Riv. Mat. Univ. Parma, Vol. **8** (2017), 161-169

Fairouz Beggas, Margherita Maria Ferrari
and Norma Zagaglia Salvi

Combinatorial interpretations and enumeration of particular bijections

Abstract. Let n be a nonnegative integer. We call *widened permutation* a bijection between two $(n+1)$-sets having n elements in common. A *widened derangement* is a widened permutation without fixed points. In this paper we determine combinatorial interpretations of these functions in the context of the theory of species of Joyal. In particular, we prove that the species of the widened permutations is isomorphic to the derivative of the species of permutations. Looking at the generating series we obtain enumerative results, which are also obtained in a direct way. Finally, we prove that the sequence of widened derangement numbers turns out to coincide with the integer sequence A000255 of the On-Line Encyclopedia of Integer Sequences.

Keywords. Permutation, derangement, species, linear species, permutation species, uniform species, derivative of a species, isomorphic species.

Mathematics Subject Classification (2010): 05A05, 05A15, 05A19.

1 - Introduction

In this paper we consider and investigate a generalization of permutations, namely the functions that we call widened permutations. We also define the particular case of widened derangements in analogy with classical derangements (i.e. fixed-point-free permutations) which are extensively studied [**1**, **7**].

The formal definition is the following:

Received: June 23, 2015; accepted in revised form: July 19, 2016.

This research was partially supported by MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca). The first author was funded by PALSE Mobility Project.

Definition 1.1. Let n be a nonnegative integer, A an n-set and u, v distinct elements that do not belong to A. A *widened permutation* (w-permutation for short) of A is a bijection $f : A \cup \{u\} \to A \cup \{v\}$.

The function f can be represented by the sequence

$$l(f) = (u, f(u), f^2(u), \dots, f^{h-1}(u), v),$$

where h is the minimum positive integer such that $f^h(u) = v$, and a permutation $\pi(f)$ on the set $A \setminus \{f(u), f^2(u), \dots, f^{h-1}(u)\}$.

For instance, if $A = \{1, 2, 3, 4, 5\}$ and $u = 6$ and $v = 7$, then the function f represented in two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 7 & 4 & 3 & 5 & 1 \end{pmatrix} \tag{1}$$

can be decomposed into the sequence $l(f) = (6, 1, 2, 7)$ and the permutation $\pi(f) = (3, 4)(5)$.

The particular case in which $l(f)$ contains all the elements of A has applications in a coloring problem related to the direct product of two graphs [**3**].

Recall that a derangement of a finite set A is a permutation of A without fixed points [**1**]. In a similar way, we call *widened derangement* (w-derangement for short) a widened permutation without fixed points.

This paper is organized as follows. In Section 2, we determine combinatorial interpretations of the w-permutations in the context of the theory of species of Joyal. In Section 3, we derive similar results in relation to the w-derangements. In Sections 4 and 5, we consider the exponential generating functions and obtain enumerative results which are also obtained in a direct way. In Section 6, we prove that the sequence of widened derangement numbers turns out to be the integer sequence $A000255$ of the On-line Encyclopedia of Integers Sequences [**6**].

2 - A combinatorial interpretation of widened permutations

It follows from Definition 1.1 that a widened permutation $f : A \cup \{u\} \to A \cup \{v\}$ can be decomposed into a sequence $l(f) = (u, y_1, \dots, y_k, v)$, where $k \geq 0$ and $Q = \{y_1, \dots, y_k\} \subseteq A$, and a permutation $\pi(f)$ on $A \setminus Q$.

We recall some particular species and some related notions, referring the reader to [**4**, **5**] for further details. A linearly ordered set (or a linear order), is a set l together with a linear order relation (i.e. a reflexive, antisymmetric and transitive relation in which all elements are comparable) [**7**].

Let *Lin* be the species of linear orders and *Perm* the species of permutations.

The cardinalities of $Perm$ and Lin are the same exponential generating functions

$$gen(Perm;t) = gen(Lin;t) = \Sigma_{k\geq 0} k!\frac{t^k}{k!} = \frac{1}{1-t}.$$

The species of widened permutations, denoted $WPerm$, is defined as the species which associates to a finite set A the set $WPerm[A]$ of all the widened permutations of A.

Note that giving a structure of species $WPerm$ on a set A is equivalent to giving a linear order l on a subset A_1 of A and a structure of permutation on $A_2 = A \setminus A_1$. At the beginning and at the end of l, we insert the elements u and v respectively. Therefore the species $WPerm$ is the product of the species Lin and $Perm$

$$WPerm = Lin.Perm.$$

Recall that the derivative of the species $Perm$ is the species $Perm'$ defined for every finite set A by

$$Perm'[A] = Perm[A+1],$$

where $A+1$ denotes the set obtained from A by the addition of a new element, denoted $*$.

Let $\pi \in Perm[A+1]$. By removing the new element $*$, we obtain a subset S of A having a linear order, obtained from the cycle C which contains $*$ and the permutation on $A \setminus S$, obtained from π by removing C. In other words

$$Perm' = Lin.Perm.$$

Then the species $WPerm$ and $Perm'$ are isomorphic. In particular they are equipotent. Since the cardinality of the derivative of the species $Perm$ is the derivative of the cardinality of $Perm$, we obtain:

$$gen(Perm';t) = D(\Sigma_{n\geq 0} n!\frac{t^n}{n!}) = \Sigma_{n\geq 0}(n+1)!\frac{t^n}{n!}.$$

It follows that the exponential generating function of the species $WPerm$ is

$$gen(WPerm;t) = \Sigma_{n\geq 0}(n+1)!\frac{t^n}{n!},$$

and, denoted by w_n the number of widened permutations of an n-set, we obtain

(2) $$w_n = (n+1)!.$$

We can also determine (2) from the product of the species Lin and $Perm$ which consists in the species defined by

$$Lin.Perm[A] = \Sigma_{A_1+A_2=A} Lin[A_1] \times Perm[A_2],$$

with respect to every partition (A_1, A_2) of A.

Recall that the product of species is preserved when taking cardinalities. Thus, considering the cardinalities of the previous species, we obtain the cardinality of $WPerm$:

$$gen(Lin.Perm;t) = \frac{1}{1-t}.\frac{1}{1-t} \tag{3}$$

$$= (\frac{1}{1-t})^2$$

$$= (\Sigma_{k\geq 0} k!\frac{t^k}{k!})(\Sigma_{k\geq 0} k!\frac{t^k}{k!})$$

$$= \Sigma_{k\geq 0} c_k \frac{t^k}{k!},$$

where

$$c_k = \Sigma_{r=0}^{k} \binom{k}{r} r!(k-r)! = (k+1)!,$$

which coincides with formula (2).

3 - A combinatorial interpretation of widened derangements

In this section we determine combinatorial interpretations of w-derangements again in the context of the theory of species of Joyal. The species of widened derangements, denoted $WDer$, is defined as the species which associates to a finite set A the set $WDer[A]$ of all widened derangements of A. In [**2**], the authors investigated another particular type of derangements again in the context of the theory of species.

Recall that the geometric species, or uniform species, U is the species defined by $U[A] = \{*\}$ for every set A.

The cardinality of U is the exponential generating function

$$gen(U;t) = \Sigma_{n\geq 0}\frac{t^n}{n!} = e^t. \tag{4}$$

Note that giving a structure of species $WPerm$ on a set B is equivalent to giving a w-permutation without fixed points on a set $B_1 \subseteq B$ and a set $B_2 = B \setminus B_1$ of fixed points.

Thus the relation

$$WPerm = WDer.U \tag{5}$$

is satisfied. The product of the species *WDer* and U is the species defined by

$$WDer.U[A] = \Sigma_{A_1+A_2=A} WDer[A_1] \times U[A_2]$$

with respect to every partition (A_1, A_2) of A.

Considering the cardinalities, we obtain the cardinality of $WDer.U$.

By (3) and (4) we obtain that the exponential generating function of $WDer$ is

$$gen(WDer;t)$$

$$= \frac{1}{(1-t)^2}.e^{-t} \tag{6}$$

$$= (\Sigma_{n\geq 0}(n+1)!\frac{t^n}{n!})(\Sigma_{n\geq 0}(-1)^n\frac{t^n}{n!})$$

$$= \Sigma_{n\geq 0} z_n \frac{t^n}{n!},$$

where z_n is the number of widened derangements of an n-set; we obtain

$$z_n = \Sigma_{h=0}^{n} \binom{n}{h} (-1)^h (n-h+1)!$$

$$= n!(n+1-\frac{n}{1!}+\frac{n-1}{2!}+\cdots+(-1)^n\frac{1}{n!}). \tag{7}$$

The generating function of the species of widened derangements (6) can also be represented as

$$\frac{1}{(1-t)}.\frac{e^{-t}}{(1-t)}. \tag{8}$$

This function suggests another combinatorial interpretation of the widened derangement numbers. Note that the first factor is the generating function of the linear species while the second one is the generating function of the derangement species.

Indeed giving a structure of species $WDer$ on a set C is equivalent to giving a linear order on a subset $C_1 \subseteq C$ and a derangement on $C_2 = C \setminus C_1$. Thus, denoted by *Der* the species of the derangements, we obtain the relation

$$WDer = Lin.Der. \tag{9}$$

In other words, the species of widened derangements turns out to be the product of the linear species and the derangement species.

4 - Widened permutation numbers

In this section we obtain the same enumerative result concerning w_n, without using the notion of species. The number of widened permutations of an n-set A can be obtained directly in the following way.

Let Q be a k-subset of A; then we have $k!$ linear orders with the elements of Q and $(n-k)!$ permutations of $A \setminus Q$. Since w_n denotes the number of widened permutations on A, we have

$$\begin{aligned} w_n &= \Sigma_{k=0}^{n} \binom{n}{k} k!(n-k)! \\ &= \Sigma_{k=0}^{n} n! \\ &= (n+1)! \end{aligned} \tag{10}$$

as in (2).

5 - Widened derangement numbers

In this section we determine directly the widened derangement numbers and some of their recurrence.

Let A be a fixed n-set. Denote by A_i, $1 \leq i \leq n$, the set of w-permutations of A fixing an element i and A_i' the complementary set of A_i with respect to $WPerm[A]$. From (2) we have $w_n = (n+1)!$ and since A_i can be interpreted as the set of widened permutations of an $(n-1)$-set, then $\mid A_i \mid = n!$. By a similar motivation, $A_i \cap A_j$ can be interpreted as the set of widened permutations of an $(n-2)$-set; then $\mid A_i \cap A_j \mid = (n-1)!$ and so on. By Sylvester's formula expressing the inclusion and exclusion principle, we obtain

$$\begin{aligned} z_n &= \mid A_1' \cap A_2' \cap \cdots \cap A_n' \mid \\ &= w_n - s_1 + s_2 - \cdots + (-1)^n s_n, \end{aligned}$$

where s_1 is the number of widened permutations of an n-set having one fixed element, s_2 is the number of widened permutations of an n-set having two fixed elements and so on. Then,

$$\begin{aligned} z_n &= (n+1)! - \binom{n}{1} n! + \binom{n}{2} (n-1)! - \cdots + (-1)^n \binom{n}{n} 1! \\ &= n!(n+1 - \frac{n}{1!} + \frac{n-1}{2!} - \cdots + (-1)^n \frac{1}{n!}) \end{aligned}$$

which coincides with (7).

Recall that the number d_n of derangements of an n-set is

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

Then

$$z_n - d_n$$

$$= n(n-1)!\left(n - 1 + 1 - \frac{n-1}{1!} + \frac{n-2}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right)$$

$$= n z_{n-1}.$$

An immediate consequence is the following recurrence.

Proposition 5.1. *For every positive integer $n \geq 1$, we have*

$$d_n = z_n - n z_{n-1}. \tag{11}$$

By applying (11) to the well known recurrence $d_n = n d_{n-1} + (-1)^n$, we obtain the following recurrence for the widened derangement numbers

$$z_n = n(2z_{n-1} - (n-1)z_{n-2}) + (-1)^n. \tag{12}$$

6 - An integer sequence

The first values of the sequence of widened derangement numbers are

$$1, 1, 3, 11, 53, 309, 2119, \ldots$$

that coincide with the first integers of the sequence A000255 [**6**]. This sequence is obtained from the recurrence

$$a_n = n a_{n-1} + (n-1)a_{n-2},$$

with the initial conditions $a_0 = 1$, $a_1 = 1$. In the next Proposition, we prove that the numbers z_n satisfy the same recurrence.

Proposition 6.1. *For every positive integer $n \geq 2$ we have*

$$z_n = n z_{n-1} + (n-1)z_{n-2}, \tag{13}$$

with the initial conditions $z_0 = 1$ and $z_1 = 1$.

Proof. The result is obtained by induction on n. It is clearly true for $n = 2$; the initial conditions are obtained by direct inspection in this case.

Let $n > 2$ and let us assume it is satisfied for every integer less than n. By replacing (11) into the well known recurrence of the derangement numbers $d_n = (n-1)(d_{n-1} + d_{n-2})$, we obtain

$$z_n - nz_{n-1} = (n-1)(z_{n-1} - (n-1)z_{n-2} + z_{n-2} - (n-2)z_{n-3}).$$

By the induction hypothesis, we have that

$$z_{n-1} - (n-1)z_{n-2} + z_{n-2} - (n-2)z_{n-3} = z_{n-2}.$$

Then the result follows. □

Thus the sequence of the integers z_n coincides with the integer sequence $A000255$.

Proposition 6.2. *For every positive integer $n \geq 2$ we have*

$$d_n = (n-1)z_{n-2} \tag{14}$$

Proof. It follows immediately from (11) and (13). □

References

[**1**] M. Bóna, *Combinatorics of permutations*, CRC Press, Boca Raton, FL 2012.

[**2**] L. de Francesco Albasini and N. Zagaglia Salvi, *On the adjacent cycle derangements*, ISRN Discrete Mathematics **2012** (2012), Article ID 340357, 12 pages.

[**3**] M. Horňák, D. Mazza and N. Zagaglia Salvi, *Edge colorings of the direct product of two graphs*, Graphs Combin. **31** (2015), 975–992.

[**4**] A. Joyal, *Une théorie combinatoire des séries formelles*, Adv. in Math. **42** (1981), 1–82.

[**5**] E. Munarini, *A combinatorial interpretation of the generalized Fibonacci numbers*, Adv. in Appl. Math. **19** (1997), 306–318.

[**6**] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, published electronically at http://oeis.org, 2016.

[**7**] R. P. Stanley, *Enumerative combinatorics, Vol. 1*, Cambridge University Press, Cambridge 1997.

Fairouz Beggas
University of Lyon, LIRIS UMR5205 CNRS
Claude Bernard Lyon 1 University
43 Bd du 11 Novembre 1918
Villeurbanne, F-69622, France
e-mail: fairouz.beggas@liris.cnrs.fr

Margherita Maria Ferrari
Dipartimento di Matematica, Politecnico di Milano
P.zza Leonardo da Vinci 32
Milano, 20133, Italy
e-mail: margheritamaria.ferrari@polimi.it

Norma Zagaglia Salvi
Dipartimento di Matematica, Politecnico di Milano
P.zza Leonardo da Vinci 32
Milano, 20133, Italy
e-mail: norma.zagaglia@polimi.it