}{\partial x}}_{parabolic\,part}.$$

Now, applying the idea of splitting we have the following sub-problem,

$$(11)\quad (CS)\begin{cases} \dfrac{\partial \rho}{\partial t} = 0 \\ \dfrac{\partial j}{\partial t} = -\dfrac{1}{\varepsilon^2}\left(2j + (1-\varepsilon^2\phi_\varepsilon^2)\dfrac{\partial \rho}{\partial x}\right) \end{cases} ;\ (TS)\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial j}{\partial x} = 0 \\ \dfrac{\partial j}{\partial t} + \phi_\varepsilon^2 \dfrac{\partial \rho}{\partial x} = 0. \end{cases}$$

We will introduce the spatial grid points $x_{i+1/2}$, $i = \ldots, -2, -1, 0, 1, 2, \ldots$ with uniform mesh width $\Delta x = x_{i+1/2} - x_{i-1/2}$. As usual we denote by $U_i(t) = U(x_i, t)$, the cell centered values of the function U (the nodal values in the present finite differences approach, the cell averages of U in the cell $[x_{i-1/2}, x_{i+1/2}]$ in the case of the finite volume approximation), while U_i^n denotes $U_i(t^n)$. In the following we will consider a first order splitting scheme as a prototype method for the approximation of the system (10) in the finite differences framework. The parameter $\phi_\varepsilon > 0$ is fixed, we set $F(\varepsilon, \rho, s) = (1-\varepsilon^2\phi_\varepsilon^2)\partial\rho(s)/\partial x$, and $F_i(\varepsilon, \rho) = (1-\varepsilon^2\phi_\varepsilon^2)D_i\rho$, where $D_i\rho$ denotes a finite differences formula for the approximation of the first derivative of the function ρ in the grid point x_i. A natural discretization to (11) is an implicit time

discretization for the "collision step" (CS), e.g. backward Euler:

$$
\begin{aligned}
&\frac{\rho_i^{n+1/2}-\rho_i^n}{\Delta t}=0\\
&\frac{j_i^{n+1/2}-j_i^n}{\Delta t}=-\frac{1}{\varepsilon^2}\left(2j_i^{n+1/2}+F_i(\varepsilon,\rho)\right),
\end{aligned}
\tag{12}
$$

then

$$
\begin{cases}
\rho_i^{n+1/2}=\rho_i^n,\\
j_i^{n+1/2}=\alpha\left(j_i^n-\dfrac{\Delta t}{\varepsilon^2}F_i(\varepsilon,\rho^{n+1/2})\right),
\end{cases}
\tag{13}
$$

where

$$
\alpha=\frac{\varepsilon^2}{\varepsilon^2+2\Delta t}.
$$

The transport step (TS) is discretized by an explicit scheme,

$$
\begin{aligned}
&\frac{\rho_i^{n+1}-\rho_i^{n+1/2}}{\Delta t}+\frac{j_{i+1/2}^{n+1/2}-j_{i-1/2}^{n+1/2}}{\Delta x}=0\\
&\frac{j_i^{n+1}-j_i^{n+1/2}}{\Delta t}+\frac{\rho_{i+1/2}^{n+1/2}-\rho_{i-1/2}^{n+1/2}}{\Delta x}=0,
\end{aligned}
\tag{14}
$$

with suitable numerical fluxes $j_{i\pm1/2}^{n+1/2}$, $\rho_{i\pm1/2}^{n+1/2}$. The system (TS), see (11), is a hyperbolic system with two distinct characteristic speeds $\pm\phi_\varepsilon$, then the first order upwind method gives

$$
\begin{aligned}
\rho_{i\pm1/2}^{n+1/2}&=\frac{\phi_\varepsilon^2}{2}\left(\rho_i^n+\rho_{i\pm1}^n\right)\pm\frac{\phi_\varepsilon}{2}\left(j_i^{n+1/2}-j_{i\pm1}^{n+1/2}\right),\\
j_{i\pm1/2}^{n+1/2}&=\frac{1}{2}\left(j_i^{n+1/2}+j_{i\pm1}^{n+1/2}\right)\pm\frac{\phi_\varepsilon}{2}\left(\rho_i^n-\rho_{i\pm1}^n\right).
\end{aligned}
\tag{15}
$$

For the numerical analysis, it is convenient to combine both steps into a single step,

$$
\begin{aligned}
\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}&+\frac{\alpha}{2\Delta x}\left(j_{i+1}^n-j_{i-1}^n\right)-\frac{\alpha\Delta t}{2\Delta x\varepsilon^2}\left[F_{i+1}(\varepsilon,\rho^n)-F_{i-1}(\varepsilon,\rho^n)\right]\\
&+\frac{\phi_\varepsilon}{2\Delta x}\left[2\rho_i^n-\rho_{i+1}^n-\rho_{i-1}^n\right]=0;
\end{aligned}
\tag{16}
$$

$$\frac{j_i^{n+1}-j_i^n}{\Delta t}+\frac{1-\alpha}{\Delta t}j_i^n-\frac{\alpha}{\varepsilon^2}F_i(\varepsilon,\rho^n)+\frac{\phi_\varepsilon^2}{2\Delta x}\left(\rho_{i+1}^n-\rho_{i-1}^n\right)-\frac{\alpha\phi_\varepsilon\Delta t}{2\varepsilon^2\Delta x}$$
$$[F_{i+1}(\varepsilon,\rho^n)+F_{i-1}(\varepsilon,\rho^n)-2F_i(\varepsilon,\rho^n)]-\frac{\alpha\phi_\varepsilon}{2\Delta x}\left[2j_i^n-j_{i+1}^n-j_{i-1}^n\right]=0. \tag{17}$$

Recalling the definition of the function $F_i(\varepsilon,\rho^n)$, to the scheme (16) corresponds the modified equation [**95**],

$$\frac{\partial\rho}{\partial t}+\alpha\frac{\partial j}{\partial x}-\frac{\alpha\Delta t(1-\phi_\varepsilon^2\varepsilon^2)}{\varepsilon^2}\frac{\partial^2\rho}{\partial x^2}=\frac{\phi_\varepsilon\Delta x}{2}\frac{\partial^2\rho}{\partial x^2}. \tag{18}$$

For small values of ε, $\alpha=O(\varepsilon^2)$, $(\alpha\Delta t(1-\phi_\varepsilon^2\varepsilon^2))/(\varepsilon^2)\approx 1/2$, and we have an approximation of the equilibrium heat equation with an accuracy of $O(\Delta x)$. From the von Neumann stability analysis we obtain the following CFL stability condition,

$$\frac{\Delta t(1-\phi_\varepsilon^2\varepsilon^2)}{\varepsilon^2+2\Delta t}\frac{\Delta t}{\Delta x^2}<\frac{1}{2}. \tag{19}$$

For $\varepsilon^2 << \Delta t$, condition (19) reduces to a parabolic CFL condition, here the diffusion coefficient is equal to $1/2$,

$$\frac{\Delta t}{\Delta x^2}<1.$$

Another important property of (16), (17) is that they preserve the correct diffusion limit. By passing $\varepsilon\to 0^+$, (16) becomes

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}=\frac{1}{2\Delta x^2}\left(D_{i+1}\rho^n-D_{i-1}\rho^n\right),$$

a suitable finite difference discretization (the final scheme depends on the choice of the discrete operator D_i) of the limiting heat equation (6): thus this scheme is *asymptotic-preserving* (AP) [**67**,**71**]. This means that this scheme captures the macroscopic behavior even if the numerical solution is underresolved, or even if the numerical discretization does not resolve the kinetic scaling characterized by the (mean free path) parameter ε, i.e., Δt, $\Delta x >> \varepsilon^2$. A robust AP scheme should allow an implicit discretization for better numerical stability, yet can be solved explicitly.

R e m a r k 2.2. Following the approach of S. Jin [**67**], another way to develop a class of AP numerical schemes consists in keeping the spatial derivative continuous and combine the above two steps of the splitting process. First, the backward Euler time discretization for the collision step,

$$\rho^{n+1/2}=\rho^n$$
$$\frac{j^{n+1/2}-j^n}{\Delta t}=-\frac{1}{\varepsilon^2}\left(2j^{n+1/2}+(1-\varepsilon^2\phi_\varepsilon^2)\frac{\partial\rho^{n+1/2}}{\partial x}\right), \tag{20}$$

followed by an explicit scheme for the transport step (TS),

$$\begin{aligned} &\frac{\rho^{n+1}-\rho^{n+1/2}}{\Delta t}+\frac{\partial j^{n+1/2}}{\partial x}=0\\ &\frac{j^{n+1}-j^{n+1/2}}{\Delta t}+\phi_\varepsilon^2\frac{\partial \rho^{n+1/2}}{\partial x}=0. \end{aligned} \tag{21}$$

Then, we combine the above two steps into the following system,

$$\begin{aligned} &\frac{\rho^{n+1}-\rho^{n}}{\Delta t}+\alpha\frac{\partial j^{n}}{\partial x}=\beta\frac{\partial^2 \rho^{n}}{\partial x^2}\\ &\frac{j^{n+1}-j^{n}}{\Delta t}+\gamma\frac{\partial \rho^{n}}{\partial x}=\delta j^n, \end{aligned} \tag{22}$$

where,

$$\begin{aligned} &\alpha=\frac{\varepsilon^2}{\varepsilon^2+2\Delta t},\ \beta=\frac{(1-\phi^2\varepsilon^2)\Delta t}{\varepsilon^2+2\Delta t},\\ &\gamma=1+\frac{(1-\phi^2\varepsilon^2)}{\varepsilon^2+2\Delta t},\ \delta=\frac{-2\Delta t}{\varepsilon^2+2\Delta t}. \end{aligned} \tag{23}$$

One can view (22) as the forward Euler discretization of the following hyperbolic-parabolic system,

$$\begin{aligned} &\frac{\partial \rho}{\partial t}+\alpha\frac{\partial j}{\partial x}=\beta\frac{\partial^2 \rho}{\partial x^2}\\ &\frac{\partial j}{\partial t}+\gamma\frac{\partial \rho}{\partial x}=\delta j. \end{aligned} \tag{24}$$

The hyperbolic part of system (24) has two distinct characteristic speeds $\pm\sqrt{\gamma\alpha}$. Multiplying the first equation of (24) by ρ and the second equation by j and integrating on $\mathbb{R}$ we find,

$$\frac{\partial}{\partial t}\int\limits_{\mathbb{R}}\frac{\rho^2}{2}dx+\alpha\int\limits_{\mathbb{R}}\rho\frac{\partial j}{\partial x}dx=\beta\int\limits_{\mathbb{R}}\rho\frac{\partial^2 \rho}{\partial x^2}dx$$

$$\frac{\partial}{\partial t}\int\limits_{\mathbb{R}}\frac{j^2}{2}dx+\gamma\int\limits_{\mathbb{R}}j\frac{\partial \rho}{\partial x}dx=\delta\int\limits_{\mathbb{R}}j^2dx.$$

We have assumed that ρ and j are smooth compactly supported functions. Then, using the integration by part, we obtain,

$$\int_{\mathbb{R}} \rho \frac{\partial j}{\partial x} dx = -\int_{\mathbb{R}} j \frac{\partial \rho}{\partial x} dx,$$

$$\int_{\mathbb{R}} \rho \frac{\partial^2 \rho}{\partial x^2} dx = -\int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x}\right)^2 dx.$$

But from the second equation of the above system we can state

$$\int_{\mathbb{R}} j \frac{\partial \rho}{\partial x} dx = \frac{\delta}{\gamma} \int_{\mathbb{R}} j^2 dx - \frac{1}{\gamma} \frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{j^2}{2} dx,$$

then

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} \left(\frac{\rho^2}{2} + \frac{\alpha}{\gamma} \frac{j^2}{2}\right) dx = -\int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x}\right)^2 dx + \frac{\delta}{\gamma} \int_{\mathbb{R}} j^2 dx.$$

Since $\delta < 0$, we can conclude that a suitable free energy decays in time:

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} \left(\frac{\rho^2}{2} + \frac{\alpha}{\gamma} \frac{j^2}{2}\right) dx \leq 0. \tag{25}$$

This energy estimate is consistent with the energy for its continuous counterpart for the Goldstein-Taylor system.

If we use the first order upwind scheme for the hyperbolic transport term and the central differences on the dissipation term, we obtain the following numerical scheme,

$$\begin{aligned} \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} &= -\frac{\alpha}{2\Delta x} \left(j_{i+1}^n - j_{i-1}^n\right) + \frac{\sqrt{\alpha\gamma}}{2\Delta x} \left(\rho_{i+1}^n + \rho_{i-1}^n - 2\rho_i^n\right) \\ &\quad + \frac{\beta}{\Delta x^2} \left(\rho_{i+1}^n + \rho_{i-1}^n - 2\rho_i^n\right) \\ \frac{j_i^{n+1} - j_i^n}{\Delta t} &= -\frac{\gamma}{2\Delta x} \left(\rho_{i+1}^n - \rho_{i-1}^n\right) + \frac{\sqrt{\alpha\gamma}}{2\Delta x} \left(j_{i+1}^n + j_{i-1}^n - 2j_i^n\right) + \delta j_i^n. \end{aligned} \tag{26}$$

With the above scheme (26), the numerical stability constrains are

$$\sqrt{\alpha\gamma} \frac{\Delta t}{\Delta x} < 1, \; \frac{\beta \Delta t}{\Delta x^2} < \frac{1}{2}, \; \Delta t \leq -\delta.$$

For a fixed value of the parameter ϕ_ε it is easy to show that the stability condition $\Delta t/\Delta x^2 < 1$, implies the previous conditions for all $0 < \varepsilon < 1$. By

passing to the limit $\varepsilon \to 0^+$, the scheme (26) becomes the classical centered difference discretization to the limiting heat equation, and the scheme is AP. We point out that to achieve a higher order accuracy in time, one can use the classical Runge-Kutta method to replace the forward Euler method.

Remark 2.3. Another approach to the numerical approximation of hyperbolic system (4), or the kinetic counterpart (1), is based on the so called well-balanced (WB) schemes, [**59**]. In this case, deriving the scheme consists essentially in localizing the production term by means of a Dirac comb on a certain lattice in order to accurately control its effects by means of generalized jump relations in a Godunov-type scheme. This technique is also useful for proving theoretical results.

The main reason for introducing the modified model (10), and the corresponding splitting, is given by the fact that, at variance with the the original macroscopic system (4), when $\varepsilon \to 0^+$, the relaxation step always projects the solution to the correct local equilibrium. To avoid the difficulties related with the presence of stiff source terms on the relaxation system it is possible to consider the *relaxed splitting*, that is to say we consider $\varepsilon = 0$ and we "project" j on the term $-(1/2)(\partial\rho/\partial x)$. The relaxed scheme is obtained by discretizing the equations in (10) and then taking the $\varepsilon \to 0^+$ limit. Then the relaxation step (20) becomes

$$\begin{aligned} \rho^{n+1/2} &= \rho^n \\ j^{n+1/2} &= -\frac{1}{2} D(\rho^{n+1/2}), \end{aligned} \tag{27}$$

where D represents a discretization of the spatial derivative. For example, since the limit problem is a diffusion equation, we can consider the central differences scheme,

$$j_i^{n+1/2} = -\frac{1}{2}\frac{\rho_{i+1}^{n+1/2} - \rho_{i-1}^{n+1/2}}{2\Delta x}.$$

Now the upwind scheme of the first order is written as, we have set $\phi_\varepsilon = 1$,

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{1}{2\Delta x}\left(2\rho_i^n - \rho_{i+1}^n - \rho_{i-1}^n\right) - \frac{1}{8\Delta x^2}\left(\rho_{i+2}^n - 2\rho_i^n + \rho_{i-2}^n\right) = 0 \tag{28}$$

$$\begin{aligned} \frac{j_i^{n+1} - j_i^{n+1/2}}{\Delta t} + \frac{1}{2\Delta x}\left(\rho_{i+1}^n - \rho_{i-1}^n\right) \\ + \frac{1}{8\Delta x^2}\left(2(\rho_{i+1}^n - \rho_{i-1}^n) + \rho_{i+2}^n - \rho_{i-2}^n\right) = 0. \end{aligned} \tag{29}$$

The relaxed approximation is also useful for numerical analysis although it is restricted to the case of a small value of ε. As an example, let $\hat{\rho}^n$, $\hat{j}^n$ the piecewise constant functions coinciding with ρ_i^n, and j_i^n generated by the scheme (28)-(29).

L e m m a 2.1. *Assume* $\hat{\rho}^0$,$\hat{j}^0$, $\hat{j}^{1/2} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *then under the CFL condition* $1 > \Delta t(1 + 4\Delta x)/(4\Delta x^2)$, *for all* $t > 0$ *and* $1 \leq p \leq +\infty$

$$\|\hat{\rho}^n\|_p \leq \|\hat{\rho}^0\|_p, \, \|\hat{j}^n\|_p \leq C_p$$

where $\|f\|_p$ *denotes the* $L^p(\mathbb{R})$ *norm of the function* f, *and* C_p *are constants that depend on initial data. Moreover, if* $\hat{\rho}^0$, $\hat{j}^0 \in BV(\mathbb{R})$, *under the same conditions, one has also:*

$$TV(\hat{\rho}^n) \leq TV(\hat{\rho}^0),$$

where $TV(f) = sup \sum_{j=0}^{N} |f(s_j) - f(s_{j-1})|$, *and the supremum is taken over all subdivision of the real line* $\{s_0 < s_1 < \ldots < s_N\}$.

P r o o f. The equation (28) can be rewritten as

$$\rho_i^{n+1} = \rho_i^n \left(1 - \frac{\Delta t}{\Delta x} - \frac{\Delta t}{4\Delta x^2}\right) + \frac{\Delta t}{2\Delta x}\rho_{i+1}^n + \frac{\Delta t}{2\Delta x}\rho_{i-1}^n + \frac{\Delta t}{8\Delta x^2}\rho_{i+2}^n + \frac{\Delta t}{8\Delta x^2}\rho_{i-2}^n. \tag{30}$$

The CFL condition implies that the coefficients are nonnegative with sum equal to 1: hence we obtain control on both the L^p norms and the total variation on $\mathbb{R}$ for the sequence $\hat{\rho}^n$.

Let $W^n = (\hat{\rho}_{i+1}^n - \hat{\rho}_{i-1}^n)/(2\Delta x)$, from linearity the values W^n satisfy the same recurrence (30), then, recalling that $\hat{j}_i^{n+1/2} = W^n/2$,

$$|\hat{j}_i^{n+1/2}| \leq \|\hat{j}^{1/2}\|_\infty$$

and $\hat{j}^{n+1/2}$ is bounded from the assumptions on the initial data.

We can express the transport step using the relaxed values $\hat{j}^{n+1/2}$ as follows, see (28) and (29),

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{1}{2\Delta x}\left(2\rho_i^n - \rho_{i+1}^n - \rho_{i-1}^n\right) - \frac{1}{2\Delta x}\left(j_{i+1}^{n+1/2} - j_{i-1}^{n+1/2}\right) = 0 \tag{31}$$

$$\frac{j_i^{n+1} - j_i^{n+1/2}}{\Delta t} + \frac{1}{2\Delta x}\left(\rho_{i+1}^n - \rho_{i-1}^n\right) + \frac{1}{2\Delta x}\left(2j_i^{n+1/2} - j_{i-1}^{n+1/2} - j_{i+1}^{n+1/2}\right) = 0. \tag{32}$$

Let $Z^n = \rho_n + j^{n+1/2}$, adding (31) and (32) we have

$$(\rho_i^{n+1} + j_i^{n+1}) = \left(1 - \frac{\Delta t}{\Delta x}\right) Z_i^n + \frac{\Delta t}{\Delta x} Z_{i-1}^n.$$

Then

$$\|\rho^{n+1} + j^{n+1}\|_\infty \leq \|Z^n\|_\infty$$

but $\|Z^n\|_\infty = \bar{C}_\infty < \infty$ due to the boundedness of ρ^n and $j^{n+1/2}$. Finally

$$\|j^n\|_\infty = \|j^n + \rho^n - \rho_n\|_\infty \leq \|j^n + \rho^n\|_\infty + \|\rho^n\|_\infty \leq \bar{C}_\infty + \|\rho^0\|_\infty.$$

In a similar way we can prove the estimates in L^p norm. □

R e m a r k 2.4. All the previously schemes deal with the full initial value problem and they can be extended to the initial boundary value problem with specular or periodic conditions at the boundary. If one considers the one-dimensional Goldstein-Taylor model with relaxation (2) with the space variable x in a bounded domain $\Omega = (-l, l)$, $l > 0$, the boundary conditions for the densities u_ε, v_ε must be provided. Consideration on characteristics shows that we have u_ε (respectively v_ε) outgoing wave (respectively incoming wave) at $x = -l$ and v_ε (respectively u_ε) outgoing wave (respectively incoming wave) at $x = l$ which means that this problem is well-posed if the boundary conditions for u_ε, v_ε are

$$u_\varepsilon(-l, t) = g_-(t),\ v_\varepsilon(l, t) = g_+(t). \tag{33}$$

For the of macroscopic variables ρ_ε and j_ε of the equivalent system (4) the boundary conditions are partially unknown. Anyway, it is possible to show [**108**] that the density ρ_ε converges weakly in L^2, as $\varepsilon \to 0^+$ to ρ which is a solution of the linear heat equation (6) with boundary conditions,

$$\rho(-l, t) = 2g_-(t),\ \rho(l, t) = 2g_+(t).$$

For the study of the linear stability of the schemes based on equations (28)-(29) by von Neumann analysis we follow [**28**]. Then, we consider the discrete Fourier modes $u_j^n = \rho^n e^{i(jk/N)}$ into the scheme, where k is the wave number and N the number of cells. Let $\xi = k/N$, we compute the amplification factor $Z(\xi)$ such that $u_j^{n+1} = Z(\xi) u_j^n$. We can consider ξ as a continuous variable, since the amplification factors for various choices of N all lie on the curves obtained considering the variable $\xi \in [0,\, 2\pi]$. Using first order upwind discretization in space and forward Euler time integration, the amplification factor is $Z(\xi) = 1 + M(\xi)$, where

$$M(\xi) = \frac{\Delta t}{\Delta^2 x} \left(\cos(\xi) - 1\right) \left(\cos(\xi) + 1 + \Delta x\right).$$

$M(\xi)$ takes maximum value 0 and attains its minimum at the point ξ^* such that $\cos(\xi^*) = -\Delta x/2$. Stability requires that $M(\xi^*) \geq -2$, i.e.

$$1 + \frac{\Delta t}{\Delta^2 x}\left(\frac{\Delta^2 x}{4} - 1\right) - \frac{\Delta t}{\Delta x}\left(\frac{\Delta x}{2} + 1\right) \geq -1,$$

then

$$\Delta t \leq \frac{2\Delta^2 x}{\left(1 + \frac{\Delta x}{2}\right)^2} \simeq 2\,(1 - \Delta x)\,\Delta^2 x. \tag{34}$$

This gives a CFL condition of the form $\Delta t \leq 2(1-\delta)\Delta^2 x$ where $\delta = O(\Delta x)$ (see Figure 1). As a numerical example we consider the case of the hyperbolic heat equations (4) with $\varepsilon = 10^{-8}$, by solving a Riemann problem with the initial data [**107**],

$$\begin{array}{lll} \rho_L = 2.0, & j_L = 0, & 0 < x < 0.5, \\ \rho_R = 1.0, & j_R = 0, & 0.5 < x < 1, \end{array} \tag{35}$$

on the interval $[0, 1]$. We take 100 spatial cells and $\Delta t = 0.005$. The boundary

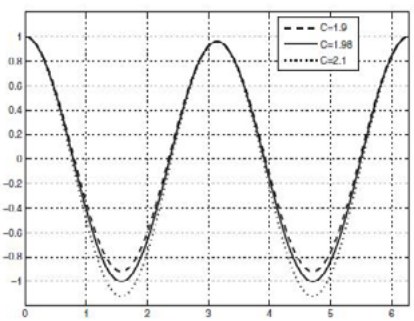

Fig. 1. Amplification factor for upwind spatial reconstruction coupled with forward Euler (left, reproduced with permission from [**28**]).

condition is numerical reflecting, and we compare the numerical results with a reference solution obtained with a fine spatial grid of 5000 cells. Because ε is very small we can consider the solution close to the solution of the limit state which is the linear diffusion equation. The solution, output at $t = 0.04$ and depicted in Figure 2, contains a right moving viscous shock wave. As it can be seen, although the relaxation time is underresolved, both numerical schemes we developed here capture the correct parabolic behavior given by the heat equation.

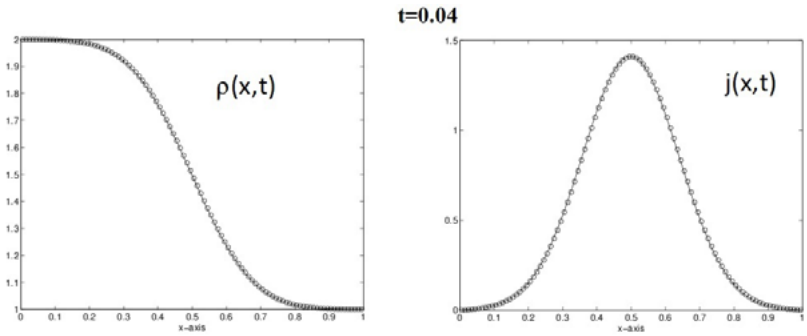

Fig. 2. Numerical solution ∘-line of the density ρ (left) and the flux j (right) at time $t = 0.04$ for a Riemann problem, the "true" solution is a computed solution on a very fine grid.

R e m a r k 2.5. We point out that there are approaches that generalize numerical schemes developed for hyperbolic system (4) in order to obtain accurate and stable methods, in particular in the diffusive limit. In such approaches, based on a systematic use of Runge-Kutta IMEX methods, it is shown that it is not necessary to add and subtract a (non-stiff) convective term to the equation, in order to capture numerically the proper diffusion limit. In some cases, the numerical method applied to the system with parabolic relaxation relaxes to an implicit scheme for the underlying diffusion limit, thus avoiding even the classical parabolic CFL type restriction.

In particular, in [**15**, **16**] the Authors presented a class of IMEX Runge-Kutta schemes that were able to handle the limit diffusive (or convection-diffusion) equation, and the resulting schemes were asymptotic preserving for the limit equation. Furthermore, in these papers the Authors overcame the classical parabolic stability restriction on the time step (dictated by the diffusive behavior of the limit equation) by a reformulation of the starting problem. An analysis of schemes applied to the reformulated problem showed that such schemes require the same stability condition derived for hyperbolic relaxation. While in [**14**] the Authors proposed the same approaches given in [**15**, **16**], but in this work the limit case is a non linear degenerate diffusion equation. In this limit, the dynamics is governed by effective systems of parabolic-type which may contain degenerate and/or fully nonlinear diffusion term. Finally, in [**13**] a general framework to construct and apply linearly implicit schemes to a large class of PDE's containing stiff terms is presented.

3 - Relaxed schemes for nonlinear (degenerate) diffusion problems

The relaxation and relaxed approximation introduced in the previous section can be extended to several partial differential equations (PDE's). In general, These methods are based on replacing the original PDE by a semi-linear hyperbolic system with stiff relaxation terms, tuned by a relaxation parameter ε. When $\varepsilon \to 0^+$, the solution of this system "relaxes" onto the solution of the original PDE. Thus a consistent discretization of the relaxation system for $\varepsilon = 0$ yields a consistent discretization of the original PDE, as can be seen, for instance, in [**74**] and [**4**]. The advantage of this procedure is that the numerical scheme obtained in this fashion does not need approximate Riemann solvers for the convective term, but possesses the robustness of upwind discretizations. Moreover, the complexity introduced by replacing the original PDE with a stiff system of equations is only apparent, because it is possible to manage the discretization in an efficient way. Relaxation approximations for conservation laws were deeply investigated in [**74**, **82**, **98**] and extended to the diffusive case of parabolic equations in [**59**, **72**, **96**]; high order numerical schemes were introduced in [**28**, **30**, **111**]. Moreover, relaxation models based on the Bhatnagar-Gross-Krook (BGK) kinetic approach were developed in [**4**, **81**]. We notice that the relaxation approximation is analogous to the regularization of the Euler equations by the Boltzmann or BGK kinetic equation [**17**, **34**, **40**, **44**, **57**, **103**]. Subsequently, the idea to approximate nonlinear PDE's by relaxation has been also extended to diffusion and convection diffusion equations, see for example [**3**, **4**, **28**, **31**, **72**, **74**, **83**, **95**, **96**].

We have considered the relaxation approximation of the linear diffusion equation starting from the microscopic description of the GoldsteinTaylor model and the Maxwell-Cattaneo system as prototype of the diffusive relaxation. Now we start directly from a nonlinear diffusion equations of the form:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta p(u) &= 0 && \text{in } \Omega \times (0, +\infty) \,, \\
u &= g_D && \text{on } \Gamma_D \times (0, +\infty) \,, \\
\nabla p(u) \cdot \boldsymbol{n}_\Omega &= g_N && \text{on } \Gamma_N \times (0, +\infty) \,, \\
u_{|_{\{t=0\}}} &= u_0 && \text{in } \Omega \,,
\end{aligned}
\tag{36}
$$

where Ω is a convex, polyhedral domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. We denote by $\boldsymbol{n}_\Omega$ the unit normal vector to $\partial\Omega$ pointing outside Ω, and $g_D = g_D(\boldsymbol{x}, t)$, $g_N = g_N(\boldsymbol{x}, t)$, $u_0 = u_0(\boldsymbol{x})$. The considered time domain is $(0, +\infty)$. Moreover, $p : \mathbb{R} \to \mathbb{R}$ is a possibly nonlinear function.

As a typical example of this general model, we might consider a homogeneous, isotropic and rigid porous medium filled with a fluid. If absorption

and chemical, osmotic and thermal effects are ignored, and if we consider for horizontal flow, it is possible to deduce the following equation [**117**]

$$\frac{\partial u}{\partial t} - \Delta u^m = 0,\ m > 0, \tag{37}$$

where $u = u(\boldsymbol{x}, t)$ models the volumetric moisture content; when $p(u) = u^m$, with $m > 1$, equation (37) is usually called the porous medium equation which describes the flow of a gas through a porous interface according to some constitutive relation like Darcy's law in order to link the velocity of the gas and its pressure. In this case the diffusion coefficient mu^{m-1} vanishes at the points where $u \equiv 0$ and the governing parabolic equation degenerates there. The set of such points is called interface. Moreover the porous media equation can exhibit a finite speed of propagation for compactly supported initial data [**6**]. The influence of the degenerate diffusion terms make the dynamics of the interfaces difficult to study from both the theoretical and the numerical point of view. Another interesting case corresponds to $0 < m < 1$ and it is referred to as the *fast diffusion equation* which appears, for example, in curvature-driven evolution and avalanches in sandpile [**117**, **118**].

Here, $p : \mathbb{R} \to \mathbb{R}$ stands for a non decreasing Lipschitz continuous function such that

$$0 \leq l_p \leq p'(s) \leq L_p < +\infty \qquad \text{for a.e. } s \in \mathbb{R}\,, \tag{38}$$

for given constants L_p and l_p, $p(0) = 0$, and there exists $s_0 > 0$ for which,

$$p'(s) > 0 \qquad \text{for a.e. } s \geq s_0\,. \tag{39}$$

In the following, we denote by $H^k = H^k(\Omega)$, k is a positive integer, the Sobolev space of all function u defined in Ω such that u and its distributional derivatives of order k all belong to $L^2(\Omega)$, H^{-1} stands for the dual space of H^1. Furthermore, let W be a separable Banach space, we introduce the following Banach space

$$L^p(0,T;W) = \{u : (0,T) \to W \text{ measurable } : t \mapsto \|u(t)\|_W \text{ belongs to } L^p(0,T)\}$$

endowed with the following norm (in case $p \in [1, \infty)$)

$$\|u\|_{L^p(0,T;W)} = \left(\int_0^T \|u(t)\|_W^p dt \right)^{1/p},$$

and in case $p = \infty$, we take

$$\|u\|_{L^\infty(0,T;W)} = \sup ess_{t\in(0,T)} \|u(t)\|_W.$$

Then, we can define the space $H^1(0,T;W)$ as the space of functions $u \in L^2(0,T;W)$ such that $u' \in L^2(0,T;W)$.

In the case $\Gamma_N = \varnothing$, the variational formulation of problem (36) reads as follows: find u with

$$u \in L^\infty(0,T;L^\infty(\Omega)) \cap H^1(0,T;H^{-1}), \;\; u(\cdot,0) = u_0,$$

such that, for a.e. $t \in (0,T)$ and all $\varphi \in H_0^1(\Omega)$, the following equation holds:

$$\int_\Omega u_t \, \varphi \, d\boldsymbol{x} + \int_\Omega \nabla\theta \cdot \nabla\varphi \, d\boldsymbol{x} = 0,$$

where $\theta(x,t) = p(u(x,t))$, a.e. $x \in \Omega$, $t \in (0,T)$. Well-posedness of this problem is discussed, for example, in [**55**,**66**,**84**], together with the additional regularity result

$$\theta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^{-1}).$$

In general the numerical analysis of equation (36) is difficult for at least two reasons: the appearance of singularities for compactly supported solutions and the growth of the size of the support as time increases (*retention property*).

From the numerical viewpoint, an usual technique to approximate (36) involves implicit discretization in time: it requires, at each time step, the discretization of a nonlinear elliptic problem. However, when dealing with nonlinear problems one generally tries to linearize them in order to take advantage of efficient linear solvers. Linear approximation schemes based on the so-called nonlinear Chernoff's formula with a suitable relaxation parameter have been studied for example in [**11**,**85**,**99**,**100**] where also some energy error estimates have been investigated. Other linear approximation schemes have been introduced by Jäger, Kačur and Handlovičová [**65**, **77**]. More recently, different approaches based on kinetic schemes for degenerate parabolic systems have been considered and analyzed by Aregba-Driollet, Natalini and Tang in [**4**]. Other approaches were investigated in the work of Karlsen et al. [**48**,**49**] based on a suitable splitting technique with applications to more general hyperbolic-parabolic convection-diffusion equations. Finally, a new scheme based on the maximum principle and on a perturbation and regularization approach was proposed by Pop and Yong in [**105**].

3.1 - *Relaxes and conquest*

A relaxation system, for nonlinear diffusion equations like (36), can be obtained introducing two auxiliary variables, as described in [**95**]. The first step

consists in rewriting the second order differential equation as a first order system through the vector auxiliary variable $\boldsymbol{v}$ and the relaxation parameter ε, obtaining

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{v} = 0 \, , \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \dfrac{1}{\varepsilon^2} \nabla(p(u)) = -\dfrac{\boldsymbol{v}}{\varepsilon^2} \, . \end{cases} \tag{40}$$

where $\nabla\cdot$ denotes the divergence operator. Formally, in the small relaxation limit $\varepsilon \to 0^+$, the second equation of (40) reduces to $\boldsymbol{v} = -\nabla(p(u))$, which substituted in the first equation allows to recover the leading order equation (36).

Since (40) is still nonlinear, we need to further relax the second equation. Introducing the scalar auxiliary variable w and a positive constant a, we obtain

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{v} = 0 \, , \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \dfrac{1}{\varepsilon^2} \nabla w = -\dfrac{\boldsymbol{v}}{\varepsilon^2} \, , \\ \dfrac{\partial w}{\partial t} + a^2 \, \nabla \cdot \boldsymbol{v} = -\dfrac{1}{\varepsilon^2} \left(p(u) - w \right) . \end{cases} \tag{41}$$

It is easy to see that when $\varepsilon \to 0^+$ we formally retrieve (36), which is now approximated by a semilinear hyperbolic system. If, for small values of ε, a Chapman-Enskog expansion is performed, it is easy to see that the original equation (36) with a negative fourth order additional term of order $O(\varepsilon^2)$ is retrieved, which results in a stable perturbation of the diffusion equation. For more details on Chapman-Enskog expansion, see [**35**].

Appropriate boundary conditions for system (41) can be deduced from those of (36) and are

$$\begin{aligned} u &= g_D && \text{on } \Gamma_D \times (0, +\infty) \, , \\ \boldsymbol{v} \cdot \boldsymbol{n}_\Omega &= -g_N && \text{on } \Gamma_N \times (0, +\infty) \, , \\ w &= p(g_D) && \text{on } \Gamma_D \times (0, +\infty) \, ; \end{aligned}$$

similarly, suitable initial conditions are

$$\begin{aligned} u_{|_{\{t=0\}}} &= u_0 && \text{in } \Omega \, , \\ \boldsymbol{v}_{|_{\{t=0\}}} &= -\nabla u_0 && \text{in } \Omega \, , \\ w_{|_{\{t=0\}}} &= p(u_0) && \text{in } \Omega \, . \end{aligned}$$

We are interested in developing a numerical approximation for (41) in the relaxed limit, i.e. when $\varepsilon = 0$ (the so called relaxation schemes), but the characteristic velocities of system (43) become stiff as $\varepsilon \to 0^+$. As described in [**96**],

this numerical issue can be dealt with by introducing a constant (dimensional) vector $\boldsymbol{\phi} = (\phi_i)_{i=1,..,d}$, and the $d \times d$ diagonal matrix

$$\Phi = \mathrm{diag}(\boldsymbol{\phi}) \,, \tag{42}$$

whose diagonal elements coincide with the components of $\boldsymbol{\phi}$ (for the linear diffusion we have introduced only the scalar parameter ϕ^2, see (10). The relaxation system can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{v} = 0 \,, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \Phi^2 \, \nabla w = -\dfrac{1}{\varepsilon^2} \left(\boldsymbol{v} - (\varepsilon^2 \Phi^2 - Id) \nabla w \right) \,, \\ \dfrac{\partial w}{\partial t} + a^2 \, \nabla \cdot \boldsymbol{v} = -\dfrac{1}{\varepsilon^2} \left(p(u) - w \right) \,, \end{cases} \tag{43}$$

where Id is the identity matrix. In the previous systems, the parameter ε^2 has the physical dimension of a time, while the dimension of w is equal to the dimension of u times length$\times$length over time, and each component of $\boldsymbol{v}$ has the dimension of u times a velocity; finally the dimension of the diagonal elements ϕ_i^2 of Φ^2 is time^{-1}. In the following, we will set $a = 1$; and consider $\alpha_i \geq 0$, $i = 1, \ldots, d$.

We notice that, since for sufficiently small values of the relaxation parameter ε, the relaxation system (43) gives a "good" approximation of the original equation (36), integrating (43) becomes a convenient way to develop numerical approximation of (36). In fact, thanks to the simple linear structure of characteristic fields and the localized lower order term, one can easily develop numerical schemes that are simple, general and that deal with a wide class of non linearities. In the works (see [**28**,**31**]), high order methods both in time and space were developed using finite difference schemes, while in [**32**] the Authors investigated the possibility of using finite element methods, in order to consider more general domains.

R e m a r k 3.1. As in the linear case, several nonlinear diffusion models are the hydrodynamical limit of a kinetic system. For example, in the kinetic theory of rarefied gases, the two velocity models of the Boltzmann equation are supposed to describe the evolution of the velocity distribution of a fictitious gas composed of two kinds of particles that move parallel to the x axis with constant and equal speeds either in the positive x direction with a density U or in the negative x direction with a density V. The most general two velocity

gas which is in local equilibrium when $U = V$ is described by the equations

$$\begin{aligned} &\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = k(U,V,x)(V-U) \\ &\frac{\partial v}{\partial t} - c\frac{\partial V}{\partial x} = k(U,V,x)(U-V) \end{aligned} \tag{44}$$

where $t \geq 0$, $x \in \mathbb{R}$, c is the modulus of the constant speed of the particles and k is a nonnegative rate coefficient. The most famous example of these models was proposed by Carleman in the 1930's and published in 1957 [**21**]. In Carleman's model $k(U,V,x) = U + V$, so that the collision term on the right-hand side of (44) describes binary interactions between particles. Choosing $k(U,V,x) \equiv 1$, we obtain a linear Goldstein-Taylor model. The system represents the forward equation for the density of a molecule moving with constant speed along the x axis subject to spontaneous reversals of directions at the jump times of a standard Poisson process of unit rate. The macroscopic variables for these models are the mass density $\rho = U+V$, and the flux $j = c(U-V)$. We point out that since U and V can be expressed in terms of ρ and j so that $k = k(\rho, j, x)$, and system (44) is equivalent to the following macroscopic equations for the mass density and the flux

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \\ &\frac{\partial j}{\partial t} + c^2\frac{\partial \rho}{\partial x} = -2k(\rho,j,x)j. \end{aligned} \tag{45}$$

Let us assume that the mean free path is not normalized to unity but is left in the equation as a small parameter ε: we replace k by k/ε^2. The limit $\varepsilon \to 0^+$ corresponds to the transition from a kinetic description of the gas to that of a gas as a continuum, hydrodynamic limit associated with the kinetic system (44). For the Carleman's model, with the scaling $c \to c/\varepsilon$, $k \to k/\varepsilon^2$, we have

$$\begin{aligned} &\frac{\partial U_\varepsilon}{\partial t} + \frac{1}{\varepsilon}\frac{\partial U_\varepsilon}{\partial x} = \frac{1}{\varepsilon^2}(V_\varepsilon^2 - U_\varepsilon^2) \\ &\frac{\partial V_\varepsilon}{\partial t} - \frac{1}{\varepsilon}\frac{\partial V_\varepsilon}{\partial x} = \frac{1}{\varepsilon^2}(U_\varepsilon^2 - V_\varepsilon^2). \end{aligned} \tag{46}$$

The corresponding asymptotic problem was first investigated by Kurtz [**80**]. Using the theory of nonlinear semigroups he proved that starting with a zero flux $U_0 = V_0 \in L^1(\mathbb{R})$, the mass density $\rho_\varepsilon(x,t) = U_\varepsilon(x,t) + V_\varepsilon(x,t)$ converges in L^1_x for all $t \geq 0$ to ρ satisfying the following nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x}\right), \tag{47}$$

while $j_\varepsilon(x,t)$ converges to zero. Subsequently McKean [**89**] generalized the previous result by removing the restriction that the initial flux has to be taken equal to zero. Further results are due to Kaper, Leaf and Reich [**76**] who studied the problem treated by Kurtz with ε dependent initial data and to Fitzgibbon [**54**] who studied the problem in a bounded domain with specular reflecting boundary conditions.

More recently, Toscani and Pulvirenti [**106**] have extended the result of McKean to the system

$$\begin{aligned} &\frac{\partial U_\varepsilon}{\partial t} + \frac{1}{\varepsilon}\frac{\partial U_\varepsilon}{\partial x} = \frac{1}{\varepsilon^2}\rho_\varepsilon^\alpha (V_\varepsilon - U_\varepsilon) \\ &\frac{\partial V_\varepsilon}{\partial t} - \frac{1}{\varepsilon}\frac{\partial V_\varepsilon}{\partial x} = \frac{1}{\varepsilon^2}\rho_\varepsilon^\alpha (U_\varepsilon - V_\varepsilon), \end{aligned} \tag{48}$$

with $0 \le \alpha \le 1$. Formally the following nonlinear diffusion equation for the limit density is the hydrodynamical limit

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right). \tag{49}$$

Lions and Toscani [**83**] solved the case $\alpha \in (-\infty, 1]$ with integrable data plus some regularity or decay conditions. Subsequently, Salvarani and Vázquez [**110**] improved the result of [**83**] in the case $|\alpha| \le 1$, by considering only L^1 data and studied the cases $\alpha > 1$ with integrable data. Moreover, Salvarani and Toscani [**109**] considered the subrange $1 < \alpha < 4/3$ by compactness methods. We emphasize that the case $\alpha < -1$ is of particular interest since we obtain in the limit the well known porous media equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2(1+|\alpha|)}\frac{\partial^2}{\partial x^2}\rho^{1+|\alpha|}. \tag{50}$$

3.2 - *The semidiscrete schemes*

In this section we survey the analysis that has been developed in [**28**] for the diffusive relaxation schemes for the numerical approximation of nonlinear parabolic equations. We suppose that the matrix Φ^2 in (41) is the $d \times d$ identity matrix times the scalar ϕ^2, then the diffusive system (41) can be written in the form:

$$z_t + \nabla \cdot f(z) = \frac{1}{\varepsilon^2} g(z), \tag{51}$$

where (as mentioned above we set $a = 1$)

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad f(z) = \begin{bmatrix} v^T \\ \Phi^2 w \\ v^T \end{bmatrix} \quad g(z) = \begin{pmatrix} 0 \\ -v + (\Phi^2\varepsilon^2 - \mathbb{I})\nabla w \\ p(u) - w \end{pmatrix}. \tag{52}$$

We start discretizing the system in time using, for simplicity, a uniform time step Δt. Let $z^n(x) = z(x, t^n)$, with $t^n = n\Delta t$. Since equation (51) involves both stiff and non-stiff terms, it is a natural idea to employ different time-discretization strategies for each of them, as in [**7**, **102**]. In this work we integrate (51) with a Runge-Kutta IMEX scheme [**7**, **30**, **102**], obtaining the following semidiscrete formulation

$$z^{n+1} = z^n - \Delta t \sum_{i=1}^{\nu} \tilde{b}_i \nabla\cdot f(z^{(i)}) + \frac{\Delta t}{\varepsilon^2} \sum_{i=1}^{\nu} b_i g(z^{(i)}), \tag{53}$$

where the $z^{(i)}$'s are the stage values of the Runge-Kutta scheme which are given by

$$z^{(i)} = z^n - \Delta t \sum_{k=1}^{i-1} \tilde{a}_{i,k} \nabla\cdot f(z^{(k)}) + \frac{\Delta t}{\varepsilon^2} \sum_{k=1}^{i} a_{i,k} g(z^{(k)}), \tag{54}$$

where $\tilde{b}_i$, $\tilde{a}_{ij}$ and b_i, a_{ij} denote the coefficients of the explicit and implicit Runge-Kutta schemes, respectively. We assume that the implicit scheme is of diagonally implicit type. To find the $z^{(i)}$'s it is necessary in principle to solve a non linear system of equations which however can be easily decoupled. The system for the first stage $z^{(1)}$ at time t^n is:

$$\begin{pmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \end{pmatrix} = \begin{pmatrix} u^n \\ v^n \\ w^n \end{pmatrix} + \frac{\Delta t}{\varepsilon^2} a_{11} \begin{pmatrix} 0 \\ -v^{(1)} + (\Phi^2\varepsilon^2 - D)\nabla w^{(1)} \\ p(u^{(1)}) - w^{(1)} \end{pmatrix}. \tag{55}$$

The first equation yields $u^{(1)} = u^n$, substituting in the third equation we immediately find $w^{(1)}$ and finally, substituting $w^{(1)}$ in the second equation, we compute $v^{(1)}$. In other words the system can be written in triangular form. For the following stage values, we group the already computed terms in the vector $B^{(i)}$ given by

$$B^{(i)} = z^n - \Delta t \sum_{k=1}^{i-1} \tilde{a}_{i,k} \nabla\cdot f(z^{(k)}) + \frac{\Delta t}{\varepsilon^2} \sum_{k=1}^{i-1} a_{i,k} g(z^{(k)}), \tag{56}$$

then the new stage values are given by

$$\begin{pmatrix} u^{(i)} \\ v^{(i)} \\ w^{(i)} \end{pmatrix} = B^{(i)} + \frac{\Delta t}{\varepsilon^2} a_{ii} \begin{pmatrix} 0 \\ -v^{(i)} + (\Phi^2\varepsilon^2 - \mathbb{I})\nabla w^{(i)} \\ p(u^{(i)}) - w^{(i)} \end{pmatrix}, \tag{57}$$

which is again a triangular system.

Following [**74**] we set $\varepsilon^2 = 0$ thus obtaining the so called *relaxed scheme*. The computation of the first stage reduces to

$$\begin{aligned} u^{(1)} &= u^n \\ w^{(1)} &= p(u^{(1)}) \\ v^{(1)} &= -\nabla w^{(1)}. \end{aligned} \tag{58}$$

For the following stages the first equation is

$$u^{(i)} = u^n - \Delta t \sum_{k=1}^{i-1} \tilde{a}_{i,k} \nabla \cdot v^{(k)}. \tag{59}$$

In the other equations the convective terms are dominated by the source terms and thus $v^{(i)}$ and $w^{(i)}$ are given by

$$\begin{aligned} v^{(i)} &= -\nabla w^{(i)}, \\ w^{(i)} &= p(u^{(i)}). \end{aligned} \tag{60}$$

We see that only the explicit part of the Runge-Kutta method is involved in the updating of the solution. Then, in the relaxed schemes we use only the explicit part of the tableaux. In particular the second and third order Strong Stability-Preserving Runge-Kutta (SSRK) schemes [**62**], are defined as in the following

`SSRK1` (1^{st} order)

0	
1	

`SSRK2` (2^{nd} order)

0	0	
1	0	
$\frac{1}{2}$	$\frac{1}{2}$	

`SSRK3` (3^{rd} order)

0	0	0
1	0	0
$\frac{1}{4}$	$\frac{1}{4}$	0
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

3.2.1 - `Convergence of the semidiscrete relaxed scheme`

The aim of this section is to show the L^1 convergence of the solution of the semidiscrete in time relaxed scheme defined by equations (58), (59) and

(60). We summarize the work [**28**] in which the Authors have extended the results proved in [**11**], where only the case of forward Euler time stepping was considered.

Eliminating v from (58) and (59) using (60), we rewrite the relaxed scheme as

$$\begin{aligned} u^{(1)} &= u^n, \\ w^{(1)} &= p(u^n) \end{aligned} \tag{61}$$

for the first stage, and (in the following Δw represents the Laplace operator of the function w)

$$\begin{aligned} u^{(i)} &= u^n + \Delta t \textstyle\sum_{k=1}^{i-1} \tilde{a}_{i,k} \Delta w^{(k)} \\ w^{(i)} &= p(u^{(i)}), \end{aligned} \tag{62}$$

for subsequent stages. We recall that a Runge-Kutta scheme for the ordinary differential equation $y' = R(y)$ can also be written in the form [**62**]

$$\begin{aligned} &y^{(1)} = y^n \\ &y^{(i)} = \textstyle\sum_{k=1}^{i-1} \alpha_{ik} \left(y^{(k)} + \Delta t \frac{\beta_{ik}}{\alpha_{ik}} R(y^{(k)}) \right) \qquad i = 2, \ldots, \nu, \end{aligned} \tag{63}$$

where $y^{n+1} = y^{(\nu)}$. We point out that we will use a different notation than the classical definition of the SSP schemes: we start numbering from one in (63) instead of zero and for $(\nu - 1)$ stages. For consistency, $\sum_{k=1}^{i-1} \alpha_{ik} = 1$ for every $i = 1, \ldots, \nu$. Moreover we assumed that $\alpha_{ik} \geq 0$, $\beta_{ik} \geq 0$ and that $\alpha_{ik} = 0$ implies $\beta_{ik} = 0$. Under these assumptions, each stage value $y^{(i)}$ can be written as a convex combination of forward Euler steps. This remark allows us to study the convergence of the Runge-Kutta scheme in terms of the convergence of the explicit forward Euler scheme applied to the non-linear diffusion problem. This latter was studied in [**11**] via a nonlinear semigroup argument. In the following we review the approach of [**11**] and next we extend the proof to the case of a ν-stages explicit Runge-Kutta scheme.

We wish to solve the evolution equation

$$\frac{du}{dt} + Lp(u) = 0 \qquad u(\cdot, t = 0) = u_0, \tag{64}$$

on the domain Ω, where $L = -\Delta$ and $p : \mathbb{R} \to \mathbb{R}$ is a non decreasing locally Lipschitz function such that $p(0) = 0$. Under these hypotheses, the nonlinear operator $Au = Lp(u)$ with domain $D(A) = \{u \in L^1(\Omega) : p(u) \in D(L)\}$, where $D(L)$ is the domain of the operator L, is m-accretive in $L^1(\Omega)$, that is $\forall \varphi \in \mathrm{L}^1(\Omega)$ and $\forall \lambda > 0$ there exists a unique solution $u \in D(A)$ such that $u + \lambda Lp(u) = \varphi$ and the application defined by $\varphi \mapsto u$ is a contraction [**41**].

Moreover $D(A)$ is dense in $L^1(\Omega)$, so it follows that

$$S_A(t)u_0 = \lim_{m\to\infty} \left(\mathbb{I} + \frac{t}{m}A\right)^{-m} u_0 \tag{65}$$

is a contraction semigroup on $L^1(\Omega)$ and $S_A(t)u_0$ is the generalized solution of (64) in the sense of Crandall-Liggett [**41**]. Let $S(t)$ be the linear contraction semigroup generated by $-L$, that is $u(t) = S(t)u_0$ is the solution of the initial value problem $u_t = -L(u)$ and $u(\cdot, t = 0) = u_0$. In the algorithm proposed in [**11**] a corresponding linear problem is solved

$$\frac{u^{n+1} - u^n}{\tau} + \left[\frac{\mathbb{I} - S(\sigma_\tau)}{\sigma_\tau}\right] p(u^n) = 0, \tag{66}$$

where τ is the time step and $\sigma_\tau \downarrow 0$. This can be written as

$$u^{n+1} = F_E(\tau)u^n \quad \text{where } F_E(\tau)\varphi = \varphi + \frac{\tau}{\sigma_\tau}\left[S(\sigma_\tau) - \mathbb{I}\right] p(\varphi). \tag{67}$$

Hence

$$u^n = (F_E(\tau))^n u_0. \tag{68}$$

The proof of the convergence of the algorithm in [**11**] is based on the following argument. Note that formally $S(\sigma_\tau)\varphi \sim e^{-\sigma_\tau L}\varphi$. Let $t = \tau n$

$$\begin{aligned} u(t) &= \left[\mathbb{I} + \tfrac{t}{n\sigma_\tau}\left(S(\sigma_\tau) - \mathbb{I}\right) \circ p\right]^n u_0 \\ &= \left[\mathbb{I} + \tfrac{t}{n\sigma_\tau}\left(e^{-\sigma_\tau L} - \mathbb{I}\right) \circ p\right]^n u_0 \qquad \text{if } \sigma_\tau \to 0 \\ &= \left[\mathbb{I} - \tfrac{t}{n} L \circ p\right]^n u_0 \\ &\to S_A(u_0) \qquad \text{when } n \to \infty\ . \end{aligned} \tag{69}$$

The convergence proof requires that $\mu\frac{\tau}{\sigma_\tau} \leq 1$ where μ is the Lipschitz constant of $p(u)$. We point out that σ_τ is linked to the spatial approximation of the operator L and this requirement will be reflected in the stability condition of the fully discrete scheme.

Now we are going to describe the case of a ν-stages Runge-Kutta scheme, proving its convergence.

Let $t > 0$ and $\tau = t/n$ with $n \geq 1$; let $\sigma_\tau : (0, \infty) \to (0, \infty)$ be a function such that $\lim_{\tau\to 0} \sigma_\tau = 0$.

$$\begin{aligned} &u^{(1)} = u^n, \\ &u^{(i)} = \textstyle\sum_{k=1}^{i-1} \alpha_{ik} \left[u^{(k)} + \tau\frac{\beta_{ik}}{\alpha_{ik}} A(u^{(k)})\right] \qquad i = 2, \ldots, \nu \end{aligned} \tag{70}$$

and proceeding as in (69), this becomes

$$
\begin{aligned}
&u^{(1)} = u^n, \\
&u^{(i)} = \textstyle\sum_{k=1}^{i-1} \alpha_{ik} \left[u^{(k)} + \tau \frac{\beta_{ik}}{\alpha_{ik}} (S(\sigma_\tau) - \mathbb{I}) \circ p(u^{(k)}) \right] \qquad i = 2, \dots, \nu \\
&u^{n+1} = u^{(\nu)}.
\end{aligned} \tag{71}
$$

We now extend (67) to the Runge-Kutta scheme defined by equation (71). Define, for $\phi \in L^1(\Omega)$,

$$
\begin{aligned}
&F^{(1)}(\tau)\phi = \phi, \\
&F^{(i)}(\tau)\phi = \textstyle\sum_{k=1}^{i-1} \alpha_{ik} F^{(k)}(\tau)\phi + \frac{\tau \beta_{ik}}{\sigma_\tau} \left[S(\sigma_\tau) - \mathbb{I} \right] p(F^{(k)}(\tau)\phi), \\
&F(\tau)\phi = F^{(\nu)}(\tau)\phi
\end{aligned} \tag{72}
$$

and therefore

$$
u^n(t) = [F(\tau)]^n u_0. \tag{73}
$$

Let $u(t)$ be the generalized solution of (64). The following theorem (see [**28**]) proves the convergence of the semidiscrete solution to $u(t)$.

Theorem 3.1. *Assume $u^0 \in L^\infty(\Omega)$, and $\|u^0\|_\infty = M$; let p be a non-decreasing Lipschitz continuous function on $[-M, M]$ with Lipschitz constant μ. Assume that the following conditions hold*

$$
\begin{cases}
\alpha_{ik} \geq 0, \\
\beta_{ik} \geq 0, \\
\alpha_{ik} = 0 \Rightarrow \beta_{ik} = 0, \\
\displaystyle\sum_{k=1}^{i-1} \alpha_{ik} = 1 \ (\textit{consistency}), \\
\dfrac{\mu\tau}{\sigma_\tau} \leq \min \dfrac{\alpha_{ik}}{\beta_{ik}}, \quad \textit{for } \tau > 0, \alpha_{ik} \neq 0 \qquad (\textit{stability}),
\end{cases} \tag{74}
$$

then $\lim_{n\to\infty} u^n(t) = u(t)$ in L^1. Moreover the convergence is uniform for t in any given bounded interval.

The proof follows the steps of [**11**]: first we show that u^n verifies a maximum principle (Lemma 3.1) and that F is a contraction (Lemma 3.2) and finally we apply the non linear Chernoff formula [**20**].

Lemma 3.1. *If (74) is verified, then $-M \leq u^n \leq M \quad \forall n$.*

The proof of this lemma is based on an induction argument [**28**]. Now we can replace p by $\bar{p}$, where $\bar{p}(x) = p(x)$ in $-M \leq x \leq M$, $\bar{p}(x) = p(M)$ for $x \geq M$ and $\bar{p}(x) = p(-M)$ for $x \leq -M$: the algorithm is the same and in what follows we can assume that p is Lipschitz continuous with constant μ on all $\mathbb{R}$.

L e m m a 3.2. *If the hypotheses of Theorem 3.1 hold, then $F(\tau)$ is a contraction on $L^1(\Omega)$, i.e.*

$$\|F(\tau)\phi - F(\tau)\psi\|_1 \leq \|\phi - \psi\|_1 \qquad \forall \psi, \phi \in L^1. \tag{75}$$

P r o o f. We start showing that the result holds for a single forward Euler step. Recalling the definition of F_E from (67)

$$\begin{aligned} &\|F_E(\tau)\phi - F_E(\tau)\psi\|_1 \\ &\qquad \leq \tfrac{\tau}{\sigma_\tau} \|S(\sigma_\tau)[p(\phi) - p(\psi)]\|_1 + \left\|(\phi - \psi) - \tfrac{\tau}{\sigma_\tau}[p(\phi) - p(\psi)]\right\|_1 \\ &\qquad \leq \tfrac{\tau}{\sigma_\tau} \|p(\phi) - p(\psi)\|_1 + \left\|\left(\phi - \tfrac{\tau}{\sigma_\tau} p(\phi)\right) - \left(\psi - \tfrac{\tau}{\sigma_\tau} p(\psi)\right)\right\|_1 \\ &\qquad = \|\phi - \psi\|_1 \end{aligned} \tag{76}$$

where we used the contractivity of S. The last equality relies on the fact that p and the function $x \mapsto x - \frac{\tau}{\sigma_\tau} p(x)$ are non-decreasing, which in turn is guaranteed by the stability condition, that in this case reduces to $\mu\tau/\sigma_\tau \leq 1$ [**11**].

In the general case we have:

$$\begin{aligned} &\|F^{(i)}(\tau)\phi - F^{(i)}(\tau)\psi\|_1 \\ &\qquad \leq \textstyle\sum_{k=1}^{i-1} \alpha_{ik} \left\|F_E\left(\frac{\tau\beta_{ik}}{\alpha_{ik}}\right) F^{(k)}(\tau)\phi - F_E\left(\frac{\tau\beta_{ik}}{\alpha_{ik}}\right) F^{(k)}(\tau)\psi\right\|_1 \\ &\qquad \leq \textstyle\sum_{k=1}^{i-1} \alpha_{ik} \left\|F^{(k)}(\tau)\phi - F^{(k)}(\tau)\psi\right\|_1 \\ &\qquad \leq \|\phi - \psi\|_1 . \end{aligned} \tag{77}$$

In the second inequality we used the contractivity of F_E and the stability condition, while in the third one we apply an induction argument on the contractivity of $F^{(k)}$, the positivity constraint on α_{ik} and β_{ik}, as well as the consistency condition $\sum_k \alpha_{ik} = 1$. Setting $i = \nu$ yields the result. □

P r o o f [Proof of Theorem 3.1]. Let ψ_τ and ψ be respectively

$$\psi_\tau = \left(I + \frac{\lambda}{\tau}(I - F(\tau))\right)^{-1} \phi \qquad \text{and} \qquad \psi = (I + \lambda A)^{-1} \phi. \tag{78}$$

The function ψ exists since the operator A is m-accretive, whereas the existence of the function ψ_τ is guaranteed by the following fixed-point argument. Let

$$G(y) = \frac{1}{1+\eta}\phi + \frac{\eta}{\eta+1} F(\tau) y,$$

where $\phi \in \mathrm{L}^1$, $y \in \overline{D(A)}$ and $\eta \geq 0$. We have,

$$\|G(y) - G(x)\| = \frac{\eta}{\eta+1}\|F(\tau)y - F(\tau)x\| \leq \frac{\eta}{\eta+1}\|y - x\|$$

since F is a contraction, as proved in Lemma 3.2. Thus G is also a contraction and therefore it possesses a unique fixed point which coincides with ψ_τ.

We want to show that

$$\psi_\tau \to \psi \qquad \text{in } \mathrm{L}^1$$

as $\tau \to 0$ for each fixed $\lambda > 0$. Let

$$\phi_\tau = \psi + \frac{\lambda}{\tau}(\mathbb{I} - F(\tau))\psi.$$

We want to estimate $\psi_\tau - \psi$ in terms of $\phi_\tau - \phi$.

$$\phi_\tau - \phi = (1 + \frac{\lambda}{\tau})(\psi - \psi_\tau) - \frac{\lambda}{\tau}(F(\tau)\psi - F(\tau)\psi_\tau)$$

Therefore

$$(1 + \frac{\lambda}{\tau})(\psi - \psi_\tau) - (\phi_\tau - \phi) = \frac{\lambda}{\tau}(F(\tau)\psi - F(\tau)\psi_\tau)$$

and taking norms and using the fact that F is contraction we have

$$\left|(1 + \frac{\lambda}{\tau})\|\psi - \psi_\tau\| - \|\phi_\tau - \phi\|\right| \leq \|(1 + \frac{\lambda}{\tau})(\psi - \psi_\tau) - (\phi_\tau - \phi)\| \leq \frac{\lambda}{\tau}\|\psi - \psi_\tau\|.$$

In particular

$$(1 + \frac{\lambda}{\tau})\|\psi - \psi_\tau\| - \|\phi_\tau - \phi\| \leq \frac{\lambda}{\tau}\|\psi - \psi_\tau\|$$

and therefore $\|\psi - \psi_\tau\| \leq \|\phi - \phi_\tau\|$.

Now we estimate $\|\phi - \phi_\tau\|$ in the simple case of a forward Euler scheme. Note that

$$\phi - \phi_\tau = \lambda A\psi - \frac{\lambda}{\tau}(\mathbb{I} - F(\tau))\psi$$

and thus $\|\phi - \phi_\tau\|$ measures a sort of consistency error. For a single forward Euler step, $F = F_E$ where F_E is defined in (67). Thus

$$\|\phi - \phi_\tau\| = \lambda \left\| A\psi - \frac{1}{\sigma_\tau}(\mathbb{I} - S(\sigma_\tau))p(\psi) \right\| \to 0 \tag{79}$$

as $\tau \to 0$ since $\frac{\mathbb{I} - S(\sigma_\tau)}{\sigma_\tau} p(\psi) \to Lp(\psi) = A\psi$.

The more general case of a ν-stages Runge-Kutta scheme can be carried out by induction following the procedure already applied in the proofs of the previous lemmas.

We now use Theorem 3.2 of [**20**] which, specialized to our case, can be written as follows. Assume that $F(\tau) : \mathrm{L}^1 \to \mathrm{L}^1$ for $\tau > 0$ is a family of contractions. Assume further that an m-accretive operator A is given and let $S(t)$ be the semigroup generated by A. In addition, assume further that the family $F(\tau)$ and the operator A are linked by the following formula

$$\psi_\tau = \left(I + \frac{\lambda}{\tau}(I - F(\tau)) \right)^{-1} \phi \to \psi = (I + \lambda A)^{-1} \phi \tag{80}$$

for each $\phi \in \mathrm{L}^1$. Then

$$\lim_{n\to\infty} F\left(\frac{t}{n}\right)^n \phi = S(t)\phi \qquad \forall \phi \in \mathrm{L}^1.$$

□

4 - Fully discrete relaxed scheme

In order to complete the description of the scheme, we need to specify the space discretization. We will first describe the discretizations based on finite differences, then we will sketch approximations based on a discontinuous Galerkin method in space. Note that the IMEX technique reduces the integration to a cascade of relaxation and transport steps. The former are the implicit parts of (55) and (57), while the transport steps appear in the evaluation of the explicit terms $B^{(i)}$ in (56). Since (55) and (57) involve only local operations, the main task of the space discretization is the evaluation of $\nabla\cdot(f)$, where we will exploit the linearity of f in its arguments.

4.1 - *Finite differences approximation*

We will start with the one-dimensional (in space) case. Let us introduce a uniform grid on $[a, b] \subset \mathbb{R}$, $x_j = a - \frac{h}{2} + jh$ for $j = 1, \ldots, n$, where $h = (b-a)/N$ is the grid spacing and N the number of cells. The fully discrete scheme may be written as

$$z_j^{n+1} = z_j^n - \Delta t \sum_{i=1}^{\nu} \tilde{b}_i \left(F^{(i)}_{j+1/2} - F^{(i)}_{j-1/2} \right) + \frac{\Delta t}{\varepsilon} \sum_{i=1}^{\nu} b_i g(z_j^{(i)}), \tag{81}$$

where $F^{(i)}_{j+1/2}$ are the numerical fluxes, which are the only item that we still need to specify. For convergence it is necessary to write the scheme in conservation form. Thus, following [**113**], we introduce the function $\hat{F}$ such that

$$f(z(x,t)) = \frac{1}{h}\int_{x-h/2}^{x+h/2} \hat{F}(s,t)\mathrm{d}s \Rightarrow \frac{\partial f}{\partial x}(z(x_j,t)) = \frac{1}{h}\left(\hat{F}(x_{j+1/2},t) - \hat{F}(x_{j-1/2},t)\right).$$

The numerical flux function $F_{j+1/2}$ approximates $\hat{F}(x_{j+1/2})$.

In order to compute these numerical fluxes, for each stage value, we reconstruct boundary extrapolated data $z^{(i)\pm}_{j+1/2}$ with a non-oscillatory interpolation method from the point values $z^{(i)}_j$ of the variables at the center of the cells. Next we apply a monotone numerical flux to these boundary extrapolated data.

To minimize numerical viscosity we choose the Godunov flux, which in the present case of a linear system of equations reduces to the upwind flux. In order to select the upwind direction we write the system in characteristic form. The characteristic variables relative to the eigenvalues $\phi, -\phi, 0$ (in one space dimension Φ reduces to a scalar parameter) are respectively

$$U = \frac{v+\phi w}{2\phi} \qquad V = \frac{\phi w - v}{2\phi} \qquad W = u - w. \tag{82}$$

Note that $u = U + V + W$. Therefore the numerical flux in characteristic variables is $F_{j+1/2} = (\phi U^-_{j+1/2}, -\phi V^+_{j+1/2}, 0)$.

The accuracy of the scheme depends on the accuracy of the reconstruction of the boundary extrapolated data. For a first order scheme we use a piecewise constant reconstruction such that $U^-_{j+1/2} = U_j$ and $V^+_{j+1/2} = V_{j+1}$. For higher order schemes, we use ENO or WENO reconstructions of appropriate accuracy ([**112**]).

For $\varepsilon \to 0$ we obtain the relaxed scheme. Recall from equation (60) that the relaxation steps reduce to

$$w^{(i)}_j = p(u^{(i)}_j), \qquad v^{(i)}_j = -D\widehat{\nabla} w^{(i)}_j, \tag{83}$$

where $\widehat{\nabla}$ is a suitable approximation of the one-dimensional gradient operator. Thus the transport steps need to be applied only to $u^{(i)}$

$$u^{(i)}_j = u^n_j - \lambda \sum_{k=1}^{i-1} \tilde{a}_{i,k}\left[\phi\left(U^{(k)-}_{j+1/2} - U^{(k)-}_{j-1/2}\right) - \phi\left(V^{(k)+}_{j+1/2} - V^{(k)+}_{j-1/2}\right)\right]. \tag{84}$$

Finally, taking the last stage value and going back to conservative variables,

$$u_j^{n+1} = u_j^n - \tfrac{\lambda}{2}\sum_{i=1}^{\nu}\tilde{b}_i \quad \Big([v_{j+1/2}^{(i)-} + v_{j+1/2}^{(i)+} - (v_{j-1/2}^{(i)-} + v_{j-1/2}^{(i)+})] \\ +\phi[w_{j+1/2}^{(i)-} - w_{j+1/2}^{(i)+} - (w_{j-1/2}^{(i)-} - w_{j-1/2}^{(i)+})]\Big)\,. \tag{85}$$

We wish to emphasize that the scheme reduces to the time advancement of the single variable u. Although the scheme is based on a system of three equations, the construction is used only to select the correct upwinding for the fluxes of the relaxed scheme and the computational cost of each time step remains moderate.

The relaxed scheme in the first order case reduces to:

$$u_j^{n+1} = u_j^n + \frac{\lambda}{2}\left(\partial_x p(u^n)|_{j+1} - \partial_x p(u^n)|_{j-1}\right) \\ + \frac{\lambda}{2}\phi\left(p(u_{j+1}^n) - 2p(u_j^n) + p(u_{j-1}^n)\right). \tag{86}$$

We wish to compute the restrictions on λ and φ so that the scheme is total variation non-increasing. We select the centered finite difference formula to approximate the partial derivatives of $p(u)$; we drop the index n and write p_j for $p(u_j^n)$. Define $\Delta_{j+1/2} = \frac{p_{j+1}-p_j}{u_{j+1}-u_j}$ and observe that these quantities are always nonnegative since p is nondecreasing. We obtain

$$\begin{aligned} TV(u^{n+1}) &= \sum_j |u_j^{n+1} - u_{j-1}^{n+1}| = \\ &\le \sum_j \Big\{ \frac{\lambda}{4h}\Delta_{j+3/2}|u_{j+2} - u_{j+1}| + \frac{\lambda}{2}\phi\Delta_{j+1/2}|u_{j+1} - u_j| \\ &\quad + \left(1 - \lambda(\frac{1}{2h} + \phi)\Delta_{j-1/2}\right)|u_j - u_{j-1}| \\ &\quad + \frac{\lambda}{2}\phi\Delta_{j-3/2}|u_{j-1} - u_{j-2}| + \frac{\lambda}{4h}\Delta_{j-5/2}|u_{j-2} - u_{j-3}|\Big\} \end{aligned} \tag{87}$$

provided that

$$(1 - \lambda(\frac{1}{2h} + \phi)\Delta_{j-1/2}) \ge 0 \qquad \forall j. \tag{88}$$

Assuming that the data have compact support, we can rescale all sums and finally get $TV(u^{n+1}) \le TV(u^n)$. Taking into account the Lipschitz condition on p, the scheme is total variation stable provided that (88) is satisfied, i.e. that

$$\Delta t \le \frac{2h^2}{\mu}\frac{1}{1+2h\phi} \simeq \frac{(2-\delta)}{\mu}h^2 \tag{89}$$

where δ vanishes as h does. We point out that the stability condition is of parabolic type. Finally, we observe that using one-sided approximations for the partial derivatives of p in the scheme (86), one gets a stability condition involving the relation $\phi > 1/h$. This would reintroduce in the scheme the constraint due to the stiffness in the convective term that prompted the introduction of Φ in (43).

4.1.1 - Boundary conditions

Different boundary conditions can be implemented. Here we describe how to implement Neumann boundary conditions, considering for simplicity the one-dimensional case. We first add L_g ghost points on each side of the computational domain $[a, b]$, where L_g depends on the order of the spatial reconstruction. We find a polynomial $q(x)$ of degree d passing through the points (x_i, u_i) for $i = 1, \ldots, d$ and having prescribed derivative at the boundary point $x_{1/2} = a$. (The degree d is determined by the accuracy of the scheme that one wants to obtain and should match the degree of the reconstruction procedures used to obtain $U_j^{\pm}$ and $V_j^{\pm}$.) This polynomial is then used to set the values $u_{-i} = q(x_{-i})$ of the ghost points for $i = 0, 1, L_g - 1$. One operates similarly at the right edge of the computational domain. It is also possible to use periodic boundary conditions, which can be implemented with obvious choice of the values u_i at the ghost points.

4.1.2 - Multi-dimensional scheme

An appropriate numerical approximation of (43) in $\mathbb{R}^d$ that generalizes the $1D$ scheme can be obtained by additive dimensional splitting. We consider the relaxed scheme, i.e. $\varepsilon = 0$ and for the sake of simplicity, let us focus on the square domain $[a, b] \times [a, b] \subset \mathbb{R}^2$. Here we shall describe the generalization of the scheme defined by equations (83), (82), (84) and (85) to the case of two space dimensions.

Without loss of generality, we consider a uniform grid in $[a, b] \times [a, b] \subset \mathbb{R}^2$ such that $\vec{x}_{i,j} = (x_i, y_j) = (a-h/2, a-h/2)+i(h, 0)+j(0, h)$ for $i, j = 1, 2, \ldots, N$ and $h = (b - a)/N$.

In the present case, u and w are one-dimensional variables, while $\vec{v} = (v_{(1)}, v_{(2)})$ is now a field in $\mathbb{R}^2$. First we observe that the relaxation steps (83) are easily generalized for $d > 1$. For the transport steps, one has to evolve

in time the system

(90)
$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v_{(1)} \\ v_{(2)} \\ w \end{pmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \phi^2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{pmatrix} u \\ v_{(1)} \\ v_{(2)} \\ w \end{pmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi^2 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} u \\ v_{(1)} \\ v_{(2)} \\ w \end{pmatrix} = 0.$$

The semidiscretization in space of the above equation can be written as

$$\frac{\partial z_{i,j}}{\partial t} = -\frac{1}{h}\left(F_{i+1/2,j} - F_{i-1/2,j}\right) - \frac{1}{h}\left(G_{i,j+1/2} - G_{i,j-1/2}\right),$$

where F and G are the numerical fluxes in the x and y direction respectively and can be written as

$$F_{i+1/2,j} = F(z^+_{i+1/2,j}, z^-_{i+1/2,j}) \qquad G_{i,j+1/2} = G(z^+_{i,j+1/2}, z^-_{i,j+1/2}).$$

The fluxes in the two directions are computed separately. We illustrate the computation of the flux F along the x direction. We note that only the field $v_{(1)}$ appears in the differential operator along this direction. The third component of the flux is zero and thus we have three independent characteristic variables, namely

$$U_{(1)} = \frac{\phi w + v_1}{2\phi} \qquad V_{(1)} = \frac{\phi w - v_1}{2\phi} \qquad W = u - w,$$

which correspond respectively to the eigenvalues $\phi, -\phi, 0$. At this point the numerical fluxes can be easily evaluated by upwinding. We proceed similarly for the numerical flux G that depends on the characteristic variables $U_{(2)}, V_{(2)}, W$.

Denote by $U^{\pm}_{i+1/2,j}$ the reconstructions of $U_{(1)}(\cdot, y_j)$ at the point $(x_i + h/2, y_j)$. This involves a reconstruction of the restriction of $U_{(1)}$ to the line $y = y_i$ and can be obtained with any of the one-dimensional techniques. Similarly, denote $U^{\pm}_{i,j+1/2}$ the reconstructions of $U_{(2)}(x_i, \cdot)$ at the point $(x_i, y_j + h/2)$. Now, formulas (84) and (85) become respectively

(91)
$$u^{(l)}_{i,j} = u^n_{i,j} - \lambda \textstyle\sum_{m=1}^{l-1} \tilde{a}_{l,m} \quad \left[\phi\left(U^{(m)-}_{i+1/2,j} - U^{(m)-}_{i-1/2,j}\right) - \phi\left(V^{(m)+}_{i+1/2,j} - V^{(m)+}_{i-1/2,j}\right)\right.$$
$$\left.\phi\left(U^{(m)-}_{i,j+1/2} - U^{(m)-}_{i,j-1/2}\right) - \phi\left(V^{(m)+}_{i,j+1/2} - V^{(m)+}_{i,j-1/2}\right)\right]$$

and

(92)
$$u^{n+1}_{i,j} = u^n_{i,j} - \lambda \textstyle\sum_{l=1}^{\nu} \tilde{b}_l \quad \left[\phi\left(U^{(l)-}_{i+1/2,j} - V^{(l)+}_{i+1/2,j}\right) - \phi\left(U^{(l)-}_{i-1/2,j} - V^{(l)+}_{i-1/2,j}\right)\right.$$
$$\left.\phi\left(U^{(l)-}_{i,j+1/2} - V^{(l)+}_{i,j+1/2}\right) - \phi\left(U^{(l)-}_{i,j-1/2} - V^{(l)+}_{i,j-1/2}\right)\right]$$

	N=60	N=180	N=540	N=1620
ENO2, RK1	2.6365e-04	1.9898e-05	2.049e-06	2.076e-07
ENO3, RK2	1.9605e-05	6.0423e-07	2.4141e-08	8.9729e-10
ENO4, RK2	1.2127e-05	2.967e-07	9.9925e-09	3.5781e-10
ENO5, RK3	4.694e-06	1.719e-07	6.3248e-09	2.4447e-10
ENO6, RK3	4.1099e-06	1.4711e-07	5.3992e-09	2.0849e-10
WENO3, RK2	1.5871e-04	1.0448e-05	4.3463e-07	8.8767e-09
WENO5, RK3	7.5662e-06	4.6049e-07	7.4746e-09	2.7985e-10

	N=60	N=180	N=540	N=1620
ENO2, RK1	2.8243	2.352	2.0692	2.084
ENO3, RK2	5.1899	3.1672	2.931	2.9968
ENO4, RK2	5.6271	3.3774	3.0865	3.0307
ENO5, RK3	6.491	3.0103	3.006	2.9611
ENO6, RK3	6.612	3.0311	3.0083	2.962
WENO3, RK2	3.2863	2.4765	2.8942	3.5418
WENO5, RK3	6.0565	2.5479	3.7509	2.9902

Table 1. L^1 *norms of the error and convergence rates for the porous media equation periodic boundary conditions, with initial data of class* C^1 (*results reproduced from* [**28**] *with permission*).

The generalization to $d > 2$ and rectangular domains is now trivial. We stress once again that no two-dimensional reconstruction is used, but only d one-dimensional reconstructions are needed. Finally, boundary conditions can be implemented direction-wise with the same techniques used in the one-dimensional case.

4.1.3 - A numerical test: the porous media equation

The porous media equation corresponds to the choice $p(u) = u^m$ in the model (36). We reproduce the numerical results obtained in [**28**], see also [**30**], for a test proposed in [**63**] taking $m = 2, 3$ and initial data of class C^1 as follows:

$$u(x,0) = \begin{cases} \cos^2(\pi x/2) & |x| \le 1 \\ 0 & |x| > 1. \end{cases} \tag{93}$$

The computational domain is $\{|x| \le 3\} \subset \mathbb{R}$ and the boundary conditions are periodic; the CFL constant is taken as $C = 0.25$.

As was shown in [**6**], see also [**9**], the solution with the initial condition (93) has a front that does not move for $t < 0.034$. We therefore chose a final time of the simulation $t_{\text{fin}} = 0.03$ to prevent the formation of the singularity of u_x from affecting the order of convergence. We used as reference solution the one

obtained numerically with $N = 4860$ grid points and computed the L^1 norms of the errors of the solutions with $N = 60, 180, 540, 1620$ grid points. The results are presented in Table 1.

We point out that the degree of regularity of the solution poses a limit on the order of convergence of the schemes: therefore the schemes we tested perform at best as third order schemes, as confirmed by the data in Table 1. Still, high order schemes yield smaller error on a given grid. This can be of practical importance in problems where one does not have the freedom of choosing the number of grid points, as in digital image analysis, where non-linear degenerate diffusion equations are sometimes used as filters for contour enhancement (see [**10**]).

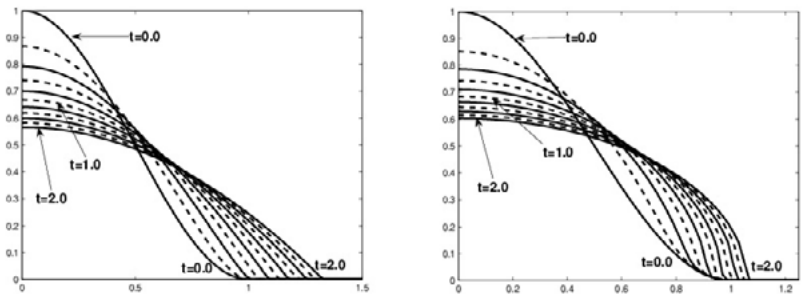

Fig. 3. Snapshots of the numerical solutions for the porous media equation with $p(u) = u^2$ (left) and $p(u) = u^3$ (right). Initial data are chosen according to (93) and the numerical solutions are represented at times $t = 0, 0.2, \ldots, 2.0$. The solutions are obtained with the spatial WENO reconstruction of order 5 and the RK3 time integrator.

In Figure 3 we show the numerical solution for the porous media equation with $p(u) = u^2$ and $p(u) = u^3$, with the initial data (93) and $t \in [0, 2]$. It can be appreciated that a front develops in finite time and then it travels at finite speed.

We present a numerical simulation for the two-dimensional porous media equation with $p(u) = u^2$. We chose an initial data $u_0(x, y)$ given by two bumps with periodic boundary conditions on $[-10, 10] \times [-10, 10]$. The large domain ensures that the compact support of the solution is still contained in the computational domain at the final time of the calculation. The numerical approximation at different time levels is shown in Figure 4. We note that the symmetries of the initial data are preserved and the solution seems to be unaffected by the dimensional splitting of the two-dimensional scheme.

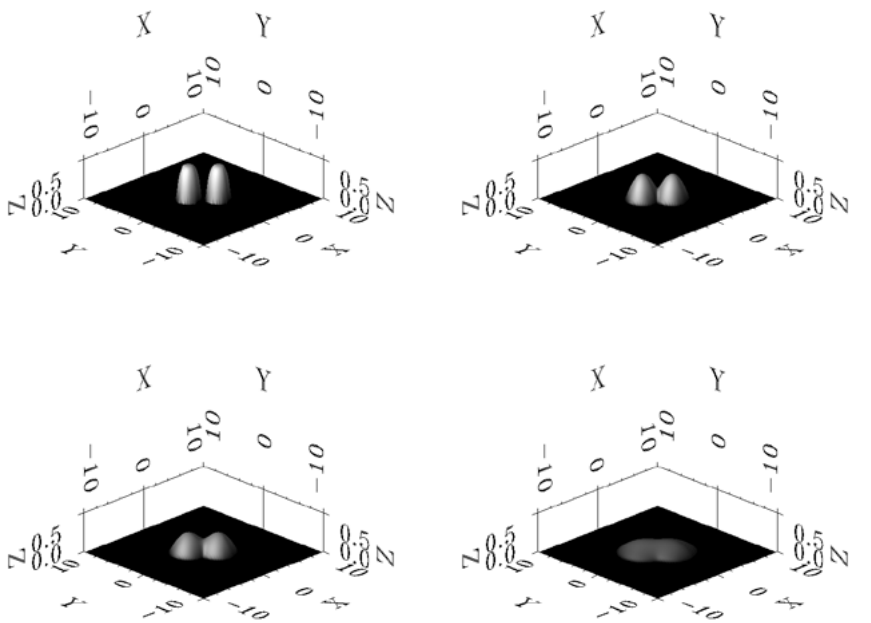

Fig. 4. The numerical solution of the porous media equation on a square regular grid with compactly supported initial data. From top left to bottom right, we show the numerical solution at times $t = 0, 0.5, 1.0, 4.0$.

4.2 - *Discontinuous Galerkin Approximation*

For nonlinear problems, some of the advantages of the relaxed methods are that we do not need to solve nonlinear systems, and that the numerical solutions inherit positivity and monotonicity properties of the analytical solutions. On the other hand, its stability requires the standard parabolic CFL condition, which constrains the time step to be proportional to the square of the mesh size. In order to handle complex geometry we recently introduced [**32**, **33**] a finite element method to approximate the semidiscrete diffusive relaxed system (71). For the time integration we consider a class of high order TVD (total variation diminishing) Runge-Kutta time discretization schemes that are suitable for solving hyperbolic conservation laws with stable spatial discretizations [**61**]. These methods guarantee that the total variation of the solution does not increase, so that no new extrema are generated. Indeed, it was shown by Gottlieb and Shu in [**61**], using numerical examples, that non-TVD methods typically produce oscillations around the points of discontinuity. For convenience we redefine the ν-stages Runge-Kutta scheme of order q, and we start defining the initial value

$$u^{(0)} = u^n, \tag{94a}$$

and then we perform the intermediate stages for $i = 1, \dots, \nu$

$$w^{(i-1)} = p(u^{(i-1)}), \tag{94b}$$

$$\boldsymbol{v}^{(i-1)} = -\nabla w^{(i-1)}, \tag{94c}$$

$$u^{(i)} = \sum_{k=0}^{i-1} \left[\alpha_{ik} u^{(k)} + \Delta t \, \beta_{ik} \nabla \cdot \boldsymbol{v}^{(k)} \right], \tag{94d}$$

where, as explained in [**61**], the coefficients α_{ik} and β_{ik} satisfy

$$\alpha_{ik} \geq 0 \,, \qquad \beta_{ik} \neq 0 \Rightarrow \alpha_{ik} \neq 0 \,, \qquad \sum_{k=0}^{i-1} \alpha_{ik} = 1 \,.$$

Finally, the updated value of u a time t^{n+1} is

$$u^{n+1} = u^{(\nu)}. \tag{94e}$$

R e m a r k 4.1. We only consider here the family of explicit time schemes (94). Better choices for the time integration consist of implicit-explicit (IMEX) techniques [**7**], which allow to treat implicitly the diffusion operator and explicitly the reaction one. The IMEX methods have the advantage of avoiding

a stability constraint on the time step Δt, but give rise to non linear implicit systems. On the other hand, a linearization technique like that introduced in [**25**] could be applied, obtaining a linearly implicit scheme. Recently in [**13**] a general framework to construct and apply linearly implicit schemes to a large class of PDE's containing stiff terms is presented. In particular, the Authors have reformulated several IMEX schemes for problems such as

$$\frac{du(t)}{dt} = \mathcal{H}(t, u(t), u(t)/\varepsilon),$$

where the function $\mathcal{H}$ is sufficiently smooth with a stiff dependence only on the last argument, emphasized by the small parameter ε. In all semi-implicit schemes described in [**13**], the second argument of $\mathcal{H}$ is treated explicitly, while the occurrence of u in the third argument, is treated implicitly.

For the space discretization of the time semidiscrete system (94), we use a discontinuous Galerkin (DG) method [**36**]. For this purpose let $\mathcal{T}_h = \{K\}$ be a triangulation of Ω with mesh parameter h defined by $h = \max_{K\in\mathcal{T}_h} h_K$, where $h_K = \mathrm{diam}(K)$. We assume the elements K to be triangles in 2D and, obviously, intervals in 1D. We denote by $\boldsymbol{n}_K$ the unit normal vector to ∂K pointing outside K. In 2D we make the assumption that $\mathcal{T}_h$ be a shape-regular, conformal, triangulation of the domain Ω. We denote by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the sets of internal and boundary "faces" (nodes in 1D, edges in 2D), respectively, of $\mathcal{T}_h$, and set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$.

In the classical DG methods, a stabilization is performed on each edge of the mesh. In the present work we follow the approach used in [**32**], where it was shown that it is possible to choose a direction $\boldsymbol{\sigma} = (\sigma_i)_{i=1,..,d}^T$, and to stabilize only the edges that are not parallel to $\boldsymbol{\sigma}$.

Given a face $f \in \mathcal{F}_h^I$ which is not parallel to $\boldsymbol{\sigma}$, we define $\boldsymbol{n}_f$ as the unit normal vector to the face f such that $\boldsymbol{\sigma}\cdot\boldsymbol{n}_f > 0$; see Figure 5. Otherwise, given a face $f \in \mathcal{F}_h^I$ parallel to $\boldsymbol{\sigma}$, the normal $\boldsymbol{n}_f$ is chosen indifferently as either one of the two normal vectors to f. To fix ideas, in this latter case, one can chose as $\boldsymbol{n}_f$ the unit normal vector pointing from the element with smaller index to the one with larger index in the list of elements. Given $f \in \mathcal{F}_h^I$, let K_f^L and K_f^R be the two elements sharing f. We define on f, see e.g. [**46**],

$$\begin{aligned} \text{the averages:} \quad & \{\!\{\psi\}\!\} := (\psi^L + \psi^R)/2\,, \quad \{\!\{\boldsymbol{\varphi}\}\!\} := (\boldsymbol{\varphi}^L + \boldsymbol{\varphi}^R)/2\,, \\ \text{the jumps:} \quad & [\![\psi]\!]_N := \psi^L\boldsymbol{n}^L + \psi^R\boldsymbol{n}^R\,, \quad [\![\boldsymbol{\varphi}]\!]_N := \boldsymbol{\varphi}^L\cdot\boldsymbol{n}^L + \boldsymbol{\varphi}^R\cdot\boldsymbol{n}^R\,, \end{aligned}$$

where ψ and $\boldsymbol{\varphi}$ are piecewise smooth function and vector field, respectively, on $\mathcal{T}_h$, and the indices L and R denote their restrictions to K_f^L and K_f^R, respectively.

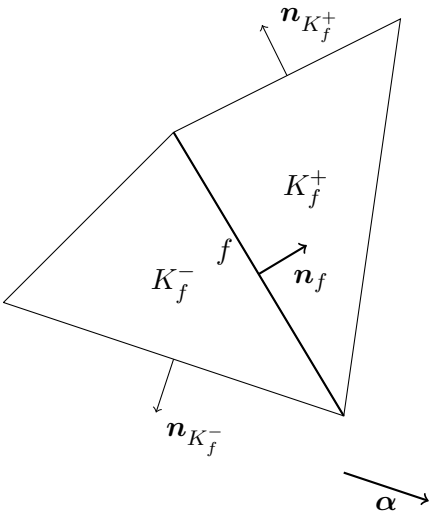

Fig. 5. The vector $\boldsymbol{\sigma}$, a face f and the two elements K_f^L and K_f^R sharing f.

Finally, let us introduce the discontinuous finite element space

$$\mathcal{V}_h = \{v \in L^2(\Omega) \,:\, v|_K \in \mathcal{P}^\ell(K),\ \forall K \in \mathcal{T}_h\}\,, \tag{95}$$

where $\mathcal{P}^\ell(K)$ is the space of polynomials of degree at most ℓ on K. In the following, we denote by ∇_h the elementwise application of the ∇ operator.

For the detailed derivation of the DG-spatial approximation, we refer to [**32**], we report here the complete scheme.

Initialize: Define u_h^0 as the L^2–projection of the initial datum u_0 onto $\mathcal{V}_h$: find $u_h^0 \in \mathcal{V}_h$ such that, for all $\psi_h \in \mathcal{V}_h$,

$$\int_\Omega u_h^0 \psi_h \, d\boldsymbol{x} = \int_\Omega u_0 \psi_h \, d\boldsymbol{x}. \tag{96}$$

Time stepping: For $n = 0, 1, \ldots,$

(a) Set $u_h^{(0)} = u_h^n$;

(b) For $i = 1, \ldots, \nu$ (time stages),
compute $w_h^{(i-1)} \in \mathcal{V}_h$ such that, for all $\psi_h \in \mathcal{V}_h$,

$$\int_\Omega w_h^{(i-1)} \psi_h d\boldsymbol{x} = \int_\Omega p(u_h^{(i-1)}) \psi_h \, d\boldsymbol{x}; \tag{97}$$

compute $\boldsymbol{v}_h^{(i-1)} \in \mathcal{V}_h^d$ such that, for all $\boldsymbol{\varphi}_h \in \mathcal{V}_h^d$,

(98)
$$\int_\Omega \boldsymbol{v}_h^{(i-1)} \cdot \boldsymbol{\varphi}_h \, d\boldsymbol{x} = -\int_\Omega \nabla_h w_h^{(i-1)} \cdot \boldsymbol{\varphi}_h \, d\boldsymbol{x} + \int_{\mathcal{F}_h^I} [\![w_h^{(i-1)}]\!]_N \cdot \{\!\{ \boldsymbol{\varphi}_h \}\!\} \, ds;$$

compute $u_h^{(i)} \in \mathcal{V}_h$ such that, for all $\psi_h \in \mathcal{V}_h$,

$$\int_\Omega u_h^{(i)} \psi_h \, d\boldsymbol{x} = \sum_{k=0}^{i-1} \left[\alpha_{ik} \int_\Omega u_h^{(k)} \psi_h \, d\boldsymbol{x} + \Delta t \, \beta_{ik} \mathcal{B}_h(\boldsymbol{v}_h^{(k)}, \psi_h) \right]. \tag{99}$$

(c) Update: $u_h^{n+1} = u_h^{(\nu)}$.

The expression for $\mathcal{B}_h(\boldsymbol{v}_h^{(k)}, \psi_h)$ in (99) is the following:

$$\begin{aligned} \mathcal{B}_h(\boldsymbol{v}_h^{(k)}, \psi_h) = & \int_\Omega \nabla_h \cdot \boldsymbol{v}_h^{(k)} \, \psi_h \, d\boldsymbol{x} - \int_{\mathcal{F}_h^I} [\![\boldsymbol{v}_h^{(k)}]\!]_N \{\!\{ \psi_h \}\!\} \, ds \\ & + \int_{\mathcal{F}_h^I} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{n}_f}{2} [\![w_h^{(k)}]\!]_N \cdot [\![\psi_h]\!]_N \, ds - \int_{\mathcal{F}_h^N} \boldsymbol{v}_h^{(k)} \cdot \boldsymbol{n}_\Omega \, \psi_h \, ds \, . \end{aligned} \tag{100}$$

We consider the case of linear diffusion $p(u) = u$ and forward Euler time-stepping, and perform stability analysis of the scheme. For stability reasons [**32**] we choose $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = \ell^2 h^{-1} \boldsymbol{s} \, , \tag{101}$$

with $\boldsymbol{s}$ independent of the mesh size h and the polynomial approximation degree ℓ.

We proceed as in [**32**], and we reformulate the method in a more compact form, eliminating the $\boldsymbol{v}$ unknown from the system. In order to do that, we introduce the so-called *lifting operators* (see [**5**]): for w piecewise smooth on $\mathcal{T}_h$, we define *lifting* $\mathcal{L}(w) \in \mathcal{V}_h^d$ by

$$\int_\Omega \mathcal{L}(w) \cdot \boldsymbol{\varphi}_h \, d\boldsymbol{x} = \int_{\mathcal{F}_h^I} [\![w]\!]_N \cdot \{\!\{ \boldsymbol{\varphi}_h \}\!\} \, ds \tag{102}$$

for all $\boldsymbol{\varphi}_h \in \mathcal{V}_h^d$. With this definition, equation (98) gives, at any given time stage k

$$\boldsymbol{v}_h^{(k)} = -(\nabla_h w_h^{(k)} - \mathcal{L}(w_h^{(k)})).$$

Thus, after integrating by parts (100), substituting the previous expression for $\boldsymbol{v}_h^{(k)}$, and taking into account that $w_h^{(k)} = p(u_h^{(k)}) = u_h^{(k)}$, we can write the form $\mathcal{B}_h(\cdot,\cdot)$ as

$$\mathcal{B}_h(\boldsymbol{v}_h^{(k)}, \psi_h) = \mathcal{A}_h(u_h^{(k)}, \psi_h),$$

where

$$\begin{aligned} \mathcal{A}_h(u_h^{(k)}, \psi_h) := & \int_\Omega (\nabla_h u_h^{(k)} - \mathcal{L}(u_h^{(k)})) \cdot (\nabla_h \psi_h - \mathcal{L}(\psi_h))\, d\boldsymbol{x} \\ & + \int_{\mathcal{F}_h^I} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{n}_f}{2} [\![u_h^{(k)}]\!]_N \cdot [\![\psi_h]\!]_N \, ds. \end{aligned} \tag{103}$$

The complete method in the case of linear diffusion and forward Euler time stepping reads as follows.

Initialize: Let u_h^0 be the L^2–projection of u_0 onto $\mathcal{V}_h$ (see (96)).

Time stepping: For $n = 0, 1, \ldots$ (time steps), compute $u_h^{n+1} \in \mathcal{V}_h$ such that, for all $\psi_h \in \mathcal{V}_h$,

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\Delta t}\, \psi_h \, d\boldsymbol{x} = -\mathcal{A}_h(u_h^n, \psi_h). \tag{104}$$

Define the following seminorm and norm on $\mathcal{V}_h$:

$$|v|_{DG}^2 = \mathcal{A}_h(v, v), \tag{105}$$

$$\|v\|_{DG}^2 = |v|_{DG}^2 + \mathrm{diam}(\Omega)^2 \|v\|_{0,\Omega}^2, \tag{106}$$

where $\mathrm{diam}(\Omega)$ is the diameter of Ω, and $\|\cdot\|_{0,\Omega}$ denote the norm in $L^2(\Omega)$.

From the definition (105), we have

$$\mathcal{A}_h(w, v) \le |w|_{DG}\, |v|_{DG} \qquad \forall\, w, v \in \mathcal{V}_h. \tag{107}$$

Moreover, from [**32**, Lemma 4.4], we have the following inverse inequality: there exists a constant $C_{\mathrm{inv}} > 0$ independent of h and ℓ such that

$$|v|_{DG} \le C_{\mathrm{inv}} \ell^2 h^{-1} \|v\|_{0,\Omega} \qquad \forall v \in \mathcal{V}_h\ . \tag{108}$$

We have the following stability result.

Theorem 4.1. *Provided that*

$$\Delta t \leq \min\left\{1,\ \frac{C_{\rm inv}^2 \ell^4}{h^2}\right\}, \tag{109}$$

we have

$$\|u_h^{n+1}\|_{0,\Omega}^2 \leq \|u_h^0\|_{0,\Omega}^2. \tag{110}$$

Proof. Taking $\psi_h = u_h^n = \dfrac{u_h^{n+1}+u_h^n}{2} - \dfrac{\Delta t}{2}\dfrac{u_h^{n+1}-u_h^n}{\Delta t}$ in (104) gives

$$\|u_h^{n+1}\|_{0,\Omega}^2 - \|u_h^n\|_{0,\Omega}^2 = -2\Delta t\, \mathcal{A}_h(u_h^n, u_h^n) + \Delta t \left\|u_h^{n+1}-u_h^n\right\|_{0,\Omega}^2. \tag{111}$$

In order to bound the last term on the right-hand side of (111), we take $\psi_h = \dfrac{u_h^{n+1}-u_h^n}{\Delta t}$ in (104), use (107) and the Cauchy-Schwarz inequality, and obtain

$$\begin{aligned}\left\|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right\|_{0,\Omega}^2 &\leq |u_h^n|_{DG}\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|_{DG}\\ &\leq \left(C_{\rm inv}\ell^2 h^{-1}\,|u_h^n|_{DG}\right)\left\|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right\|_{0,\Omega},\end{aligned}$$

where in the last step we used the inverse inequality (108). Dividing by $\left\|\dfrac{u_h^{n+1}-u_h^n}{\Delta t}\right\|_{0,\Omega}$ and squaring the result, we get

$$\left\|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right\|_{0,\Omega}^2 \leq 2\, C_{\rm inv}^2 \ell^4 h^{-2}\,|u_h^n|_{DG}^2$$

which, substituted into (111), gives

$$\begin{aligned}\|u_h^{n+1}\|_{0,\Omega}^2 \leq& \|u_h^n\|_{0,\Omega}^2 - 2\Delta t\,|u_h^n|_{DG}^2 + 2\, C_{\rm inv}^2\ell^4 h^{-2}\Delta t^2\,|u_h^n|_{DG}^2\\ \leq& \|u_h^n\|_{0,\Omega}^2 + 2\Delta t\left(C_{\rm inv}^2\ell^4 h^{-2}\Delta t - 1\right)|u_h^n|_{DG}^2,\end{aligned} \tag{112}$$

where in the first inequality we have applied the definition (105). Assuming the parabolic CFL condition

$$\Delta t \leq \frac{C_{\rm inv}^2\ell^4}{h^2},$$

the coefficients into brackets on the right-hand side of (112) is less than or equal to zero, thus the bound (112) becomes

$$\|u_h^{n+1}\|_{0,\Omega}^2 \leq \|u_h^n\|_{0,\Omega}^2. \tag{113}$$

Then,

$$\|u_h^{n+1}\|_{0,\Omega}^2 \leq \|u_h^0\|_{0,\Omega}^2.$$

□

Remark 4.2. For more general $p(u)$ we refer to [**32**], where a stability condition as in Theorem 4.1 is shown. Moreover in [**33**] a reaction term is added, in particular the Fisher reaction term $r(u) = u(1-u)$ is considered. We point out that in this case we must pay attention to how the reaction term is treated in the numerical scheme. In [**33**] a modified reaction function is introduced in order to have that the cell averages stay positive.

We show a $2D$ example in $\Omega = [-4, 4]^2$ for the porous media equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \qquad \text{in } \Omega \times (0, T)\ , \tag{114}$$

with homogeneous Dirichlet boundary conditions. Equation (114) degenerates for $u = 0$, since $p'(u) = 0$; thus, compactly supported initial data give rise to solutions with interfaces that travel with finite speeds, as the well-known similarity solution studied by Barenblatt (see, for example, [**9**]). Simulations show that our scheme applied to problem (114) is stable and accurate. In Figure 6, we report the solution of equation (114) with $m = 2$ obtained evolving the Barenblatt solution until final time $T = 0.5$. The simulation uses piecewise discontinuous cubic elements, and forward Euler for the time integration, we choose the stability parameter $\boldsymbol{\alpha} = h^{-1}(1, 0)$. As we can see, the shape and the symmetry of the solution is correctly represented, and the speed of the traveling front is correctly approximated by the numerical scheme.

5 - Looking for a general framework

For the sake of simplicity, in this section we consider a scalar equation in one space dimension of the form (nonlinear transport-diffusion equation)

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial^2 p(u)}{\partial x^2}, \tag{115}$$

and we will limit ourselves to first order schemes. The construction of a relaxation system for (115) is by no means unique and each of these may result in a

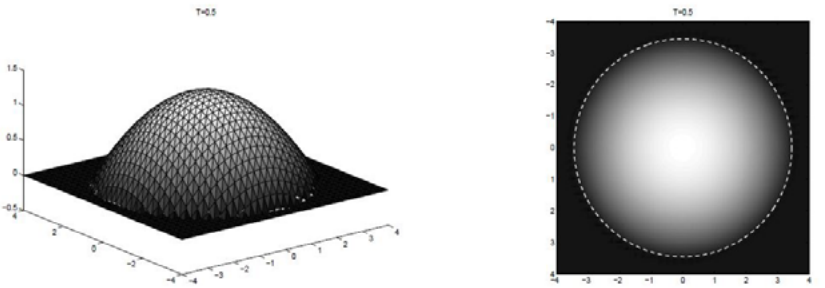

Fig. 6. Numerical solution of problem (114) with Barenblatt initial datum at final time $T = 0.5$ seen in isometric perspective (left plot) and from above (right plot). In the right plot, the white line represents the contour of the support of the analytical solution (from [**32**]).

different consistent discretization for the original convection-diffusion equation. We observe that consistent relaxation discretizations can yield good schemes, but also numerical junk.

A three-velocities model. We rewrite equation (115) in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\underbrace{f(u) - \frac{\partial p(u)}{\partial x}}_{G(u)}) = 0. \tag{116}$$

Different approximations can be obtained by considering relaxation systems that introduce at least one auxiliary variable v that relaxes onto $G(u)$ in the limit $\varepsilon \to 0^+$. In order to simplify the notations, from here on we consider the rescaling with ε and not ε^2. This choice is not essential for the development of the numerical scheme, but now the parameter ε has the physical dimension of a time.

Starting from the relaxed scheme for the purely diffusive case a relaxation approximation can be formally stated as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial G(u)}{\partial x} = \dfrac{1}{\varepsilon} v. \end{cases} \tag{117}$$

From the definition of the function $G(u)$ follows that

$$\frac{1}{\varepsilon}\frac{\partial G(u)}{\partial x} = \frac{1}{\varepsilon}\left(f(u) - \frac{\partial p(u)}{\partial x}\right),$$

then the relaxation system becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial p(u)}{\partial x} = \dfrac{1}{\varepsilon}(f(u) - v). \end{cases} \tag{118}$$

Finally, introducing another auxiliary variable w we rewrite the system as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial w}{\partial x} = \dfrac{1}{\varepsilon}\left(f(u) - v\right) \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial v}{\partial x} = \dfrac{1}{\varepsilon}(p(u) - w). \end{cases} \tag{119}$$

Formally, as $\varepsilon \to 0^+$, $w \to p(u)$, $v \to \left(f(u) - \dfrac{\partial p(u)}{\partial x}\right)$ and the original transport-diffusion equation is recovered. The system which describes the relaxation approximation can be written in the following general matrix form:

$$\frac{\partial s}{\partial t} + \left(A + \frac{1}{\varepsilon}B\right)\frac{\partial s}{\partial x} = \frac{1}{\varepsilon}g(s), \tag{120}$$

where, in the present case, $s = (u, v, w)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad g(s) = \begin{pmatrix} 0 \\ f(u) - v \\ p(u) - w \end{pmatrix}.$$

The relaxed scheme is obtained by discretizing the equations in (120) and then taking the $\varepsilon \to 0^+$ limit.

To avoid the stiffness in the convective part of the equations, we add and subtract the term $\phi^2 \dfrac{\partial w}{\partial x}$ in the second equation of (120), as in [**96**] and [**28**]. In this fashion, the stiffness appears only in the source term which is stiff (when $\varepsilon \to 0^+$) and must be integrated implicitly in time. Therefore we consider a first order Implicit-Explicit Euler scheme, which will be implicit in the source term,

and explicit in the (non stiff) transport operator in the left hand side of the system. In order to obtain a fully discrete numerical scheme, let us introduce a uniform grid on $[a,b] \subset \mathbb{R}$, with grid spacing $h = (b-a)/N$. The numerical solution of the diffusive system at time t^{n+1} will be given by:

$$s^{n+1} = s^n - \Delta t(A + \phi^2 B)\frac{\partial s^{(1)}}{\partial x} + \frac{\Delta t}{\varepsilon} H\left(s^{(1)}\right)$$

where H is the modified source term, and

$$s^{(1)} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{(1)} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{n} + \frac{\Delta t}{\varepsilon} \begin{pmatrix} 0 \\ (\varepsilon\phi^2 - 1)\frac{\partial w}{\partial x} + f(u) - v \\ p(u) - w \end{pmatrix}^{(1)}.$$

From the first equation, $u^{(1)} = u^n$ while, when $\varepsilon \to 0^+$, the third equation reduces to $w^{(1)} = p\left(u^{(1)}\right) = p(u^n)$ and the second equation gives $v^{(1)} = f\left(u^{(1)}\right) - \dfrac{\partial w^{(1)}}{\partial x} = f(u^n) - \dfrac{\partial w^{(1)}}{\partial x}$.

Now the space discretization is introduced. For the sake of illustration, we consider a first order Godunov scheme, which is the least diffusive monotone first order numerical flux. Note that, as in all relaxation schemes, the (stiff) convective part has been replaced by a constant coefficient linear advection, which makes the implementation of Godunov scheme straightforward. In particular, in this case, the matrix $A + \phi^2 B$ has eigenvalues $0, \pm\phi$. Let R be the matrix of the right eigenvectors. The time discretized scheme becomes:

$$s^{n+1} = s^n + \Delta t R \begin{pmatrix} \phi & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} R^{-1}\frac{\partial s^{(1)}}{\partial x} + \frac{\Delta t}{\varepsilon} g\left(s^{(1)}\right).$$

Applying upwinding in each characteristic direction, the fully discrete scheme is obtained. The first equation is:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2h}\left(v_{j+1}^{(1)} - v_{j-1}^{(1)}\right) + \phi\frac{\Delta t}{2h}\left(w_{j+1}^{(1)} - 2w_j^{(1)} + w_{j-1}^{(1)}\right) \tag{121}$$

while the other equations, in the stiff limit, simply project v and w to equilibrium, namely: $w_j^{(1)} = p(u_j^n)$ and $v_j^{(1)} = f(u_j^n) - \widehat{\nabla}_j w^{(1)}$ where $\widehat{\nabla}_j$ is a suitable finite difference discretization of the first derivative in space (in the following test we consider central difference formulas).

We thus observe that, in the degenerate case, $w_j^{(1)} = p(u_j^n) = 0$ and $v_j^{(1)} = f(u_j^n)$; thus the scheme reduces to an unconditionally unstable central

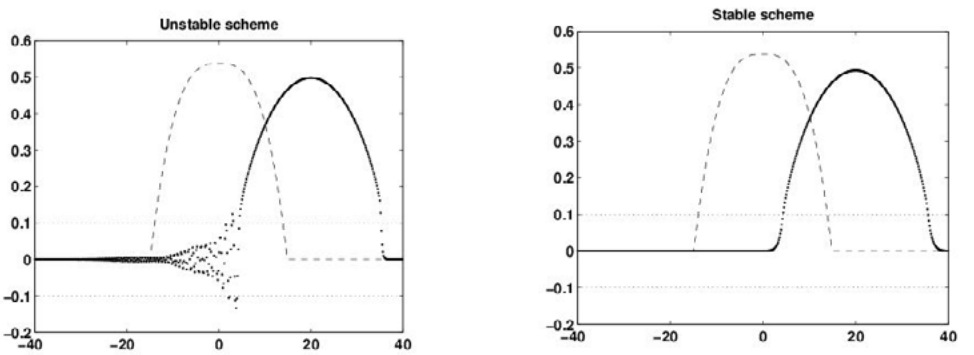

Fig. 7. Numerical integration of the advection-diffusion equation with strong degenerate diffusion (122). The dashed line is the initial data, the dots are the numerical solution with 500 points while the dotted lines mark the region where diffusion is degenerate. On the left the results obtained with the unstable scheme (121); on the right panel shows the result obtained with the stable scheme (127) (from [**31**]).

scheme for hyperbolic equations. In figure 7 we show the result of the numerical integration of the diffusive system with the scheme (121) in a strongly degenerate case for which:

$$p(u) = \begin{cases} 0 & u < 0.1 \\ (u - 0.1)^2 & u > 0.1 \end{cases} \tag{122}$$

while the advection term is linear, $f(u) = u$. We remark that the stable version of the scheme mentioned in the figure 7 will be derived later.

Another three-velocities model. Let us start again from equation (116). We first apply a Jin-Xin relaxation step of the term $\dfrac{\partial G(u)}{\partial x}$ obtaining the system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + a^2 \dfrac{\partial u}{\partial x} = \dfrac{1}{\varepsilon} \left(G(u) - v \right). \end{cases} \tag{123}$$

Then we can relax the non linear term $p(u)$ introducing a third variable w and a third equation, as in [**28**]. We obtain

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t}+\dfrac{\partial v}{\partial x}=0 \\ \\ \dfrac{\partial v}{\partial t}+a^2\dfrac{\partial u}{\partial x}=\dfrac{1}{\varepsilon}\left(f(u)-\dfrac{\partial w}{\partial x}-v\right) \\ \\ \dfrac{\partial w}{\partial t}+b^2\dfrac{\partial v}{\partial x}=\dfrac{1}{\varepsilon}(p(u)-w) \end{array}\right. \tag{124}$$

where a and b are constants that must satisfy a generalization of the Jin-Xin subcharacteristic condition [**74**]. We proceed as above adding the term $\phi^2\dfrac{\partial w}{\partial x}$ to both sides of the second equation of (124) to get

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t}+\dfrac{\partial v}{\partial x}=0 \\ \\ \dfrac{\partial v}{\partial t}+a^2\dfrac{\partial u}{\partial x}+\phi^2\dfrac{\partial w}{\partial x}=\dfrac{1}{\varepsilon}\left(f(u)+(\phi^2\varepsilon-1)\dfrac{\partial w}{\partial x}-v\right) \\ \\ \dfrac{\partial w}{\partial t}+b^2\dfrac{\partial v}{\partial x}=\dfrac{1}{\varepsilon}(p(u)-w). \end{array}\right. \tag{125}$$

The left hand side of this system is a linear convective operator with eigenvalues $0,\pm\mu$, where $\mu=\sqrt{a^2+b^2\phi^2}$. Applying the same first order discretization as in the previous subsection we obtain the scheme

$$\begin{aligned} u_j^{n+1} = u_j^n &- \frac{\Delta t}{2h}\left(v_{j+1}^n-v_{j-1}^n\right) \\ &+\frac{a^2\Delta t}{2h\mu}\left(u_{j-1}^n-2u_j^n+u_{j+1}^n\right)-\frac{\phi^2\Delta t}{2h\mu}\left(w_{j-1}^n-2w_j^n+w_{j+1}^n\right). \end{aligned} \tag{126}$$

Substituting $v_j^n=f(u_j^n)-(w_{j+1}^n-w_{j-1}^n)/2h$, we get

$$\begin{aligned} u_j^{n+1} = u_j^n &- \frac{\Delta t}{2h}\left(f\left(u_{j+1}^n\right)-f\left(u_{j-1}^n\right)\right)+\frac{a^2\Delta t}{2h}\left(u_{j-1}^n-2u_j^n+u_{j+1}^n\right) \\ &+\frac{\Delta t}{2h}\left[\frac{w_{j-2}^n-2w_j^n+w_{j+2}^n}{2h}+\frac{\phi^2}{\mu}\left(w_{j-1}^n-2w_j^n+w_{j+1}^n\right)\right]. \end{aligned} \tag{127}$$

Now, even in the degenerate case, where $w_j^n=p(u_j^n)=0$, the scheme reduces to a Lax-Friedrichs scheme and is thus stable provided that the parameter a is

large enough. In Figure 7 we show that the scheme obtained from the improved three-velocities relaxation system is stable even in the strongly degenerate case, with $p(u)$ assigned by equation (122).

5.1 - *A unified framework*

The relaxation systems introduced above can be cast in the general form

$$\frac{\partial s}{\partial t} + A\frac{\partial s}{\partial x} + \frac{1}{\varepsilon}B\frac{\partial s}{\partial x} = \frac{1}{\varepsilon}(h(u) - Cs), \tag{128}$$

with $s \in \mathbf{R}^d$ and suitable constant matrices $A, B, C \in \mathbf{R}^{d\times d}$.

In this section, we provide conditions on the matrices A, B and C to ensure that the relaxation system (128) in the limit $\varepsilon \to 0^+$ relaxes onto the convection diffusion equation (115). We will show that the BGK relaxation systems of [**3**, **4**, **81**] can be also written in the form (128), so that in this case, we can exploit the convergence results obtained in [**18**].

To fix ideas, we suppose that the first equation of the system (128) is the evolution equation for the variable u of the convection-diffusion PDE. Thus $s_1 = u$, $h_1(u) = u$ and $(Cs)_1 = u$. Note that the first equation is not stiff. In more general cases, it is possible to rotate the system with a change of variables, so that the first equation is the evolution equation for u. As $\varepsilon \to 0^+$, the stiff terms must balance, so that:

$$B\frac{\partial s}{\partial x} = h(u) - Cs. \tag{129}$$

This equation can be solved formally for the variable s:

$$s = \sum_{k=0}^{\infty} \left(-C^{-1}B\frac{\partial}{\partial x}\right)^k C^{-1}h(u). \tag{130}$$

Expanding (130), we find:

$$s = C^{-1}h(u) - C^{-1}B\frac{\partial}{\partial x}(C^{-1}h(u)) + (C^{-1}B)^2\frac{\partial}{\partial x^2}(C^{-1}h(u)) + \dots \tag{131}$$

For consistency, the first equation must satisfy (here and in the following we denote the first row of a matrix or vector by $[\,]_1$),

$$\left[(A + \frac{1}{\varepsilon}B)\frac{\partial s}{\partial x}\right]_1 = \frac{\partial f(u)}{\partial x} - \frac{\partial^2 p(u)}{\partial x^2}.$$

Since there are no stiff terms in the first equation, the first row of B must be zero. Differentiating s and substituting in the equation above, one finds:

$$\left[AC^{-1}\frac{\partial h}{\partial x} - AC^{-1}BC^{-1}\frac{\partial^2 h}{\partial x^2} + A(C^{-1}B)^2C^{-1}\frac{\partial^3 h}{\partial x^3} + \dots\right]_1 = \frac{\partial f(u)}{\partial x} - \frac{\partial^2 p(u)}{\partial x^2}.$$

The right hand side does not contain third or higher order derivatives, so $(C^{-1}B)^l = 0,\ l \geq 2$. Therefore the expansion for s in the stiff limit (131) reduces to:

$$s = C^{-1}h(u) - C^{-1}B\frac{\partial(C^{-1}h(u))}{\partial x}. \tag{132}$$

This approach can be generalized to higher order PDEs: the expansion of s would contain more terms to take into account higher order derivatives. Note that as $\varepsilon \to 0^+$, the vector s is completely determined by u through (131). The equation for h is satisfied provided that:

$$\begin{aligned} \left[(AC^{-1})h_x\right]_1 &= \frac{\partial f(u)}{\partial x} \\ \left[(AC^{-1}BC^{-1})h_{xx}\right]_1 &= \frac{\partial p(u)}{\partial x^2}. \end{aligned} \tag{133}$$

System (128) has the correct relaxation limit, but is not suited for numerical integration, because of the stiffness contained in the linear convective term. As in [**96**] and [**28**], we remove the stiffness from the convective term adding and subtracting the quantity $\phi^2 B\frac{\partial s}{\partial x}$. We obtain the relaxation system which will be integrated numerically:

$$\frac{\partial s}{\partial t} + (A + \phi^2 B)\frac{\partial s}{\partial x} = \frac{1}{\varepsilon}\left(h(u) + (\varepsilon\phi^2 - 1)B\frac{\partial s}{\partial x} - Cs\right). \tag{134}$$

Now we need that the matrix $(A + \phi^2 B)$ is diagonalizable because we shall approximate in space with a Godunov scheme.

We summarize the consistency conditions that ensure relaxation towards the correct limit:

$$\begin{aligned} &A + \phi^2 B \qquad \text{diagonalizable with real eigenvalues } \forall\phi \\ &s_1 = u \qquad [C^{-1}h(u)]_1 = u \\ &[B]_1 = \vec{0} \\ &\left[AC^{-1}h\right]_1 = f(u) \\ &\left[AC^{-1}BC^{-1}h\right]_1 = p(u) \\ &(C^{-1}B)^l = 0\ l \geq 2. \end{aligned} \tag{135}$$

5.1.1 - The first order relaxed scheme

The system (134) can be integrated with any order of accuracy. Here we address the construction of a first order scheme from the general framework of (134). From this discussion it will be apparent that the numerical complexity of the scheme does not depend necessarily on the number of equations composing system (134): what matters most is the complexity of the eigenstructure of the matrix $A + \phi^2 B$.

Let us rewrite system (134) as

$$\frac{\partial s}{\partial t} + A_\phi \frac{\partial s}{\partial x} = \frac{1}{\varepsilon} H_\phi(s), \tag{136}$$

where $A_\phi = A + \phi^2 B$ and $H_\phi(s) = h(u) + (\varepsilon\phi^2 - 1)B\frac{\partial s}{\partial x} - Cs$. Consider the trivial first order IMEX scheme [**7**,**78**], with the backward Euler scheme applied to the source term. We obtain the following time-discretized scheme:

$$s^{n+1} = s^n - \Delta t A_\phi \frac{\partial s^{(1)}}{\partial x} + \frac{\Delta t}{\varepsilon} H_\phi(s^{(1)}) \tag{137a}$$

$$s^{(1)} = s^n + \frac{\Delta t}{\varepsilon} H_\phi(s^{(1)}). \tag{137b}$$

Consider now the relaxed scheme, that is the time discretized scheme, in the limit $\varepsilon \to 0^+$. The source term becomes dominant in both equations, and the scheme reduces to $H_\phi(s^{(1)}) = 0$. However, the first component of H_ϕ is zero, due to the compatibility conditions: in fact, the first row of B is zero, and $[h(u) - Cs]_1 = 0$. Thus the first component of (137b) gives simply $u^{(1)} = u^n$. Substituting this value into (132), one immediately finds $s^{(1)}$.

Now, it is clear that only the first component of s must be updated, because all other components will be obtained, in the limit $\varepsilon \to 0$, from (131). For the relaxed scheme, (137a) gives

$$u^{n+1} = u^n - \Delta t \left[A_\phi \frac{\partial s^{(1)}}{\partial x} \right]_1 . \tag{138}$$

The space discretization can be carried out with Godunov scheme. Let j denote the generic grid node in space and let Λ_ϕ^+, Λ_ϕ^- denote the positive and negative parts of the diagonal matrix Λ_ϕ of eigenvalues for A_ϕ. Let R be the matrix of right eigenvectors of A_ϕ, which is diagonalizable. Then Godunov scheme applied to the equation above can be written as:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h} \left[R\Lambda_\phi^+ R^{-1}(s_j^{(1)} - s_{j-1}^{(1)}) + R\Lambda_\phi^- R^{-1}(s_{j+1}^{(1)} - s_j^{(1)}) \right]_1 . \tag{139}$$

From this equation, we note that the fields corresponding to zero eigenvalues need not be advected, and therefore they will not be reconstructed. Moreover, a zero in the first row of the matrix R, means that the corresponding characteristic field will not contribute to the computation of u, and its evolution will also not be computed. Thus the efficiency of the scheme is due to the number of zero eigenvalues of Λ_ϕ and the number of zeros in the first row of R.

R e m a r k 5.1. We observe that different relaxation schemes may be characterized by different eigenstructures, then these schemes may have a different behavior regardless of the number of the computed numerical fluxes. Moreover, schemes with the same number of auxiliary variables may have different conditions of stability and robustness with respect to changes in the value of the diffusivity of the problem, see [**30**, **31**].

5.2 - *Connection with diffusive BGK approximations*

In [**81**] and [**4**] Natalini et al. introduced a class of diffusive BGK schemes for the approximation of strongly degenerate convection diffusion equations. We recall that the relaxation systems described in [**81**] are of the following form

$$\frac{\partial \xi_i}{\partial t} + \left(\lambda_i + \frac{\theta_i}{\sqrt{\varepsilon}}\right) \frac{\partial \xi_i}{\partial x} = \frac{1}{\varepsilon} \left(M_i(u) - \xi_i\right), \quad i = 1, .., N \tag{140}$$

where λ_i, θ_i are the microscopic velocities of the system which do not depend on ε. The M_i are the Maxwellian functions of the system and they depend only on u which is defined by

$$u = \sum_{i=1}^{N} \xi_i. \tag{141}$$

The Maxwellians must verify the following compatibility conditions

$$\begin{aligned} &\sum_{i=1}^{N} M_i = u \quad \sum_{i=1}^{N} \lambda_i M_i = f(u) \\ &\sum_{i=1}^{N} \theta_i M_i = 0 \quad \sum_{i=1}^{N} \theta_i^2 M_i = p(u). \end{aligned} \tag{142}$$

We wish to show that this relaxation system can be recast into the form (128), and that the compatibility conditions (141) and (142) imply the compatibility conditions (135). Thus, the unified framework includes also diffusive BGK approximations. Let $\Lambda = \mathrm{diag}\{\lambda_1, ..., \lambda_N\}, \Theta = \mathrm{diag}\{\theta_1, ..., \theta_N\}$ and $\xi = [\xi_1, ..., \xi_N]^T$.

As a matter of fact (135) in this case should be modified allowing a weak dependence of A on ε: $A = A_0 + A_1(\varepsilon)$, where $\lim_{\varepsilon\to 0^+} \varepsilon A_1 = 0$ and A_0 is a constant matrix. We remark that the term $A_1(\varepsilon)$ does not affect the numerical schemes in their relaxed form.

L e m m a 5.1. *There exist matrices A, B, C and a function $h(u)$ satisfying the compatibility conditions* (135) *such that* (140) *is a diagonalization of* (128)

P r o o f. Multiplying (140) by a constant coefficient invertible matrix Q:

$$Q\frac{\partial \xi}{\partial t} + Q\left(\Lambda + \frac{1}{\sqrt{\varepsilon}}\Theta\right) Q^{-1} Q \frac{\partial \xi}{\partial x} = \frac{1}{\varepsilon}(QM(u) - Q\xi). \tag{143}$$

Setting $s = Q\xi$, $h(u) = QM(u)$, we obtain a system of the form (128) with $C = I$, provided Q can be found so that:

$$Q^{-1}\left(A + \frac{1}{\varepsilon}B\right) Q = \Lambda + \frac{1}{\sqrt{\varepsilon}}\Theta, \tag{144}$$

with A and B satisfying the consistency conditions (135). Then, for any choice of ϕ, the matrix $A + \phi B$ is similar to a diagonal matrix because the matrices Θ and Λ are diagonal: thus the first requirement of (135) is satisfied.

In order to show that system (140) can be cast in the form (128) we start from the four-velocities case and choose Q of the form

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \theta_1/\sqrt{\varepsilon} & \theta_2/\sqrt{\varepsilon} & \theta_3/\sqrt{\varepsilon} & \theta_4/\sqrt{\varepsilon} \\ \theta_1^2 & \theta_2^2 & \theta_3^2 & \theta_4^2 \end{pmatrix} = H + \frac{J}{\sqrt{\varepsilon}} \tag{145}$$

where the third row of H is zero, while being the only nonzero one in J. Thanks to (142), we expect the microscopic velocities λ_i's and θ_i's to be such that the matrix Q is nonsingular for nontrivial transport and diffusion terms. One may thus compute its inverse, observing that it has the structure

$$Q^{-1} = H' + \sqrt{\varepsilon}J' = \begin{pmatrix} \vdots & \vdots & 0 & \vdots \end{pmatrix} + \sqrt{\varepsilon}\begin{pmatrix} 0 & 0 & \vdots & 0 \end{pmatrix},$$

namely the third column is zero in H' and it is the only non-zero one in J'.

In the general case, we choose

$$Q = \left[\begin{array}{c|c} Q_1 & Q_2 \\ \hline \mathbf{1} & 0 \end{array}\right] = H + \frac{J}{\sqrt{\varepsilon}} \tag{146}$$

where Q_1 is $4 \times (n-4)$, Q_2 is 4×4 and $\mathbf{1}$ is the $(n-4) \times (n-4)$ identity matrix. One may see that

$$Q^{-1} = \left[\begin{array}{c|c} 0 & \mathbf{1} \\ \hline Q_2^{-1} & -Q_2^{-1}Q_1 \end{array}\right] = H' + \sqrt{\varepsilon}J' \,. \tag{147}$$

Since the first row of Q is given by $[Q]_1 = \vec{1}$, we have that

$$[Qx]_1 = \sum x_i \quad \forall x = [x_1, \ldots, x_N].$$

Recalling that $s_1 = [Q\xi]_1$ and using (141) we have that $s_1 = u$. Moreover, applying the first compatibility condition of (142), we have that

$$[h(u)]_1 = [QM(u)]_1 = u$$

so that both conditions of the second equation of (135) are satisfied. By using the splitting $Q = H + J/\sqrt{\varepsilon}$ and $Q^{-1} = H' + J'\sqrt{\varepsilon}$ in (144) and equating terms of the same order with respect to ε, we find the following conditions:

$$\begin{aligned} &A = A_0 + A_1(\varepsilon) \\ &A_0 = H\Lambda H' + H\Theta J' + J\Lambda J' \\ &A_1(\varepsilon) = \sqrt{\varepsilon}H\Lambda J' + \tfrac{1}{\sqrt{\varepsilon}}\left[H\Theta H' + J\Lambda H' + J\Theta J'\right] \\ &B = J\Theta H' \,. \end{aligned} \tag{148}$$

Note that since

$$QQ^{-1} = (H + J/\sqrt{\varepsilon})(H' + J'\sqrt{\varepsilon}) = Q^{-1}Q = I$$

we state the following identities

$$\begin{aligned} &HJ' = 0 \qquad && J'H = 0 \\ &JH' = 0 \qquad && H'J = 0 \\ &HH' + JJ' = I \quad && H'H + J'J = I; \end{aligned}$$

moreover it is easy to prove that $JJ' = e_3 \otimes e_3$, where e_3 denotes the third vector of the canonical basis of $\mathbb{R}^N$, and consequently $HH' = \mathbf{1} - e_3 \otimes e_3$. Then it is possible to check the validity of the other conditions in (135). □

R e m a r k 5.2. The high order numerical approximation of system (136) can be obtained generalizing the first order scheme described in section 5.1. It is possible to achieve this following ideas already exploited above: first we obtain a semidiscrete scheme applying a high order IMEX time integrator, then we couple it with a corresponding high order non-oscillatory space discretization.

Remark 5.3. We remark that our numerical approach based on relaxation approximation is different from those proposed in [**4**], where the Authors present two different kinds of approximation. The first one is based on a kinetic splitting and requires that $\varepsilon = O(\Delta x^2)$, while in our schemes ε can be chosen independently from Δx and so can also be used to discretize relaxing ($\varepsilon \neq 0$) schemes. The second one is a relaxed approach similar to [**72**] that does not require any relation between ε and Δx and it still relies on a splitting technique, while we will make use of IMEX time integrator to obtain high order schemes.

6 - Conclusions or some new developments

We have surveyed and analyzed relaxation approximation and relaxed schemes for nonlinear (degenerate) partial differential equations. By using suitable discretization in space and time, namely ENO/WENO non-oscillatory reconstructions for numerical fluxes or Discontinuous Galerkin approximation, and IMEX Runge-Kutta schemes for time integration, we have obtained a class of high order schemes. We have developed some topics about the theoretical convergence analysis for the semidiscrete scheme; furthermore we studied stability for the fully discrete schemes in some cases. Finally, we point out that these schemes can be easily implemented on parallel computers. Some preliminary results and details are reported in [**29**]. In particular the schemes involve only linear matrix-vector operations and the execution time scales linearly when increasing the number of processors which means that the execution time is inversely proportional to the number of processors. Our numerical approach can be easily extended also to more general problems, such as nonlinear convection-diffusion equations or nonlinear parabolic systems. As a conclusion we will make quick notes on some recent developments involving the relaxation framework.

6.1 - *From AP (Asymptotic Preserving) to AS (All Speed) schemes*

As we have seen, see Section 2, multiscale hyperbolic equations contain small scales [**68**, **97**] that lead to various different asymptotic regimes, in which the classical numerical approximations become prohibitively expensive (Δt, $h \approx \varepsilon$). Moreover, in many physical applications, the scaling parameter (mean free path/time, relaxation or reaction time, etc.), may vary over several orders of magnitude for different regimes: from the rarefied regime to the hydrodynamic (or diffusive) regime within the same problem. Here, we are interested in numerical techniques that works uniformly at different scale: from kinetic to fluid

regimes. The direct numerical solution of multiple scale problems is difficult to obtain due to the scale of computation: the ratio between the largest scale and the smallest scale could be as large. In physical systems, it is often sufficient to predict the macroscopic properties of the multiple-scale systems. Therefore, it is desirable to develop a method that captures the small scale effect on the large scales, but does not require resolving all the small scale features which can be prohibitively expensive. These asymptotic regime usually yield asymptotic expansions on some parameters. If the macroscopic equations are valid in the whole domain of interest, it is more efficient to solve them. However, there some problems where the microscopic models are needed, at least locally [**68**]. Moreover, it is also interesting to develop suitable numerical schemes able to follow the behaviour of the model between different regime: this is a challenge for effective numerical computations.

Following a forthcoming paper with F. Cavalli [**27**], we consider, as a prototypical example of multiscale hyperbolic model, an Isentropic Euler equation with stiff source/reaction terms,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0, \\ \\ \dfrac{\partial \rho u}{\partial t} + \nabla \left(\rho u^2 + \dfrac{p(\rho)}{\varepsilon^2} \right) = -\dfrac{1}{\varepsilon^2} R(\rho, \rho u), \end{cases} \tag{149}$$

where ρ represents the mass, ρu is the momentum, $p(\rho)$ is the pressure function, and $R(\rho, \rho u)$ is some kind of friction term. For the pressure p we usually assume that,

$$p(\rho) = D\rho^\gamma,$$

where $D > 0$ and $\gamma \geq 1$ are constants which depend on the physical problem. The constant $\varepsilon > 0$, which represents in the Isentropic Euler equation the scaled Mach number, is a multiscale parameter. According to $R(\rho, \rho u)$, we have different kind of equations. Here, to simplify the analysis, we usually consider the linear friction.

R e m a r k 6.1. In [**86**] the Authors study the singular convergence of solutions to a damped compressible Euler flow in one dimension of space and with a polytropic equation of state, when the inertial term tend to zero in a suitable rescaling. With reference to the system (149) they consider the pressure function $p(\rho) = C\rho^\gamma$, where $C > 0$ and $\gamma = 1 + 2/n$, $n \geq 3$ denoting the number of degrees of freedom of the molecules. Moreover, the rescaled momentum equation reads

$$\varepsilon \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x} \left(\varepsilon \rho u^2 + p(\rho) \right) = -ku,$$

where $k > 0$. As $\varepsilon \to 0^+$, they show that there exist limit functions ρ and u such that ρ satisfies, in the sense of distribution, Darcy's law

$$\frac{\partial p(\rho)}{\partial x} = -ku,$$

and it is a weak solution of the following non-linear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{\gamma C}{k(\gamma+1)} \frac{\partial^2 \rho^{\gamma+1}}{\partial x^2}.$$

Here, we focus on the numerical approximation of following system of balance laws

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0, \\ \dfrac{\partial \rho u}{\partial t} + \nabla \left(\rho u^2 + \dfrac{p(\rho)}{\varepsilon^2} \right) = -\dfrac{1}{\varepsilon^2} \rho u. \end{cases} \tag{150}$$

Moreover, we assume periodic boundary conditions for both ρ and ρu, remarking that general Dirichlet or Neumann boundary conditions can be taken into account as well. System (150) consists of a hyperbolic part, the isentropic Euler equation in the left hand side, and of the linear, possibly stiff, reaction term $-1/\varepsilon^2 \rho u$ in the right hand side of the momentum equation. We focus on the difficulties related to the time integration for the numerical approximation of the system (150), see e.g. [**19**] for a finite volume approximation of a similar system.

R e m a r k 6.2. A very interesting (and challenging) relaxation is with no source term, that is, putting $R(\rho, \rho u) \equiv 0$ in the system (149), that corresponds to the isentropic Euler equation with small Mach number. The occurrence of low Mach number regions in a globally compressible flow may be caused by the boundary or initial conditions, by the geometry of the problem, or by the underlying physical phenomena (e.g. in the case of phase changes). When the Mach number tends to zero, the system would relax to the incompressible Euler equations. This convergence has been studied mathematically by several Authors (see e.g. [**79**]). However, in numerical simulations, it is very difficult to shift from compressible flow equations to incompressible ones in the regions where the Mach number becomes very small. Therefore, it is necessary to design numerical methods for compressible flows that can handle both the compressible regime (i.e. local Mach-number of order unity) and the incompressible one (i.e. very small local Mach-number). This is the purpose of the so called *All-Speed schemes*. A vast literature has been developed in the last decade about the numerical methods for low Mach number problem, see e.g. [**38**, **39**, **43**, **90**, **92**, **101**] as the references cited therein.

We note that the hyperbolic part of (150) has stiff characteristic velocities when ε is small, as we have

$$\lambda_\varepsilon = u \pm \frac{\sqrt{p'(\rho)}}{\varepsilon}. \tag{151}$$

This means that if system (150) is approximated using a standard approach based on an explicit integration together with fluxes reconstruction, the CFL stability requirement on the time integration step Δt would impose

$$\Delta t \leq \frac{h}{\max\{|\lambda_\varepsilon|\}} = O(\varepsilon),$$

where h is the mesh size. Such requirement is by far too restrictive for vanishing values of the parameter ε. Moreover, in order to use nonlinear non-oscillatory reconstructions of the nonlinear term in high order approximations, a fully implicit approximation of $\partial_x(p(\rho))/\varepsilon^2$ should be avoided. A numerical scheme which is able to deal with multiscale ε avoiding to undergo too restrictive constraints on the time step Δt is usually referred as *all speed accurate* (AS) scheme [**64**]. We remark that characteristic velocities (151) may be large because of u and $p'(\rho)$, as the physical phenomenon is very fast. Anyway, when we will speak of stiff characteristic velocities, we always mean that the origin of stiffness is the scaling parameter ε.

In the limit $\varepsilon \to 0^+$ in the momentum equation, we have, at least formally, $\rho u \to \nabla p(\rho)$, so that the first equation shows an asymptotic behavior which is described by the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\nabla p(\rho)) = 0. \tag{152}$$

In particular, if the initial datum is far from the equilibrium $\rho u = -\nabla p(\rho)$, by means of Chapman-Enskog expansion, it is possible to show that the solution relaxes toward the equilibrium in a time that is proportional to ε. Therefore, the numerical scheme has to be able to project any approximated solution $(\rho u)^n$ on the approximate the equilibrium $\partial_x p(u^n)/\varepsilon^2$ with an accuracy of $O(\varepsilon)$, which means, for $\Delta t \gg \varepsilon$, in just one iteration. We recall that a scheme endowed with such property is an *asymptotic preserving* (AP) scheme. Finally, since system (150) exhibits an asymptotic diffusive behavior when $\varepsilon \to 0$, a fully time explicit approximation would lead to a classical parabolic constraint like $\Delta t \leq Ch^2$, where h is the mesh size, as for example in the schemes proposed in [**28**]. Conversely, a fully or linearly implicit approximation of the diffusive terms [**25**, **53**] would provide unconditionally stable numerical approximations. Here, to properly treat each term, we use Implicit-Explicit (IMEX) technique [**7**, **102**], in order to obtain an *hyperbolic stable* (HS) numerical schemes, that only require the hyperbolic stability constraint $\Delta t \leq Ch$.

First order All Speed Asymptotic Preserving numerical scheme. We describe a way to obtain a first order AS,HS and AP scheme. We remark that when λ_ε is small, the classical fully explicit approximation of (150) is enough to obtain efficient approximations. For this reason, without loss of generality, we focus on $\varepsilon \leq 1$. Similarly, we have that ε mainly affects the size of Δt only when $p'(\rho)/\varepsilon$ "controls" the term u. For this reason, we will assume that $\max|u| < \sqrt{p'(\rho)}/\varepsilon$, i.e.

$$\varepsilon < \frac{\sqrt{p'(\rho)}}{u}, \tag{153}$$

that is true for ε small. If (153) is not satisfied, a fully explicit scheme can be used again to approximate system (150). In order to develop the numerical schemes and to focus on the significant aspects, we consider only the one-dimensional spatial case.

We first deal with AS issue. To avoid stiff characteristic velocities, we proceed as proposed in [**72**, **95**] for the approximation of diffusive relaxation systems. Then, we split the stiff convective term $\partial_x p(u)/\varepsilon^2$ in the momentum equation into a non-stiff part $a(t)\partial_x p(u)$, which is treated along with the hyperbolic operators, and a stiff one $(a(t) - 1/\varepsilon^2)\partial_x p(u)$, which is treated along with the reaction operator. The term $a(t)\partial_x p(u)$ is non-stiff provided that $a(t)$ is sufficiently small, so we impose $0 < a(t) \leq 1$. Then, system (150) is rewritten as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho}{\partial x} = 0, \\ \dfrac{\partial (\rho u)}{\partial t} + \dfrac{\partial}{\partial x}(\rho u^2 + a(t)p(u)) = \dfrac{1}{\varepsilon^2}\left(\rho u + (1 - a(t)\varepsilon^2)\dfrac{\partial p(\rho)}{\partial x}\right). \end{cases} \tag{154}$$

Now, the hyperbolic part has characteristic velocities

$$\lambda_a = u \pm \sqrt{a(t)p'(\rho)},$$

which no more depend on ε. Then, system (154) can integrated instead of (150). To this end, let us define a possibly variable time step Δt^n, with $t^n = \sum_{i=1}^n \Delta t^n$ and let z^n be an approximation of the continuous variable $z(x,t)$ at time t^n. In the approach we are going to propose, the time integration step is usually adapted at each time step n. However, for the sake of simplicity, from here on we avoid to make explicit the dependence of Δt on n.

First, we detail the time integration of the momentum equation. The nonlinear term $\partial_x(\rho u^2 + a(t)p(u))$ are indeed explicitly treated. While the reaction part $-\rho u/\varepsilon^2$ can be discretize with an implicit scheme, the term $\partial_x p(\rho)/\varepsilon^2$ is in general nonlinear. A possible strategy to avoid nonlinear implicit problems

is, as in [**72**], to perform an operator splitting, which allows to treat this term in implicit without having to solve any nonlinear problem. The drawback is that the resulting fully discrete scheme is explicit and the time step is subdued to the parabolic constraint $\Delta t \leq Ch^2$. To avoid this, we approximate the term $\partial_x p(\rho)/\varepsilon$ in implicit, being aware of the nonlinear implicit problem that arises. A different approach based on linearly implicit approximations is proposed in [**25**, **26**].

In the mass equation, as in [**64**], we split the linear flux of the first equation into $\partial_x \rho u = \alpha(t)\partial_x \rho u + (1-\alpha(t))\partial_x \rho u$, with $0 \leq \alpha(t) \leq 1$ in order to treat term $\alpha(t)\partial_x \rho u$ explicitly and term $(1-\alpha(t))\partial_x \rho u$ implicitly. This is rather natural since, as ε varies, a system like (150) shows a behavior which ranges from hyperbolic, when $\varepsilon \gg 0$, to the leading order diffusive approximations (152) as $\varepsilon \approx 0$. The goal is to have a fully implicit scheme with $\alpha(t) \approx 0$ when ε is small, increasing $\alpha(t)$ to emphasize the explicit term $\alpha(t)\partial_x \rho$ in the hydrodynamic regime, $\varepsilon \approx 1$.

Then, the resulting semi-discrete scheme is

$$
\begin{cases}
\dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \alpha^n \dfrac{\rho^n u^n}{\partial x} + (1-\alpha^n)\dfrac{\partial \rho^{n+1}u^{n+1}}{\partial x} = 0, \\
\dfrac{\rho^{n+1}u^{n+1}-\rho^n u^n}{\Delta t} + \dfrac{\partial}{\partial x}(\rho^n (u^n)^2 + a^n p(\rho^n)) = -\dfrac{1}{\varepsilon^2}(\rho^{n+1}u^{n+1} \\
\qquad\qquad +(1-a^n\varepsilon^2)\dfrac{\partial}{\partial x}p(\rho^{n+1}))\,.
\end{cases} \tag{155}
$$

The characteristic velocities of hyperbolic part of (155) are now

$$
\lambda = u^n \pm \sqrt{(1-\alpha^n)(u^n)^2 + \alpha^n a^n p'(\rho^n)}. \tag{156}
$$

We remark that parameters a^n, α^n do not worsen the constraint on Δt induced by the new discrete characteristic velocities (156). In fact, since $a^n \leq 1 \leq 1/\varepsilon^2$ and thanks to (153), we have that $\max|\lambda| \leq \max|\lambda_\varepsilon|$.

Scheme (155) is AP. In fact, we can rewrite the scheme as

$$
\begin{aligned}
\rho^{n+1}u^{n+1} + \frac{\partial}{\partial x}p(\rho^{n+1}) =& \frac{\varepsilon^2}{\varepsilon^2+\Delta t}\left(\rho^n u^n - \Delta t \frac{\partial}{\partial x}\rho^n (u^n)^2\right) \\
&+ \frac{\varepsilon^2(1+a^n\Delta t)}{\varepsilon^2+\Delta t}\left(\frac{\partial}{\partial x}p(\rho^{n+1}) - \frac{\partial}{\partial x}p(\rho^n)\right) \\
&+ \frac{\varepsilon^2}{\varepsilon^2+\Delta t}\frac{\partial}{\partial x}p(\rho^n),
\end{aligned}
$$

and, assuming that all the functions are sufficiently smooth (in particular, the

derivative of $p(u)$ has to be continuous), gives the asymptotic preserving property

$$\rho^{n+1}u^{n+1} + \frac{\partial}{\partial x}p(\rho^{n+1}) = C_1\varepsilon^2\left(\rho^n u^n + \frac{\partial}{\partial x}p(\rho^n)\right) + C_2\varepsilon^2.$$

Summarizing, semi-discrete scheme (155) is both AS and AP. We point out that the scheme (155) is different from both that proposed in [**42**], as in such scheme no parameter α is used, and from the approximation introduced in [**64**], as a different strategy is used to remove stiffness from the characteristic velocities.

The last HS property is related to the fully discrete scheme, so let us introduce a uniform grid on $[a, b] \subset \mathbb{R}$, with mesh size h. To obtain a generic high order spatial discretization, the explicit hyperbolic fluxes are approximated at cell boundaries by means of conservative finite difference fluxes $F_{i+1/2}$, computed through

$$\hat{F}_{i+1/2} = \hat{F}_i^+(x_{i+1/2}) + \hat{F}_{i+1}^-(x_{i+1/2}),$$

where $\hat{F}_i^+$ and $\hat{F}_{i+1}^-$ are reconstructions of the solution in $i, i+1$-cells. To compute the reconstructions at the cell boundary $i+1/2$, we split the fluxes in (43) using the Local Lax-Friedrichs splitting

$$F^{\pm}(s) = \frac{1}{2}\left(F(s) \pm A_{i+1/2}\, s\right), \tag{157}$$

where $A_{i+1/2}$ is the local maximal wave speed, computed on the neighboring cells. Then, starting from the values $F^{\pm}(s_j)$ at the cell centers, the fluxes at cell boundaries are reconstructed using (non-oscillatory) reconstructions. In the first order scheme, numerical reconstructions (158) $\hat{F}^n_{k,i+1/2}$ are obtained using constant approximations, so that the resulting fluxes are

$$\begin{aligned}
\frac{\hat{F}^n_{1,i+1/2} - \hat{F}^n_{1,i-1/2}}{h} &= D_{x,i}(\rho^n u^n + \frac{1}{2}(A^n_{i-1/2}D_i^{x,-} - A^n_{i+1/2}D^+_{x,i})\rho^n,\\
\frac{\hat{F}^n_{2,i+1/2} - \hat{F}^n_{2,i-1/2}}{h} &= D_{x,i}(\rho^n(u^n)^2 + a^n p(\rho^n))\\
&\quad + \frac{1}{2}(A^n_{i-1/2}\mathrm{D}^-_{x,i} - A^n_{i+1/2}D^+_{x,i})\rho^n u^n,
\end{aligned}$$

where the upwind difference operators $D_x^{\pm} : \mathbb{R}^N \to \mathbb{R}^N$ are defined by

$$\begin{aligned}
D_x^+ z = (D^+_{x,i}\, z)_{i=1,\dots,N} &= \left(\frac{z_{i+1} - z_i}{h}\right)_{i=1,\dots,N},\\
D_x^- z = (D^-_{x,i}\, z)_{i=1,\dots,N} &= \left(\frac{z_i - z_{i-1}}{h}\right)_{i=1,\dots,N},
\end{aligned}$$

and

$$A^n_{i+1/2} = \max\{|\lambda_i|, |\lambda_{i+1}|\}.$$

To discretize implicit terms $\partial_x \rho^{n+1} u^{n+1}$ and $\partial_x p(\rho^{n+1})$ we introduce the second order central difference operator $D_2 : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$D_{2,i} z = \left(\frac{z_{i+1} - z_{i-1}}{2h} \right) \; i = 1, \ldots, N.$$

Then, the conservative IMEX finite difference approximation of system (155) is

$$\text{(158)} \quad \begin{cases} \dfrac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \alpha^n \dfrac{\hat{F}^n_{1,i+1/2} - \hat{F}^n_{1,i-1/2}}{h} + (1-\alpha^n) D_{2,i}((\rho u)^{n+1}) = 0 \\ \dfrac{\rho_i^{n+1} u_i^{n+1} - \rho_i^n u_i^n}{\Delta t} + \dfrac{\hat{F}^n_{2,i+1/2} - \hat{F}^n_{2,i-1/2}}{h} = \\ \qquad\qquad - \dfrac{1}{\varepsilon^2} \left(\rho_i^{n+1} u_i^{n+1} + (1 - a^n \varepsilon^2) D_{2,i}(p(\rho^{n+1})) \right). \end{cases}$$

Periodic boundary conditions are handled by setting $s_0 = s_N$ and $s_{N+1} = s_1$. To solve (158), as observed in [**42**], since the reaction term in the second equation is linear in ρu, we can obtain $\rho^{n+1} u^{n+1}$ from the momentum equation and insert it in the first equation, in order to obtain an implicit equation with respect to ρ^{n+1}.

$$\text{(159a)} \quad \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} - (1-\alpha^n) \frac{\Delta t}{\varepsilon^2 + \Delta t} (1 - a^n \varepsilon^2) (D_{2,i})^2 (p(\rho^{n+1})) =$$
$$- \alpha^n \frac{\hat{F}^n_{1,i+1/2} - \hat{F}^n_{1,i-1/2}}{h} - \frac{\varepsilon^2 (1-\alpha^n)}{\varepsilon^2 + \Delta t} \Big(D_{2,i}(\rho^n u^n) - \frac{\Delta t}{h} D_{2,i}(\hat{F}^n_{2,i+1/2} - \hat{F}^n_{2,i-1/2}) \Big).$$

After having obtained ρ^{n+1} from such equation, we put it into the second equation

$$\text{(159b)} \quad \rho_i^{n+1} u_i^{n+1} = \frac{\varepsilon^2}{\varepsilon^2 + \Delta t} \left(\rho_i^n u_i^n - \frac{\Delta t}{h} (\hat{F}^n_{2,i+1/2} - \hat{F}^n_{2,i-1/2}) \right)$$
$$- \frac{\Delta t}{\varepsilon^2 + \Delta t} (1 - a^n \varepsilon^2) D_i^x p(\rho^{n+1}),$$

to compute the updated value of the momentum In the mass equation, operator $(D_i^x)^2$ is an approximation of the second derivative ∂_{xx} which uses a five-point

stencil. To keep the stencil of the global scheme compact and to avoid unnecessary artificial diffusion, this operator can be simply replaced by the classical three-point discrete Laplace operator $L_{xx} : \mathbb{R}^N \to \mathbb{R}^N$

$$L_{xx}\, z = (L_{xx,i}\, z)_{i=1,\dots,N} = \left(\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} \right)_{i=1,\dots,N}.$$

The nonlinear implicit mass equation is solved by using Newton iterations.

The numerical scheme (159) depends on parameters a^n and α^n, which influence both stability and accuracy. We need to specify how a^n, α^n can be chosen, bearing in mind that such parameters affect the characteristic velocities and consequently the choice of the time step. In [**42**], for a similar problem, the authors proposed a simple way to obtain an estimation of parameter a^n. They noticed that discretizing the original system (150) by means of a classical forward Euler scheme with Lax-Friedrichs fluxes, the resulting scheme is stable thanks to artificial diffusion terms, even if, in practice, it is useless because of the unacceptably severe time step constraint $\Delta t \leq h/\lambda_\varepsilon$. So, they assume that the scheme they propose is stable as soon as its numerical diffusion is larger than that of the classical scheme, under a CFL condition related to the new non-stiff characteristic velocities λ. We remark that such approach is essentially equivalent to a modified equation method and can be applied with any numerical fluxes. We underline that in the equation we are studying, we also have the reaction term $-\rho u/\varepsilon^2$, which, however, since it is implicitly discretized, it is just a (linear) diagonal perturbation of the implicit operator and so it does not affect stability.

The analysis of the stability properties of the numerical scheme (159) with a recipe for the choice of parameters has been developed in [**27**] and part of these results were announced in [**26**]. As an example of application we will show the approximation of the $1D$ Euler-Poisson equation for semiconductors in the isentropic case with a more general scaling [**95**],

$$\text{(160)} \qquad \begin{cases} \dfrac{\partial N}{\partial t} + \dfrac{\partial (Nu)}{\partial x} = 0 \\ \dfrac{\partial (Nu)}{\partial t} + \dfrac{\partial}{\partial x}\left(Nu^2 + \dfrac{p\,(N)}{\varepsilon^2} \right) = -\dfrac{1}{\varepsilon^{2+\eta}}\,(Nu - NE) \\ \dfrac{\partial^2 \Phi}{\partial x^2} = N - c\,(x) \\ E = \dfrac{\partial \Phi}{\partial x}\,. \end{cases}$$

Where N is the density of electrons, Nu the current, p the pressure, and $-E$ is the electric field (Φ the electrical potential). The function $c(x)$ describes the density of positive ions in the material. The parameter η is connected to the

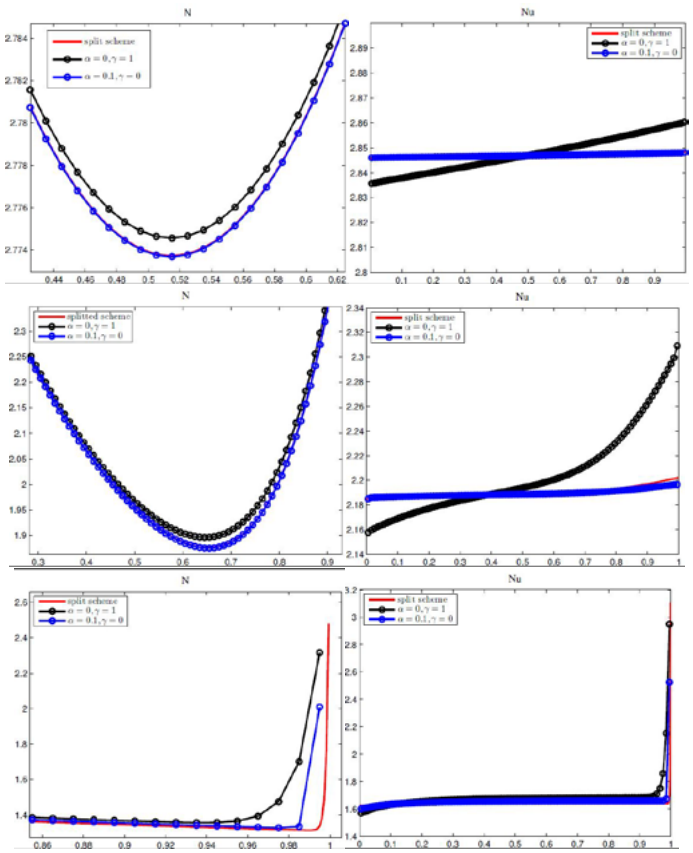

Fig. 8. Numerical simulation of the Isentropic Euler-Poisson system (160) for different scaling values, top line $\eta = 0$, $\varepsilon = 0.07$; middle line $\eta = 0.125$, $\varepsilon = 10^{-8}$; bottom line $\eta = 1$, $\varepsilon = 10^{-8}$. We report the variables N (left) and Nu (right), the discretization parameters were selected under the CFL stability condition [**27**], we used 80 space grid points.

collision rate in the gas of electrons. When $\varepsilon \to 0^+$ the limit profiles of N and E satisfy the classical drift-diffusion equation. We consider the problem of a unipolar diode of type N^+NN^+. We suppose the N^+ regions at a constant density $N = 3$ and consider the numerical solution only in the region N. The initial conditions are

$$N(x,0) = 3,\ u(x,0) = 0,\ \ x \in [0,1].$$

The doping profile was assumed to be constant equal to 1 in the N region. According to [**8**] the boundary conditions we assume are

$$N(0,t) = N(1,t) = 3,\ \Phi(0) = 0,\ \Phi(1) = 1,\ \ \frac{\partial Nu}{\partial x}(0) = \frac{\partial Nu}{\partial x}(1) = 0.$$

In our numerical computations we take $h = 0.01$, while the reference solution is a numerical solution on a very fine grid. In Figure 8 we report the asymptotic numerical approximation solution for the density N of the electrons, and the current Nu, for different value of $\eta = 0$, 0.125, 1 and the scaling parameter ε.

Remark 6.3. The scheme is robust also in the case of a non constant value of ε. In principle it is possible to automatically adjust the values of the parameters a and α in order perform local adaptation of the numerical method. In Figure 9 we display a numerical experiment using $\eta = 1$ and

$$\varepsilon(x) = 0.07\chi_{[0,0.5)} + 10^{-5}\chi_{[0.5,1]}.$$

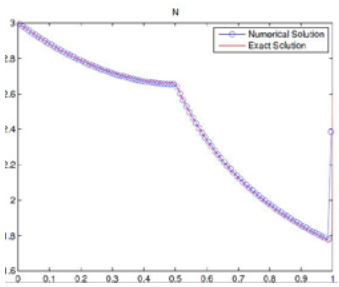

Fig. 9. A numerical experiments of the Isentropic Euler-Poisson system (160) with a non constant value of the scaling parameter ε, we show the density N, we used 80 spatial grid points.

6.2 - *Nonlocal problem*

At the end of these notes we want to show by an example related to a nonlocal problem, a possible development that seems interesting and promising. Conservation laws with nonlocal fluxes have appeared recently in the literature, arising naturally in many fields of application, such as in crowd dynamics, biology or biophysics (see e.g. [**22**, **37**, **47**, **52**, **104**] and the references therein). Here, we initiate the study of these equations from a numerical point of view. For this purpose we consider, as a significant example, the following evolution equation for the population density $\rho(x,t)$ in $\mathbb{R}^d$,

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (f(x,t,\rho)V(\rho,K)) - D\triangle\rho = 0 \\ &V(\rho,K) = g(K \star \rho) \end{aligned} \tag{161}$$

where $K : \mathbb{R}^d \to \mathbb{R}$ represents an interaction kernel and $\star$ denotes the convolution operator, $D \geq 0$ is the diffusivity coefficient, f and g are suitable functions, and $\triangle$ denotes the Laplacian. In the following we will (briefly) see three examples related to the model (161).

A first order aggregation model [52]. Equation (161) may be regarded as the continuum approximation, when the number of particles increases to infinity, of the following individual-based model (see e.g. [**23**]). Consider N particles in $\mathbb{R}^d$ whose position x_i, $i = 1, 2, \ldots, N$ evolve according to the system

$$\frac{dx_i}{dt} = v_i, \;\; v_i = -\frac{1}{N}\sum_{j\neq i} \nabla_x G(x_i - x_j), \tag{162}$$

where G denotes the interaction potential. The model can be justified and formally derived starting from the following second-order model in the Newton's form,

$$\varepsilon \frac{d^2x_i}{dt^2} + \frac{dx_i}{dt} = F_i, \;\; F_i = -\frac{1}{N}\sum_{j\neq i} \nabla_x G(x_i - x_j), \tag{163}$$

with small $\varepsilon 0$. From a biological point of view, this means to consider some small inertia/response time of individuals. By neglecting the ε-term, one can formally derive model (162). One can write (163) as

$$\frac{dx_i}{dt} = v_i, \;\; \varepsilon \frac{d^2x_i}{dt^2} = -v_i - \frac{1}{N}\sum_{j\neq i} \nabla_x G(x_i - x_j). \tag{164}$$

Using techniques reviewed in [**23**] one can formally take the limit $N \to \infty$ and associate to (164) the following kinetic equation for the density $f(t, x, v)$ of individuals at position x and velocity v:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon}\nabla_v \cdot (vf) + \frac{1}{\varepsilon}\nabla_v \cdot ((\nabla_x G \star \rho) f), \tag{165}$$

where

$$\rho(x, t) = \int_{\mathbb{R}^d} f(x, t, v) dv.$$

In [**52**] the Authors studied the measure-valued solutions of the kinetic model (165), and they considered their macroscopic limit for $\varepsilon \to 0$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \;\; u = -\nabla G \star \rho \tag{166}$$

which is a particular case of the (161) with $f(\rho) = \rho$, $g(s) = -s$, $K = \nabla G$, $D = 0$.

A stochastic model for swarm aggregation [91]. Consider a population of individuals subject to attraction/repulsion in addition to a classical Brownian random dispersal. Let $N > 0$ the constant size of the population, in the Lagrangian stochastic model proposed by the authors in [**91**] the k-th individual, out of N, is located at the random position $X_N^k(t) \in \mathbb{R}^d$, at time $t \geq 0$, so that $\left\{X_N^k(t)\right\}$ is a stochastic process in the state space $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$, on a common probability space $(\Omega, \mathcal{F}, P)$. The dynamics underlying the system of stochastic processes is given by a system of stochastic differential equations (EDSs),

$$dX_N^k(t) = F_N[X_N(t)](X_N^k(t))dt + \sigma_N dW^k(t),\ k = 1, \ldots, N, \tag{167}$$

where the randomness is modelled by additive independent standard Wiener processes W^k. Furthermore, the common variance σ_N^2 might depend on the total number of individuals and

$$\lim_{N\to\infty} \sigma_N^2 = \sigma_\infty^2 \geq 0.$$

The drift term F_N describes the mutual interaction among individuals; it depends on the relative location of the specific individual $X_k^N(t)$ with respect to all other individuals, via the empirical measure of the whole system of individuals

$$X_N(t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_k^N(t)} \in \mathcal{M}_P(\mathbb{R}^d), \tag{168}$$

where $\mathcal{M}_P(\mathbb{R}^d)$ is the space of all probability measures on $\mathbb{R}^d$. In order to find an expression for the drift operator F_N,we suppose that the interactions can be expressed in terms of the so-called "generalized gradient" of the empirical measure $X_N(t)$ convoluted with a suitable kernel G,

$$F_N(x) = \nabla G \star X_N(t)(x) = \sum_{k=1}^{N} \nabla G(x - X_N^k(t)). \tag{169}$$

Under sufficient regularity on the kernels, the stochastic process of empirical measures $\{X_N(t)\}$ has been shown to converge for $N \to \infty$ to a deterministic process $\{X_\infty(t)\}$. If the limit process $X_\infty(t)$ admits a density $\rho(\cdot, t)$ for any time t, then $\lim_{N\to\infty} X_N(t) = \rho(\cdot, t)dx$. Under some assumptions, the density satisfies the equation

$$\frac{\partial \rho}{\partial t} = \frac{\sigma_\infty^2}{2}\triangle\rho - \nabla \cdot (\rho \nabla G \star \rho) \tag{170}$$

on $\mathbb{R}^d \times (0, \infty)$, with initial condition $\rho(x, 0) = \rho_0(x)$, the density of $X_\infty(0)$. We still have a nonlocal equation of type (161).

A nonlocal crowd dynamics [37]. From a macroscopic point of view, a moving crowd is described by its density $\rho = \rho(t, x)$. If the number of individuals is constant, the conservation laws of the type

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

are a natural tool for the description of crowd dynamics. A key issue is the choice of the speed $\mathbf{v}$ in the flux of individuals. Considering the pedestrians' attitude to adapt to the crowd density they estimate to meet, in [**37**] the Authors proposed the following class of Cauchy problems,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v(\rho)(\nu(x) + \mathcal{I}(\rho))) = 0, \ \rho(0, x) = \rho_0(x). \tag{171}$$

An individual at time t and position $x \in \mathbb{R}^d$ moves at a speed with modulus $v(\rho)$, the vector $(\nu(x) + \mathcal{I}(\rho))$ describes the direction of movement of the individual given that the density is ρ. The vector ν is tangent at x to a suitable optimal path with respect to the visible geometry, in particular ν takes into consideration the discomfort felt by pedestrians. The vector $\mathcal{I}(\rho)$ describes the deviation from the direction ν due to the density ρ at time t. The operator $\mathcal{I}$ is in general nonlocal because depends on all value of the density in a neighborhood of the position x. A specific choices assumes that each individual aims at avoiding high crowd densities. Fix a mollifier η, then the convolution $(\rho \star \eta)$ is an average of the crowd density around x. This leads to the following definition,

$$\mathcal{I}(\rho) = -\varepsilon \frac{\nabla(\rho \star \eta)}{\sqrt{1 + \|\nabla(\rho \star \eta)\|^2}},$$

which states that individuals deviate from the optimal path trying to avoid entering regions with higher densities.

A numerical example. In order to show an example of approximation of the nonlocal equation (161) we modify the one-dimensional nonlocal traffic model proposed in [**2**] adding a linear diffusion term,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(f(t, x, \rho) v(\rho \star \eta)\right) = D \frac{\partial^2 \rho}{\partial x^2}. \tag{172}$$

Starting form the classical Lighthill, Whitham and Richards (LWR) model for vehicular traffic, where drivers adjust their speed according to the local traffic density, the speed law takes the functional form

$$f(\rho) = \rho(1 - \rho),\ v(s) = V_m a x(1 - r),\ \eta(x) = \alpha((x - c_1)(c_2 - x))^{5/2} \chi_{[c_1, c_2]}(x).$$

The coefficient α is chosen so that the total mass is equal to 1, the parameters $c_1 < c_2$ are the horizon of each driver, in the sense that a driver situated at x adjusts his speed according to the average vehicular density he sees on the interval $[xc_2, xc_1]$, we select the values $c_1 = 0$, $c_2 = 1/4$. We consider the initial datum as in [**2**]

$$\rho_0(x) = \frac{1}{2}\chi_{[-2.8,-1.8]}(x) + \frac{3}{4}\chi_{[-1.2,-0.2]}(x) + \frac{3}{4}\chi_{[0.6,1]}(x) + \chi_{[1.5,+\infty]}(x)$$

representing three groups of vehicles. We can choose different numerical schemes

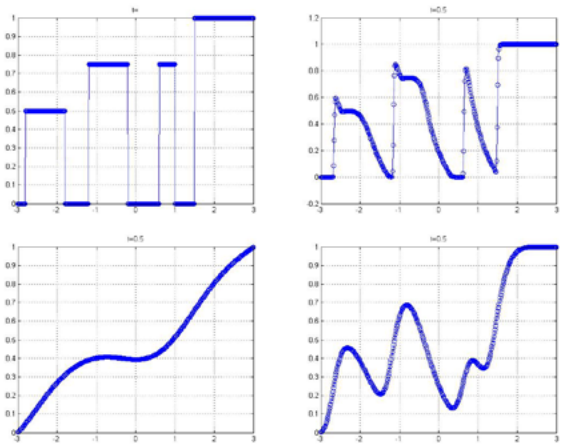

Fig. 10. A numerical test with nonlocal traffic model adding linear diffusion. Top line: initial datum (left), simulation with $D = 0$. Bottom line: numerical simulations with $D = 0.5$ (left), and $D = 0.05$ (right).

in the family of relaxed schemes, we select the modified three-velocities model (125) and the corresponding stable first order approximation (127). In Figure 10 we report the numerical simulations with different diffusion coefficient $D = 0, 0.05, 0.5$.

R e m a r k 6.4. A general finite volume method for equation with a gradient flow structure were introduced by Carrillo et al. in [**24**]. Here we do not use the gradient flow approach but the relaxation of the nonlocal transport term. The computational cost of the methods is mainly due to the calculation of the convolution at each time step.

An interesting approach based on a kinetic formulation for the $1D$ aggregation equation has been recently introduced and studied by Gosse and Vauchelet [**60**] using a new well-balanced and asymptotic preserving numerical scheme.

Acknowledgments. The Author would like to thank the co-workers in this relaxation adventure, in particular F. Cavalli, G. Toscani, L. Pareschi, G. Puppo, M. Semplice, and G. Russo.

References

[**1**] G. Aletti, A. Mentasti, and G. Naldi, *From microscopic to macroscopic models for self-organized interacting populations*, submitted.

[**2**] P. Amorim, R. M. Colombo and A. Teixeira, *On the numerical integration of scalar nonlocal conservation laws*, ESAIM Math. Model. Numer. Anal. **49** (2015), 19–37.

[**3**] D. Aregba-Driollet and R. Natalini, *Discrete kinetic schemes for multidimensional systems of conservation laws*, SIAM J. Numer. Anal. **37** (2000), no. 6, 1973–2004.

[**4**] D. Aregba-Driollet, R. Natalini and S. Tang, *Explicit diffusive kinetic schemes for nonlinear degenerate parabolic systems*, Math. Comp. **73** (2004), no. 245, 63–94.

[**5**] D. N. Arnold, F. Brezzi, B. Cockburn and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2001/02), no. 5, 1749–1779.

[**6**] D. G. Aronson, *Regularity properties of flows through porous media: A counterexample*, SIAM J. Appl. Math. **19** (1970), 299–307.

[**7**] U. M. Ascher, S. J. Ruuth and R. J. Spiteri, *Implicit-explicit Runge-Kutta methods for time dependent Partial Differential Equations*, Appl. Numer. Math. **25** (1997), 151–167.

[**8**] U. M. Ascher, P. A. Markowich, P. Pietra and C. Schmeiser, *A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model*, Math. Models Methods Appl. Sci. **1** (1991), 347–376.

[**9**] G. I. Barenblatt, *Scaling, self-similarity, and intermediate asymptotics*, Cambridge Texts in Applied Mathematics, **14** , Cambridge University Press, Cambridge 1996.

[**10**] G. I. Barenblatt and J. L. Vázquez, *Nonlinear diffusion and image contour enhancement*, Interfaces Free Bound **6** (2004), no. 1, 31–54.

[**11**] A. E. Berger, H. Brézis and J. C. W Rogers, *A numerical method for solving the problem* $u_t - \Delta f(u) = 0$, RAIRO Anal. Numér. **13** (1979), 297–312.

[**12**] M. Bisi and L. Desvillettes, *From reactive Boltzmann equations to reaction-diffusion systems*, J. Stat. Phys. **125** (2006), no. 1, 249–280.

[**13**] S. Boscarino, F. Filbet and G. Russo, *High order semi-implicit schemes for time dependent partial differential equations*, J. Sci. Comput. **68** (2016), no. 3, 975–1001.

[**14**] S. Boscarino, P. G. LeFloch and G. Russo, *High order asymptotic-preserving methods for fully nonlinear relaxation problems*, SIAM J. Sci. Comput. **36** (2014), no. 2, A377–A395.

[**15**] S. Boscarino, L. Pareschi and G. Russo, *Implicit-explicit Runge-Kutta schemes for hyperbolic systems and kinetic equations in the diffusion limit*, SIAM J. Sci. Comput. **35** (2013), no. 1, A22–A51.

[**16**] S. Boscarino and G. Russo, *Flux-explicit IMEX Runge-Kutta schemes for hyperbolic to parabolic relaxation problems*, SIAM J. Numer. Anal. **51** (2013), no. 1, 163–190.

[**17**] F. Bouchut, *Entropy satisfying flux vector splittings and kinetic BGK models*, Numer. Math. **94** (2003), no. 4, 623–672.

[**18**] F. Bouchut, F. R. Guarguaglini and R. Natalini, *Diffusive BGK approximations for nonlinear multidimensional parabolic equations*, Indiana Univ. Math. J. **49** (2000), no. 2, 723–749.

[**19**] F. Bouchut, H. Ounaissa and B. Perthame, *Upwinding of the source term at interfaces for Euler equations with high friction*, Comput. Math. Appl. **53** (2007), no. 3-4, 361–375.

[**20**] H. Brézis and A. Pazy, *Convergence and approximation of semigroups of nonlinear operators in Banach spaces*, J. Functional Analysis **9** (1972), 63–74.

[**21**] T. Carleman, *Problèmes Matématiques dans la théorie cinétique des gaz*, Almqvist & Wiksells Boktryckeri Ab, Uppsala 1957.

[**22**] J. A. Carrillo, R. M. Colombo, P. Gwiazda and A. Ulikowska, *Structured populations, cell growth and measure valued balance laws*, J. Differential Equations **252** (2012), no. 4, 3245–3277.

[**23**] J. A. Carrillo, M. Fornasier, G. Toscani and F. Vecil, *Particle, kinetic, and hydrodynamic models of swarming*, in "Mathematical modeling of collective behavior in socio-economic and life sciences", G. Naldi, L. Pareschi and G. Toscani, eds., Model. Simul. Sci. Eng. Technol., Birkhäuser, Boston, MA 2010, 297–336.

[**24**] J. A. Carrillo, A. Chertock and Y. Huang, *A finite-volume method for nonlinear nonlocal equations with a gradient flow structure*, Commun. Comput. Phys. **17** (2015), 233–258.

[**25**] F. Cavalli, *Linearly implicit approximations of diffusive relaxation systems*, Acta Appl. Math. **125** (2013), 79–103,

[**26**] F. Cavalli, *Linearly implicit schemes for convection-diffusion equations*, in "Hyperbolic problems: theory, numerics, applications", AIMS Ser. Appl. Math., **8**, F. Ancona, A. Bressan, P. Marcati and A. Marson, eds., AIMS, Springfield, MO 2014, 423–430.

[**27**] F. Cavalli and G. Naldi, *Time integration for semi-discrete approximation of multiscale hyperbolic systems*, submitted.

[**28**] F. Cavalli, G. Naldi, G. Puppo and M. Semplice, *High-order relaxation schemes for non linear degenerate diffusion problems*, SIAM J. Numer. Anal. **45** (2007), no. 5, 2098–2119.

[**29**] F. Cavalli, G. Naldi and M. Semplice, *Parallel algorithms for nonlinear diffusion by using relaxation approximation*, in "Numerical mathematics and advanced applications" (ENUMATH 2005 proc.), A. Bermúdez de Castro, D. Gómez, P. Quintela and P. Salgado, eds., Springer-Verlag, Berlin 2006, 404–411.

[**30**] F. Cavalli, G. Naldi, G. Puppo and M. Semplice, *A family of relaxation schemes for nonlinear convection diffusion problems*, Commun. Comput. Phys. **5** (2009), no. 2-4, 532-545.

[**31**] F. Cavalli, G. Naldi, G. Puppo and M. Semplice, *Relaxed schemes based on diffusive relaxation for hyperbolic-parabolic problems: some new developments*, in "Numerical methods for balance laws", Quad. Mat., **24**, G. Puppo and G. Russo, eds., Seconda Univ. Napoli, Caserta 2010, 157–195.

[**32**] F. Cavalli, G. Naldi and I. Perugia, *Discontinuous Galerkin approximation of relaxation models for linear and nonlinear diffusion equations*, SIAM J. Sci. Comput. **34** (2012), no. 1, A105–A136.

[**33**] F. Cavalli, G. Naldi and I. Perugia, *Discontinuous Galerkin approximation of porous Fisher-Kolmogorov equations*, Commun. Appl. Ind. Math. **4** (2013), e-446, 18 pp.

[**34**] C. Cercignani, *The Boltzmann equation and its applications*, Applied Mathematical Sciences, **67**, Springer-Verlag, New York 1988.

[**35**] S. Chapman and T. G. Cowling, *The mathematical theory of non-uniform gases. An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases*, 3rd edition, Cambridge Univ. Press, London 1970.

[**36**] B. Cockburn and C.-W. Shu, *Runge-Kutta discontinuous Galerkin methods for convection-dominated problems*, J. Sci. Comput. **16** (2001), no. 3, 173–261.

[**37**] R. M. Colombo, M. Garavello and M. Lécureux-Mercier, *Non-local crowd dynamics*, C. R. Math. Acad. Sci. Paris **349** (2011), 769–772.

[**38**] F. Coquel, M. Postel, N. Poussineau and Q. H. Tran, *Multiresolution technique and explicit-implicit scheme for multicomponent flows*, J. Numer. Math. **14** (2006), no. 3, 187–216.

[**39**] F. Cordier, P. Degond and A. Kumbaro, *An asymptotic-preserving all-speed scheme for the Euler and Navier-Stokes equations*, J. Comput. Phys. **231** (2012), no. 17, 5685-5704.

[**40**] F. Coron and B. Perthame, *Numerical passage from kinetic to fluid equations*, SIAM J. Numer. Anal. **28** (1991), no. 1, 26–42.

[**41**] M. G. Crandall and T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971), 265–298.

[**42**] P. Degond and M. Tang, *All speed scheme for the low Mach number limit of the isentropic Euler equations*, Commun. Comput. Phys. **10** (2011), no. 1, 1–31.

[**43**] S. Dellacherie, *Analysis of Godunov type schemes applied to the compressible Euler system at low Mach number*, J. Comput. Phys. **229** (2010), no. 4, 978–1016.

[**44**] S. M. Deshpande, P. S. Kulkarni and A. K. Ghosh, *New developments in kinetic schemes*, Comput. Math. Appl. **35** (1998), no. 1-2, 75–93.

[**45**] G. Dimarco, L. Pareschi, and V. Rispoli, *Asymptotically implicit schemes for the hyperbolic heat equation*, in "Hyperbolic problems: theory, numerics, applications" (Proc. Conference HYP2012), AIMS Ser. Appl. Math., **8**, AIMS, Springfield, MO 2014, 865–872.

[**46**] D. A. Di Pietro and A. Ern, *Mathematical aspects of discontinuous Galerkin methods*, Series: Mathématiques & Applications, **69**, Springer, Heidelberg 2012.

[**47**] R. Eftimie, *Hyperbolic and kinetic models for self-organized biological aggregations and movement: a brief review*, J. Math. Biol. **65** (2012), no. 1, 35–75.

[**48**] S. Evje and K. H. Karlsen, *Viscous splitting approximation of mixed hyperbolic-parabolic convection-diffusion equations*, Numer. Math. **83** (1999), 107–137.

[**49**] S. Evje and K. H. Karlsen, *Monotone difference approximations of BV solutions to degenerate convection-diffusion equations*, SIAM J. Numer. Anal. **37** (2000), no. 6, 1838–1860.

[**50**] W. E, *Principles of multiscale modeling*, Cambridge University Press, Cambridge 2011.

[**51**] B. Engquist, P. Lötstedt and O. Runborg, *Multiscale methods in science and engineering*, Lect. Notes Comput. Sci. Eng., **44**, Springer-Verlag, Berlin 2005.

[**52**] R. C. Fetecau and W. Sun, *First-order aggregation models and zero inertia limits*, J. Differential Equations **259** (2015), no. 11, 6774–6802.

[**53**] F. Filbet and S. Jin, *A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources*, J. Comput. Phys. **229** (2010), no. 20, 7625–7648.

[**54**] W. E. Fitzgibbon, *The fluid dynamical limit of the Carleman equation with reflecting boundary*, Nonlinear Anal. **6** (1982), no. 7, 695–702.

[**55**] A. Friedman, *The Stefan problem in several space variables*, Trans. Amer. Math. Soc. **133** (1968), 51–87.

[**56**] E. Gabetta and B. Perthame, *Scaling limits for the Ruijgrok-Wu model of the Boltzmann equation*, Math. Methods Appl. Sci. **24** (2001), no. 13, 949–967.

[**57**] E. Gabetta, L. Pareschi and G. Toscani, *Relaxation schemes for nonlinear kinetic equations*, SIAM J. Numer. Anal. **34** (1997), no. 6, 2168–2194.

[**58**] S. Goldstein, *On diffusion by discontinuous movements, and on the telegraph equation*, Quart. J. Mech. Appl. Math. **4** (1951), 129–156.

[**59**] L. Gosse, and G. Toscani, *An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 4, 337–342.

[**60**] L. Gosse and N. Vauchelet, *Numerical high-field limits in two-stream kinetic models and 1D aggregation equations*, SIAM J. Sci. Comput. **38** (2016), no. 1, A412–A434.

[**61**] S. Gottlieb and C.-W. Shu, *Total variation diminishing Runge-Kutta schemes*, Math. Comp. **67** (1998), no. 221, 73–85.

[**62**] S. Gottlieb, C.-W. Shu and E. Tadmor, *Strong stability-preserving high-order time discretization methods*, SIAM Rev. **43** (2001), no. 1, 89–112.

[**63**] J. L. Graveleau and P. Jamet, *A finite difference approach to some degenerate nonlinear parabolic equations*, SIAM J. Appl. Math. **20** (1971), 199–223.

[**64**] J. Haack, S. Jin and J.-G. Liu, *An all-speed asymptotic-preserving method for the isentropic Euler and Navier-Stokes equations*, Commun. Comput. Phys. **12** (2012), no. 4, 955–980.

[**65**] W. Jäger and J. Kačur, *Solution of porous medium type systems by linear approximation schemes*, Numer. Math. **60** (1991), no. 3, 407–427.

[**66**] J. W. Jerome and M. E. Rose, *Error estimates for the multidimensional two-phase Stefan problem*, Math. Comp. **39** (1982), no. 160, 377–414.

[**67**] S. Jin, *Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations*, SIAM J. Sci. Comput. **21** (1999), no. 2, 441–454.

[**68**] S. Jin, *Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review*, Riv. Math. Univ. Parma (N.S.) **3** (2012), no. 2, 177–216.

[**69**] S. Jin, M. A. Katsoulakis and Z. Xin, *Relaxation schemes for curvature-dependent front propagation*, Comm. Pure Appl. Math. **52** (1999), 1587–1615.

[**70**] S. Jin and C. D. Levermore, *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms*, J. Comput. Phys. **126** (1996), 449–467.

[**71**] S. Jin and L. Pareschi, *Asymptotic-preserving (AP) schemes for multiscale kinetic equations: a unified approach*, in "Hyperbolic problems: theory, numerics, applications, Vol. I, II" (Magdeburg, 2000), Internat. Ser. Numer. Math., **140**, **141**, Birkhäuser, Basel 2001, 573–582.

[**72**] S. Jin, L. Pareschi and G. Toscani, *Diffusive relaxation schemes for multiscale discrete velocity kinetic equations*, SIAM J. Numer. Anal. **35** (1998), 2405–2439.

[**73**] S. Jin, L. Pareschi and G. Toscani, *Uniformly accurate diffusive relaxation schemes for multiscale transport equations*, SIAM J. Numer. Anal. **38** (2000), no. 3, 913–936.

[**74**] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), no. 3, 235–276.

[**75**] S. Jin and Z. Xin, *Numerical passage from systems of conservation laws to Hamilton-Jacobi equations, relaxation schemes*, SIAM J. Numer. Anal. **35** (1998), no. 6, 2385–2404.

[**76**] H. G. Kaper, G. K. Leaf and S. Reich, *Convergence of semigroups with an application to the Carleman equation*, Math. Methods Appl. Sci. **2** (1980), no. 3, 303–308.

[**77**] J. Kačur, A. Handlovičová and M. Kačurová, *Solution of nonlinear diffusion problems by linear approximation schemes*, SIAM J. Numer. Anal. **30** (1993), no. 6, 1703–1722.

[**78**] C. A. Kennedy and M. H. Carpenter, *Additive Runge-Kutta schemes for convection-diffusion-reaction equations*, Appl. Numer. Math. **44** (2003), no. 1-2, 139–181.

[**79**] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math. **34** (1981), no. 4, 481–524.

[**80**] T. G. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics*, Trans. Amer. Math. Soc. **186** (1973), 259-272.

[**81**] C. Lattanzio and R. Natalini, *Convergence of diffusive BGK approximations for nonlinear strongly parabolic systems*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 2, 341–358.

[**82**] R. J. LeVeque and M. Pelanti, *A class of approximate Riemann solvers and their relation to relaxation schemes*, J. Comput. Phys. **172** (2001), no. 2, 572–591.

[**83**] P. L. Lions and G. Toscani, *Diffusive limit for finite velocity Boltzmann kinetic models*, Rev. Mat. Iberoamericana **13** (1997), no. 3, 473-513.

[**84**] E. Magenes, *Two-phase Stefan problems in several space variables* (Italian), Matematiche (Catania) **36** (1981), no. 1, 65–108.

[**85**] E. Magenes, R. H. Nochetto and C. Verdi, *Energy error estimates for a linear scheme to approximate nonlinear parabolic problems*, RAIRO Modél. Math. Anal. Numér. **21** (1987), no. 4, 655–678.

[**86**] P. Marcati and A. Milani, *The one-dimensional Darcy's law as the limit of a compressible Euler flow*, J. Differential Equations **84** (1990), no. 1, 129–147.

[**87**] P. Marcati and B. Rubino, *Parabolic relaxation limit for hyperbolic systems of conservation laws*, Rend. Circ. Mat. Palermo (2) Suppl. 45, part I (1996), 393–406.

[**88**] G. Martalò and G. Naldi, *A model for the evacuation dynamics of a crowd in bounded domain with obstacles*, submitted.

[**89**] H. P. McKean, *The central limit theorem for Carleman's equation*, Israel J. Math. **21** (1975), 54–92.

[**90**] F. Miczek, F. K. Röpke and P. V. F. Edelmann, *A new numerical solver for flows at various Mach numbers*, Astronomy & Astrophysics **576** (2015), A50.

[**91**] D. Morale, V. Capasso and K. Oelschläger, *An interacting particle system modelling aggregation behavior: from individuals to populations*, J. Math. Biol. **50** (2005), no. 1, 49–66.

[**92**] C.-D. Munz, S. Roller, R. Klein and K. J. Geratz, *The extension of incompressible flow solvers to the weakly compressible regime*, Comput. & Fluids **32** (2003), no. 2, 173–196.

[**93**] G. Naldi, *A chemo-mechanical model for the single myofibril in striated muscle contraction*, Meccanica (2017), doi: 10.1007/s11012-017-0654-9.

[**94**] G. Naldi and L. Pareschi, *Numerical schemes for kinetic equations in diffusive regimes*, Appl. Math. Lett. **11** (1998), no. 2, 29–35.

[**95**] G. Naldi and L. Pareschi, *Numerical schemes for hyperbolic systems of conservation laws with stiff diffusive relaxation*, SIAM J. Numer. Anal. **37** (2000), no. 4, 1246–1270.

[**96**] G. Naldi, L. Pareschi and G. Toscani, *Relaxation schemes for partial differential equations and applications to degenerate diffusion problems*, Surveys Math. Indust. **10** (2002), no. 4, 315–343.

[**97**] G. Naldi and G. Russo, eds., *Multiscale and adaptivity: modeling, numerics and applications*, Lecture Notes in Math., **2040**, CIME Found. Subser., Springer, Heidelberg; Fondazione CIME, Florence 2012.

[**98**] R. Natalini, *Convergence to equilibrium for the relaxation approximations of conservation laws*, Comm. Pure Appl. Math. **49** (1996), no. 8, 795–823.

[**99**] R. H. Nochetto, A. Schmidt and C. Verdi, *A posteriori error estimation and adaptivity for degenerate parabolic problems*, Math. Comp. **69** (2000), no. 229, 1–24.

[**100**] R. H. Nochetto and C. Verdi, *Approximation of degenerate parabolic problems using numerical integration*, SIAM J. Numer. Anal. **25** (1988), 784–814.

[**101**] A. Nonaka, A. S. Almgren, J. B. Bell, M. J. Lijewski, C. M. Malone and M. Zingale, *MAESTRO: An adaptive low Mach number hydrodynamics algorithm for stellar flows*, The Astrophysical Journal Supplement Series **188** (2010), no. 2, 358–383.

[**102**] L. Pareschi and G. Russo, *Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation*, J. Sci. Comput. **25** (2005), no. 1-2, 129–155.

[**103**] B. Perthame, *Second-order Boltzmann schemes for compressible Euler equations in one and two space dimensions*, SIAM J. Numer. Anal. **29** (1992), no. 1, 1–19.

[**104**] B. Piccoli and A. Tosin, *Time-evolving measures and macroscopic modeling of pedestrian flow*, Arch. Ration. Mech. Anal. **199** (2011), no. 3, 707–738.

[**105**] I. S. Pop and W.-A. Yong, *A numerical approach to degenerate parabolic equations*, Numer. Math. **92** (2002), no. 2, 357–381.

[**106**] A. Pulvirenti and G. Toscani, *Fast diffusion as a limit of a two-velocity kinetic model*, Rend. Circ. Mat. Palermo (2) Suppl. 45, part II (1996), 521–528.

[**107**] P. L. Roe and M. Arora, *Characteristic-based schemes for dispersive waves. I. The method of characteristics for smooth solutions*, Numer. Methods Partial Differential Equations **9** (1993), 459–505.

[**108**] F. Salvarani, *Diffusion limits for the initial-boundary value problem of the Goldstein-Taylor model*, Rend. Sem. Mat. Univ. Politec. Torino **57** (1999), no. 3, 209–220.

[**109**] F. Salvarani and G. Toscani, *The diffusive limit of Carleman-type models in the range of very fast diffusion equations*, J. Evol. Equ. **9** (2009), no. 1, 67–80.

[**110**] F. Salvarani and J. L. Vázquez, *The diffusive limit for Carleman-type kinetic models*, Nonlinearity **18** (2005), 1223–1248.

[**111**] M. Seaïd, *High-resolution relaxation scheme for the two-dimensional Riemann problems in gas dynamics*, Numer. Methods Partial Differential Equations **22** (2006), no. 2, 397–413.

[**112**] C.-W. Shu, *Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws*, in "Advanced numerical approximation of nonlinear hyperbolic equations" (Cetraro, 1997), Lecture Notes in Math., **1697**, CIME Found. Subser., Springer, Berlin 1998, 325–432.

[**113**] C.-W. Shu and S. Osher, *Efficient implementation of essentially nonoscillatory shock-capturing schemes. II*, J. Comput. Phys. **83** (1989), no. 1, 32–78.

[**114**] R. Spigler and D. H. Zanette, *A BGK model for chemical processes: the reaction-diffusion approximation*, Math. Models Methods Appl. Sci. **4** (1994), 35–47.

[**115**] E. B. Tadmor and R. E. Miller, *Modeling materials*, Cambridge University Press, Cambridge UK, 2011.

[**116**] G. I. Taylor, *Diffusion by continuous movements*, Proc. London Math. Soc. **S2-20** (1922), 196–212.

[**117**] J. L. Vázquez, *The Porous Medium Equation. Mathematical theory*, Oxford University Press, Oxford 2007.

[**118**] J. L. Vázquez, *Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type*, Oxford University Press, Oxford 2006.

Giovanni Naldi
Università degli studi di Milano
Dipartimento di Scienze e Politiche Ambientali
Centro ADAMSS
via Celoria 2
Milano, 20133, Italy
e-mail: giovanni.naldi@unimi.it