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Maged G. Bin-Saad

Some properties of multivariable Gegenbauer matrix polynomials

Abstract. New generalized form of the multi-variable Gegenbauer matrix polynomials are introduced using the integral representation method. Certain properties for these new generalized multi-variable Gegenbauer matrix polynomials such as differential relations, operational and hypergeometric matrix representations, generating matrix functions are derived. Further,some formulas in [**18**] have been corrected.

Keywords. Hypergeometric matrix function, Multi-variable Gegenbauer matrix polynomials, generating matrix function, recurrence relations, differential equations.

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1 - Introduction, definitions and notations

The analogous extension to the matrix framework for the classical case of Humbert (see [**1**]- [**4**], [**20**]), Hermite [**17**], Jacobi [**9**], Gegenbauer [**12**, **13**, **16**], Laguerre [**14**] and Chebyshev [**8**] polynomials has been carried out in recent years, and properties and applications of different classes for these matrix polynomials are the focus of a number of previous papers, see [**1**, **7**, **10**, **11**, **23**, **24**] and references therein, for example. Throughout this paper, for a matrix $A \in \mathbb{C}^{N\times N}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\parallel A \parallel$, is defined by

$$\parallel A \parallel = \sup_{x\neq 0} \frac{\parallel Ax \parallel_2}{\parallel x \parallel_2}$$

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where, for a vector $y \in \mathbb{C}^{N\times N}$, $\| y \|_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y. I will denote the identity matrix in $\mathbb{C}^{N\times N}$. We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ is the set of the eigenvalues of A. If $A_0, A_1, \ldots, A_n \ldots$, are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq 0$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots + A_1 x + A_0,$$

a matrix polynomial of degree n in x. If $A + nI$ is invertible for every integer $n \geq 0$ then

$$(A)_n = A(A+I)(A+2I)\cdots(A+(n-1)I); n \geq 1; (A)_0 = I. \tag{1.1}$$

Thus we have

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-n)_k}{n!}I = \frac{(-nI)_k}{n!}; 0 \leq k \leq n, \tag{1.2}$$

$$(A)_{m+n} = (A)_m (A+mI)_n, \tag{1.3}$$

and

$$(A)_{m-n} = \frac{(-1)^n (A)_m}{(I-A-m)_n}, 0 \leq n \leq m. \tag{1.4}$$

The hypergeometric matrix function is defined in [**13**] as follows:

$${}_2F_1\left[A, B; C; z\right] = \sum_{n=0}^{\infty} \frac{1}{n!}(A)_n (B)_n [(C)_n]^{-1} z^n, \tag{1.5}$$

where A, B and C are matrices in $\mathbb{C}^{N\times N}$ such that $C + nI$ is invertible for integer $n \geq 0$ and $|z| < 1$. For the purpose of this work, we introduce the following hypergeometric matrix functions:

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{cc} (A_p) : (B_q); (C_k); & \\ & x, y \\ (D_l) : (E_m); (F_n); & \end{array}\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(A_j)_{r+s}\prod_{j=1}^{q}(B_j)_r\prod_{j=1}^{k}(C_j)_s}{r!s!}$$

$$\left[\prod_{j=1}^{l}(D_j)_{r+s}\right]^{-1}\left[\prod_{j=1}^{m}(E_j)_r\right]^{-1}\left[\prod_{j=1}^{n}(F_j)_s\right]^{-1} x^r y^s, \tag{1.6}$$

where for convergence

$$p+q<l+m+1, p+k<l+n+1, |x|<\infty, |y|<\infty,$$

or

$$p+q=l+m+1, p+k=l+n+1, \text{and}$$
$$|x|^{\frac{1}{(p-l)}}+|y|^{\frac{1}{(p-l)}}<1, \text{if} \quad p>l; \qquad \max\{|x|,|y|\}<1, \text{if} \quad p\leq l;$$

(1.7)
$$F_B^{(n)}\left[A_1,...,A_n,B_1,...,B_n;C;x_1,...,x_n\right] = \sum_{m_1,...,m_n=0}^{\infty} \frac{(A_1)_{m_1}...(A_n)_{m_n}(B_1)_{m_1}...(B_n)_{m_n}}{m_1!...m_n!}\left[(C)_{m_1+\cdots+m_n}\right]^{-1} x_1^{m_1}...x_n^{m_n},$$

$$\max\{|x_1|,\cdots|x_n|\}<1;$$

(1.8)
$$H_C\left[A,B,D;C;x,y,z\right] = \sum_{m,n,p=0}^{\infty}(A)_{m+p}(B)_{m+n}(D)_{n+p}\left[(C)_{m+n+p}\right]^{-1}\frac{x^m y^n z^p}{m!n!p!},$$

$\{|x|=r<1; |y|=s<1; |z|=t<1; r+s+t-2\sqrt{[(1-r)(1-s)(1-t)]}<2\}$, where (1.6), (1.7) and (1.8) are the hypergeometric matrix versions of Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}$[**27**, p. 27, Eq. 28], Lauricella function $F_B^{(n)}$ [**27**, p. 33, Eq. 2] and Srivastava's triple series H_C [**27**, p. 69(38)] respectively and all matrices in (1.6), (1.7) and (1.8) are in $\mathbb{C}^{N\times N}$ such that

$(D_j+rI), j=1,2,...,l$ is invertible for $r\geq 0$,
$(E_j+rI), j=1,2,...,m$ is invertible for $r\geq 0$,
$(F_j+rI), j=1,2,...,n$ is invertible for $r\geq 0$,

and

$(C+nI)$ is invertible for $n\geq 0$.

Clearly for $n=2$, formula (1.7) reduces to

(1.9)
$$F_B^{(2)}=F_3\left[A,A',B,B';C;x,y\right]=\sum_{m,n=0}^{\infty}\frac{(A)_m(A')_n(B)_m(B')_n}{m!n!}\left[(C)_{m+n}\right]^{-1}x^m y^n,$$

$$\max\{|x|,|y|\}<1;$$

where Equation (1.9) is the matrix version of Appellś function of double variables F_3 (see [**27**, p. 53(6)]). The generalized hypergeometric matrix function (see (1.5) and [**21**]) is given in the form:

(1.10) $${}_pF_q\left[A_1,...,A_p;C_1,C_2,...,C_q;z\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (A_1)_n (A_2)_n \cdots (A_p)_n [(C_1)_n]^{-1} .... [(C_q)_n]^{-1} z^n,$$

where $(C_i + nI), (i = 1, 2, ..., q)$ is invertible for $n \geq 0$.

Next, we recall the following relation [**27**]:

$$(1.11)\quad (1 - x_1 - x_2 - \cdots - x_r)^{-A} = \sum_{n_1, \cdots, n_r = 0}^{\infty} (A)_{n_1 + \cdots + n_r} \frac{x^{n_1}}{n_1!} \cdots \frac{x^{n_r}}{n_r!}, \quad |x_1 + \cdots + x_r| < 1.$$

Also, we recall that if $A(k, n)$ and $B(k, n)$ are matrices in $\mathbb{C}^{N \times N}$ for $n \geq 0$ and $k \geq 0$ then it follows that [**28**]:

$$(1.12)\qquad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n + 2k),$$

and

$$(1.13)\qquad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + mk),$$

where m is positive integer.

For convenience, a few conventions and some notations are introduced here:

(1) Boldface letters denote sum of dimension r; for instance, we have

$$\mathbf{n} = (n_1 + n_2 + \cdots + n_r), \mathbf{m} = (m_1 + m_2 + \cdots + m_r),$$

$$\mathbf{k} = (k_1 + k_2 + \cdots + k_r), \mathbf{s} = (s_1 + s_2 + \cdots + s_r)$$

$$\mathbf{n} + \mathbf{m} = (n_1 + m_1 + \cdots + n_r + m_r)$$

and

$$\mathbf{n} + \mathbf{m} + \mathbf{s} + \mathbf{k} = (n_1 + m_1 + s_1 + k_1 + \cdots + n_r + m_r + s_r + k_r).$$

(2) The symbols $\mathbf{x} = (x_1, \cdots, x_r)$ and $\mathbf{y} = (y_1, \cdots, y_r)$ denote multiple (r-dimensional) variables.

Motivated by the aforementioned works in [**1**, **15**, **16**, **17**] and largely by the work of Dattoli et al. [**6**] who have used the link between Hermite and Gegenbauer polynomials to introduce generalized forms of Gegenbauer polynomials. In this paper, various generalizations of Hermite polynomials were exploited to introduce several forms of multivariable Gegenbauer matrix polynomials.

In Section 2, we introduce form of multivariable Gegenbauer matrix polynomials. In Section 3, we derive certain differential properties involving these polynomials. In Section 4, we obtain hypergeometric and operational matrix representations for multivariable Gegenbauer matrix polynomials. In Section 5, we establish certain generating matrix relations involving multivariable Gegenbauer matrix polynomials. In Section 6, concluding remarks are given.

2 - Multi-variable Gegenbauer matrix polynomials

In many situations an integral representation of matrix polynomials is more convenient to use and to establish more (new) general forms of matrix polynomials than its series representation. In this regard the the second-order Hermite Kámpe de Fériet polynomials [**28**]

$$H_m\left(x,y\right)=n!\sum_{s=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2s}y^s}{(n-2s)!s!}, \tag{2.1}$$

will be exploited here to introduce multivariable Gegenbauer polynomials $C^A_{n_1,m_1,\cdots,n_r,m_r}\left(\mathbf{x},\mathbf{y}\right)$ in form of integral formula. The multivariable Gegenbauer matrix polynomials $C^A_{n_1,m_1,\cdots,n_r,m_r}\left(\mathbf{x},\mathbf{y}\right)$ can be defined in the following form:

$$C^A_{n_1,m_1,\cdots,n_r,m_r}\left(\mathbf{x},\mathbf{y}\right)=\frac{\Gamma^{-1}(A)}{n_1!m_1!\cdots n_r!m_r!}$$

$$\times\int_0^\infty e^{-t}t^{A+(\mathbf{n}+\mathbf{m}-1)I}\prod_{j=1}^{r}\left\{H_{n_j}\left(2x_j,-t^{-1}\right)H_{m_j}\left(2y_j,-t^{-1}\right)\right\}dt, \tag{2.2}$$

where A is a matrix in $\mathbb{C}^{N\times N}$, such that $-k\notin\sigma(A),\forall$ integer $k>0$.

Now, making use of (2.1) and formula (see [**16**])

$$\Gamma(A)=\int_0^\infty e^{-t}t^{A-I}dt,\quad t^{A-I}=\exp[(A-I)lnt],$$

we find that $C^A_{n_1,m_1,\cdots,n_r,m_r}\left(\mathbf{x},\mathbf{y}\right)$ are defined by the following series

$$C^A_{n_1,m_1,\cdots,n_r,m_r}\left(\mathbf{x},\mathbf{y}\right)=\sum_{s_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_1=0}^{\left[\frac{m_1}{2}\right]}\cdots\sum_{s_r=0}^{\left[\frac{n_r}{2}\right]}\sum_{k_r=0}^{\left[\frac{m_r}{2}\right]}(A)_{\mathbf{n}+\mathbf{m}-\mathbf{s}-\mathbf{k}}$$

$$\prod_{j=1}^{r}\left\{\frac{(-1)^{s_j+r_j}(2x_j)^{n_j-2s_j}(2y_j)^{m_j-2k_j}}{(n_j-2s_{j)})!(m_j-2k_{j)})!s_j!k_j!}\right\}. \tag{2.3}$$

From (2.3), it follows that $C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})$ is a matrix polynomials in $2r$ variables x_j and y_j of degree precisely n_j in x_j and m_j in $y_j,(j=1,2,3,...,r)$.

Lemma 2.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$, such that $-k\notin\sigma(A),\forall$ integer $k>0$ and $|u_j|<1,|v_j|<1,|x_j|<1,$ and $|y_j|<1,$ $|\sum_{j=1}^{r}\Big(2u_jx_j+2v_jy_j-u_j^2$ $-v_j^2\Big)|\neq 1$. Then it follows that*

$$\left(1-\sum_{j=1}^{r}\left(2u_jx_j+2v_jy_j-u_j^2-v_j^2\right)\right)^{-A}$$

$$=\sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty}C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})\,u_1^{n_1}v_1^{m_1}\cdots u_r^{n_r}v_r^{m_r}. \tag{2.4}$$

Proof. Based on (1.11), formula (2.4) can be written in the form

$$\left(1-\sum_{j=1}^{r}\left(2u_jx_j+2v_jy_j-u_j^2-v_j^2\right)\right)^{-A}$$

$$=\sum_{n_1,m_1,s_1,k_1,\cdots,n_r,m_r,s_r,k_r=0}^{\infty}(A)_{\mathbf{n}+\mathbf{m}+\mathbf{s}+\mathbf{k}}$$

$$\prod_{j=1}^{r}\left\{\frac{(-1)^{s_j+k_j}(2x_j)^{n_j}(2y_j)^{m_j}u_j^{n_j+2s_j}v_j^{m_j+2k_j}}{n_j!m_j!s_j!k_j!}\right\}. \tag{2.5}$$

In (2.5), if we replace n_j and m_j by n_j-2s_j and $m_j-2k_j, j=(1,2,3,...,r)$, respectively, we get

$$\left(1-\sum_{j=1}^{r}\left(2u_jx_j+2v_jy_j-u_j^2-v_j^2\right)\right)^{-A}$$

$$=\sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty}\sum_{s_1=0}^{[\frac{n_1}{2}]}\sum_{k_1=0}^{[\frac{m_1}{2}]}\cdots\sum_{s_r=0}^{[\frac{n_r}{2}]}\sum_{k_r=0}^{[\frac{m_r}{2}]}(A)_{\mathbf{n}+\mathbf{m}-\mathbf{s}-\mathbf{k}}$$

$$\prod_{j=1}^{r}\left\{\frac{(-1)^{s_j+r_j}(2x_j)^{n_j-2s_j}(2y_j)^{m_j-2k_j}u_j^{n_j}v_j^{m_j}}{(n_j-2s_{j)})!(m_j-2k_{j)})!s_j!k_j!}\right\}, \tag{2.6}$$

which on view of (2.3), gives us the right-hand side of (2.5) and this complete the proof of the lemma. □

In view of the assertion (2.3), we find that

$$C^A_{n,0,\cdots,0}(x,1,\cdots,1)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(A)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}=C^A_n(x), \tag{2.7}$$

$$C^A_{n,m,0,\cdots,0}(x,y,1,\cdots,1)$$

$$=\sum_{k=0}^{\left[\frac{m}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^{k+s}(A)_{m+n-k-s}}{k!s!(m-2k)!(n-2s)!}(2x)^{m-2k}(2y)^{n-2s}=C^A_{n,m}(x,y), \tag{2.8}$$

where $C^A_n(x)$ and $C^A_{n,m}(x,y)$ are Gegenbauer matrix polynomials defined in [**25**] and [**18**] respectively, and

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(-\mathbf{x},-\mathbf{y})=(-1)^{\mathbf{n}+\mathbf{m}}C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y}). \tag{2.9}$$

Next, Putting $x_j=y_j=1,(j=1,2,3...,r)$, in (2.4), we get

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(1,\cdots,1)=\sum_{s_1=0}^{\left[\frac{n_1}{2}\right]}\sum_{k_1=0}^{\left[\frac{m_1}{2}\right]}\cdots\sum_{s_r=0}^{\left[\frac{n_r}{2}\right]}\sum_{k_r=0}^{\left[\frac{m_r}{2}\right]}(A)_{\mathbf{n}+\mathbf{m}-\mathbf{s}-\mathbf{k}}$$

$$\prod_{j=1}^{r}\left\{\frac{(-1)^{s_j+r_j}2^{n_j+m_j-2s_j-2k_j}}{(n_j-2s_{j)})!(m_j-2k_{j)})!s_j!k_j!}\right\}. \tag{2.10}$$

From (2.10) it easily seen that

$$C^A_{0,\cdots,0}(1,\cdots,1)=I, \tag{2.11}$$

$$C^A_{1,\cdots,1}(1,\cdots,1)=2^{2r}(A)_{2r}, \tag{2.12}$$

and

$$C^A_{2,\cdots,2}(1,\cdots,1)=2^{2r}(A)_{4r}+(A)_{2r}. \tag{2.13}$$

For $x_j = y_j = 0, (j = 1, 2, 3..., r)$, (2.4) gives us

$$\left(1 + u_1^2 + v_1^2 + \cdots + u_r^2 + v_r^2\right)^{-A}$$

$$= \sum_{n_1, m_1, \cdots, n_r, m_r = 0}^{\infty} C^A_{n_1, m_1, \cdots, n_r, m_r}(0, \cdots, 0)\, u_1^{n_1} v_1^{m_1} \cdots u_r^{n_r} v_r^{m_r}. \tag{2.14}$$

However,

$$\left(1 + u_1^2 + v_1^2 + \cdots + u_r^2 + v_r^2\right)^{-A}$$

$$= \sum_{n_1, m_1, \cdots, n_r, m_r = 0}^{\infty} (-1)^{\mathbf{n}+\mathbf{m}} (A)_{\mathbf{n}+\mathbf{m}}$$

$$\frac{(-u_1^2)^{n_1}(-v_1^2)^{m_1} \cdots (-u_r^2)^{n_r}(-v_r^2)^{m_1}}{n_1! m_1! \cdots n_r! m_r!}. \tag{2.15}$$

Therefore, we get

$$C^A_{2n_1, 2m_1, \cdots, 2n_r, 2m_r}(0, \cdots, 0) = (-1)^{\mathbf{n}+\mathbf{m}} (A)_{\mathbf{n}+\mathbf{m}}$$

$$\frac{(-u_1)^{n_1}(-v_1)^{m_1} \cdots (-u_r)^{n_r}(-v_r)^{m_1}}{n_1! m_1! \cdots n_r! m_r!}, \tag{2.16}$$

and

$$C^A_{2n_1+1, 2m_1+1, \cdots, 2n_r+1, 2m_r+1}(0, \cdots, 0) = 0. \tag{2.17}$$

3 - Differential relations

Let

$$\Omega = \left(1 - \sum_{j=1}^{r} \left(2u_j x_j + 2v_j y_j - u_j^2 - v_j^2\right)\right)^{-A}. \tag{3.1}$$

On differentiating (3.1) partially, with respect to $x_j, (j = 1, 2, ..., r)$, we get

$$\frac{\partial \Omega}{\partial x_1} = \frac{u_1}{(1 - \sum_{j=1}^{r}(2u_j x_j + 2v_j y_j - u_j^2 - v_j^2))} 2A\Omega, \tag{3.2}$$

$$\frac{\partial^2\Omega}{\partial x_1\partial x_2} = \frac{u_1u_2}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^2))^2}2^2A(A+I)\Omega, \tag{3.3}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial^r\Omega}{\partial x_1\cdots\partial x_r} = \frac{u_1u_2\cdots u_r}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^r))^r}2^r(A)_r\Omega. \tag{3.4}$$

Similarly,one can easily show that

$$\frac{\partial^r\Omega}{\partial y_1\cdots\partial y_r} = \frac{v_1v_2\cdots v_r}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^r))^r}2^r(A)_r\Omega, \tag{3.5}$$

$$\frac{\partial^r\Omega}{\partial u_1\cdots\partial u_r} = \frac{(x_1-u_1)(x_2-u_2)\cdots(x_r-u_r)}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^r))^r}2^r(A)_r\Omega, \tag{3.6}$$

$$\frac{\partial^r\Omega}{\partial v_1\cdots\partial v_r} = \frac{(y_1-v_1)(y_2-v_2)\cdots(y_r-v_r)}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^r))^r}2^r(A)_r\Omega. \tag{3.7}$$

So that the matrix function Ω satisfies the partial matrix differential equations

$$(v_1v_2\cdots v_r)\frac{\partial^r\Omega}{\partial x_1\cdots\partial x_r} = (u_1u_2\cdots u_r)\frac{\partial^r\Omega}{\partial y_1\cdots\partial y_r}, \tag{3.8}$$

$$\prod_{j=1}^{r}(x_j-u_j)\frac{\partial^r\Omega}{\partial v_1\cdots\partial v_r} = \prod_{j=1}^{r}(y_j-v_j)\frac{\partial^r\Omega}{\partial u_1\cdots\partial u_r} \tag{3.9}$$

and

$$\prod_{j=1}^{r}(y_j-v_j)\frac{\partial^r\Omega}{\partial y_1\cdots\partial y_r} - \prod_{j=1}^{r}(x_j-u_j)\frac{\partial^r\Omega}{\partial x_1\cdots\partial x_r}$$

$$= (v_1v_2\cdots v_r)\frac{\partial^r\Omega}{\partial v_1\cdots\partial v_r} - (u_1u_2\cdots u_r)\frac{\partial^r\Omega}{\partial u_1\cdots\partial u_r}. \tag{3.10}$$

Again, differentiating Equations (3.4) and (3.5) with respect to y_j and x_j, $(j = 1,2,...,r)$, respectively, we get

(3.11)
$$\frac{\partial^{2r}\Omega}{\partial x_1\cdots\partial x_r\partial y_1\cdots\partial y_r} = \frac{(u_1v_1\cdots u_rv_r)}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^r))^{2r}}2^{2r}(A)_{2r}\Omega,$$

and

$$\frac{\partial^{2r}\Omega}{\partial y_1\cdots\partial y_r\partial x_1\cdots\partial x_r}=\frac{(u_1v_1\cdots u_rv_r)}{(1-\sum_{j=1}^{r}(2u_jx_j+2v_jy_j-u_j^2-v_j^r))^{2r}}2^{2r}(A)_{2r}\Omega, \tag{3.12}$$

respectively.

Hence

$$\frac{\partial^{2r}\Omega}{\partial x_1\cdots\partial x_r\partial y_1\cdots\partial y_r}=\frac{\partial^{2r}\Omega}{\partial y_1\cdots\partial y_r\partial x_1\cdots\partial x_r}.$$

Similarly, we can show that

$$\frac{\partial^{2r}\Omega}{\partial x_1\cdots\partial x_r\partial u_1\cdots\partial u_r}=\frac{\partial^{2r}\Omega}{\partial u_1\cdots\partial u_r\partial x_1\cdots\partial x_r}, \tag{3.13}$$

$$\frac{\partial^{2r}\Omega}{\partial x_1\cdots\partial x_r\partial v_1\cdots\partial v_r}=\frac{\partial^{2r}\Omega}{\partial v_1\cdots\partial v_r\partial x_1\cdots\partial x_r}, \tag{3.14}$$

$$\frac{\partial^{2r}\Omega}{\partial y_1\cdots\partial y_r\partial u_1\cdots\partial u_r}=\frac{\partial^{2r}\Omega}{\partial u_1\cdots\partial u_r\partial y_1\cdots\partial y_r}, \tag{3.15}$$

(3.16).
$$\frac{\partial^{2r}\Omega}{\partial y_1\cdots\partial y_r\partial v_1\cdots\partial v_r}=\frac{\partial^{2r}\Omega}{\partial v_1\cdots\partial v_r\partial y_1\cdots\partial y_r}.$$

On comparing Equation (3.4) with (3.6) and Equation (3.5) with (3.7), we find that

$$\prod_{j=1}^{r}(x_j-u_j)\frac{\partial^r\Omega}{\partial x_1\cdots\partial x_r}=(u_1u_2\cdots u_r)\frac{\partial^r\Omega}{\partial u_1\cdots\partial u_r}, \tag{3.17}$$

and

$$\prod_{j=1}^{r}(y_j-v_j)\frac{\partial^r\Omega}{\partial y_1\cdots\partial y_r}=(v_1v_2\cdots v_r)\frac{\partial^r\Omega}{\partial v_1\cdots\partial v_r}. \tag{3.18}$$

Now, in view of (3.1), Equations (3.17) and (3.18), yield the following recurrence relations:

$$\prod_{j=1}^{r}(x_j-u_j)\,\frac{\partial^r}{\partial x_1\cdots\partial x_r}C^{A}_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})$$

$$-(n_1 \dots n_r)C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y}) = 0, \tag{3.19}$$

and

$$\prod_{j=1}^{r}(y_j - v_j)\frac{\partial^r}{\partial y_1 \cdots \partial y_r}C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})$$

$$-(m_1 \dots m_r)C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y}) = 0, \tag{3.20}$$

respectively. Also, comparing (3.19) with (3.20), yields

$$\prod_{j=1}^{r}(x_j - u_j)\frac{\partial^r}{\partial x_1 \cdots \partial x_r}C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})$$

$$+\prod_{j=1}^{r}(y_j - v_j)\frac{\partial^r}{\partial y_1 \cdots \partial y_r}C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})$$

$$= (n_1 \dots n_r) + (m_1 \dots m_r)C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y}). \tag{3.21}$$

4 - Hypergeometric and operational matrix representations

Starting from (2.3) and using the results

$$(-nI)_{2k} = (-1)^{2k}\frac{n!}{(n-2k)!}I = 2^{2k}\prod_{i=1}^{2}\left(\frac{-n+i-1}{2}I\right)_k \tag{4.1}$$

and (1.4), we get

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y}) = (A)_{\mathbf{n}+\mathbf{m}}\prod_{j=1}^{r}\left\{\frac{(2x_j)^{n_j}(2y_j)^{m_j}}{n_j!m_j!}\right\}$$

$$\times\sum_{s_1,\cdots,s_r,k_1,\cdots,k_r=0}^{\infty}\prod_{j=1}^{r}\left\{\left(\frac{-n_j}{2}\right)_{s_j}\left(\frac{-n_j+1}{2}\right)_{s_j}\left(\frac{-m_j}{2}\right)_{k_j}\left(\frac{-m_j+1}{2}\right)_{k_j}\frac{x_j^{-2s_j}y_j^{-2k_j}}{s_j!k_j!}\right\}$$

$$\times\left[(I-A-(\mathbf{n}+\mathbf{m})I)_{\mathbf{s}+\mathbf{k}}\right]^{-1}. \tag{4.2}$$

Now, according to the definition of the matrix version of Lauricella series in n variables $F_B^{(n)}$ (1.7), Equation (4.2), yields the following hypergeometric matrix representation

$$C^A_{n_1,m_1,\dots,n_r,m_r}(\mathbf{x},\mathbf{y}) = (A)_{\mathbf{n}+\mathbf{m}}\prod_{j=1}^{r}\left\{\frac{(2x_j)^{n_j}(2y_j)^{m_j}}{n_j!m_j!}\right\}$$

$$F_B^{(2r)}\left[\frac{-n_1}{2}I, \frac{-n_1+1}{2}I, \cdots, \frac{-n_r}{2}I, \frac{-n_r+1}{2}I, \frac{-m_1}{2}I, \frac{-m_1+1}{2}I, \cdots,\right.$$

$$\left.\frac{-m_r}{2}I, \frac{-m_r+1}{2}I; I-A-(\mathbf{n}+\mathbf{m})I; x_1^{-2}, \cdots, x_r^{-2}, y_1^{-2}, \cdots, y_r^{-2}\right], \tag{4.3}$$

where

$$max\left\{|x_1^{-2}|, \cdots, |x_r^{-2}|, |y_1^{-2}| \cdots, |y_r^{-2}|\right\} < 1, \tag{4.4}$$

and all matrices in $F_B^{(2r)}$ are commutative in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.1). In its special case when $r = 1$, Equation (4.3) would obviously reduces to hypergeometric matrix representation for Gegenbauer matrix polynomials of two variables (see [**18**]), in it is corrected form, involving the matrix version of Applle's double hypergeometric series F_3 (1.9), as follows:

$$C_{n,m}^{A}(x,y) = \frac{(2x)^n(2y)^m}{n!m!}(A)_{n+m}$$

$$\times F_3\left[\frac{-n}{2}I, \frac{-n+1}{2}I, \frac{-m}{2}I, \frac{-m+1}{2}I; I-A-(n+m); x, y\right]. \tag{4.5}$$

Next, we build up the right Riemann-Liouville fractional derivative operator, which plays the role of augmenting parameters in the hypergeometric functions involved (see [**28**, Chapters 4 and 5]).

The following two formulas are well-known consequences of the derivative operator $\hat{D}_x$ and the integral operator $\hat{D}_x^{-1}$ [**19**]

$$\hat{D}_x^m x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)}x^{\lambda-m}, \tag{4.6}$$

$$\hat{D}_x^{-m} x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m+1)}x^{\lambda+m}, \tag{4.7}$$

$m \in N \cup \{0\}, \lambda \in C/\{-1,-2,....\}$.

According to (4.6) and (4.7), we introduce the following symbolic operational image for the multivariable Gegenbauer matrix polynomials $C_{n_1,m_1,\cdots,n_r,m_r}^{A}(\mathbf{x},\mathbf{y})$ (for more details see [**5**]):

$$C_{n_1,m_1,\cdots,n_r,m_r}^{A}(\mathbf{x},\mathbf{y}) =$$

$$\prod_{j=1}^{r}\left\{\frac{(2x_j/u_j)^{n_j}(2y_j/v_j)^{m_j}}{n_j!m_j!}\right\}t^{A+(\mathbf{n}+\mathbf{m}-1)I}\Gamma^{-1}(A)\Gamma(A+(\mathbf{n}+\mathbf{m})I)$$

$$exp\left[-\left(\sum_{j=1}^{r}\left((2x_j/u_j)^{-2}\hat{D}_t^{-1}\hat{D}_{u_j}^{2}t^{-1}+(2y_j/v_j)^{-2}\hat{D}_t^{-1}\hat{D}_{v_j}^{2}t^{-1}\right)\right)\right]$$

$$\left\{t^{A+(\mathbf{n}+\mathbf{m}-1)I}\prod_{j=1}^{r}\left\{u_j^{n_j}v_j^{m_j}\right\}\right\}. \tag{4.8}$$

Derivation of the result (4.8)

Denoting the right-hand side of assertion (4.8) by I and using the series expansion of the exponential function, we find that

$$I=\prod_{j=1}^{r}\left\{\frac{(2x_j/u_j)^{n_j}(2y_j/v_j)^{m_j}}{n_j!m_j!}\right\}t^{-A-(\mathbf{n}+\mathbf{m}-1)I}\Gamma^{-1}(A)\Gamma(A+(\mathbf{n}+\mathbf{m})I)$$

$$\prod_{j=1}^{r}\left\{\sum_{s_j=0}^{\infty}\sum_{k_j=0}^{\infty}\frac{(-1)^{s_j+k_j}(2x_j/u_j)^{-2s_j}(2y_j/v_j)^{-2k_j}}{s_j!k_j!}\right\}$$

$$\prod_{j=1}^{r}\left\{\hat{D}_{u_j}^{2s_j}\left\{u_j^{n_j}\right\}\hat{D}_{v_j}^{2k_j}\left\{v_j^{m_j}\right\}\right\}\left\{\hat{D}_t^{-(\mathbf{s}+\mathbf{k})}t^{A+(\mathbf{n}+\mathbf{m}-\mathbf{s}-\mathbf{k}-1)I}\right\}.$$

Now, by virtue of the operators (4.6) and (4.7) and by employing (2.3), we find the left-hand side of assertion (4.8).

It may of interest to point out that the representation (4.8), in particular, $(r=1)$, yields the following interesting operational matrix representation for Gegenbauer matrix polynomials $C_{n,m}^{A}(x,y)$ (see Equation (2.8)):

$$C_{n,m}^{A}(x,y)=\left\{\frac{(2x/u)^{n}(2y/v)^{m}}{n!m!}\right\}t^{-A-(n+m-1)I}\Gamma^{-1}(A)\Gamma(A+(n+m)I)$$

$$\exp\left[-\left((2x/u)^{-2}\hat{D}_t^{-1}\hat{D}_u^{2}t^{-1}+(2y/v)^{-2}\hat{D}_t^{-1}\hat{D}_v^{2}t^{-1}\right)\right]$$

$$\left\{t^{A+(n+m-1)I}\left\{u^{n}v^{m}\right\}\right\}. \tag{4.9}$$

5 - Generating functions

In this section we aim at establishing two additional generating functions for the Gegenbauer matrix polynomials $C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})$. First, according to Equation (2.3), we get

$$\sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty} [(A)_{\mathbf{n}+\mathbf{m}}]^{-1} C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})\, u_1^{n_1} v_1^{m_1}\cdots u_r^{n_r} v_r^{m_r}$$

$$= \sum_{n_1,m_1,s_1,k_1,\cdots,n_r,m_r,s_r,k_r=0}^{\infty} (A)_{\mathbf{n}+\mathbf{m}+\mathbf{s}+\mathbf{k}} [(A)_{\mathbf{n}+\mathbf{m}+\mathbf{2s}+\mathbf{2k}}]^{-1}$$

$$\prod_{j=1}^{r}\left\{\frac{(-1)^{s_j+k_j}(2x_j)^{n_j}(2y_j)^{m_j} u_j^{n_j+2s_j} v_j^{m_j+2k_j}}{n_j! m_j! s_j! k_j!}\right\}, \tag{5.1}$$

which on using the results (1.3) and (4.1), gives us

$$\sum_{n_1,m_1,\ldots,n_r,m_r=0}^{\infty} [(A)_{\mathbf{n}+\mathbf{m}}]^{-1} C^A_{n_1,m_1,\ldots,n_r,m_r}(\mathbf{x},\mathbf{y})\, u_1^{n_1} v_1^{m_1}\ldots u_r^{n_r} v_r^{m_r}$$

$$= \sum_{n_1,m_1,\ldots,n_r,m_r=0}^{\infty} \prod_{j=1}^{r}\left\{\frac{(2x_j u_j)^{n_j}(2y_j v_j)^{m_j}}{n_j! m_j!}\right\}$$

$$\times \sum_{s_1,k_1,\ldots,s_r,k_r=0}^{\infty} (A+(\mathbf{n}+\mathbf{m})I)_{\mathbf{s}+\mathbf{k}}$$

$$\left[\left(\frac{A+(\mathbf{n}+\mathbf{m})I}{2}\right)_{\mathbf{s}+\mathbf{k}}\right]^{-1}\left[\left(\frac{A+(\mathbf{n}+\mathbf{m}+\mathbf{1})I}{2}\right)_{\mathbf{s}+\mathbf{k}}\right]^{-1}$$

$$\times \prod_{j=1}^{r}\left\{\frac{(-1)^{s_j+k_j} u_j^{2s_j} v_j^{2k_j}}{4^{s_j+k_j}! s_j! k_j!}\right\}. \tag{5.2}$$

Now, from (5.2) and with the aid of the known formula [**28**, p. 52(3)]:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty} f(m)\frac{(x_1+\ldots+x_n)^m}{m!}, \tag{5.3}$$

we can easily establish the following generating relation

$$\sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty} [(A)_{\mathbf{n}+\mathbf{m}}]^{-1} C^{A}_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})\, u_1^{n_1} v_1^{m_1} \cdots u_r^{n_r} v_r^{m_r}$$

$$= \sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty} \prod_{j=1}^{r} \left\{ \frac{(2x_j u_j)^{n_j} (2y_j v_j)^{m_j}}{n_j! m_j!} \right\}$$

$$\times {}_1F_2\left[A + (\mathbf{n}+\mathbf{m})I; \frac{A + (\mathbf{n}+\mathbf{m})I}{2}, \frac{A + (\mathbf{n}+\mathbf{m}+1)I}{2}; \right.$$

$$\left. -\left(\frac{u_1^2 + v_1^2 + \ldots + u_r^2 + v_r^2}{4} \right) \right], \tag{5.4}$$

$$\left| -\left(\frac{u_1^2 + v_1^2 + \ldots + u_r^2 + v_r^2}{4} \right) \right| < \infty,$$

where ${}_1F_2$ is a special case of the generalized matrix hypergeometric series ${}_pF_q$ (see(1.10))such that all matrices in ${}_1F_2$ commuting in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.1).

Again and directly, from (2.3), by exploiting the same procedure leading to Equation (5.3) and making use of the familiar notation for multi-variable hypergeometric functions (see [**27**, p. 38, Equation 1.4(24)]), we thus obtain the following generating functions:

$$\sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty} [(A)_{\mathbf{n}+\mathbf{m}}]^{-1} (B)_{\mathbf{n}+\mathbf{m}} C^{A}_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y})\, u_1^{n_1} v_1^{m_1} \cdots u_r^{n_r} v_r^{m_r}$$

$$= \sum_{n_1,m_1,\cdots,n_r,m_r=0}^{\infty} (B)_{\mathbf{n}+\mathbf{m}} \prod_{j=1}^{r} \left\{ \frac{(2x_j u_j)^{n_j} (2y_j v_j)^{m_j}}{n_j! m_j!} \right\}$$

$$\times {}_3F_2\left[A + (\mathbf{n}+\mathbf{m})I, \frac{B + (\mathbf{n}+\mathbf{m})I}{2}, \frac{B + (\mathbf{n}+\mathbf{m})I}{2}; \right.$$

$$\left. \frac{A + (\mathbf{n}+\mathbf{m})I}{2}, \frac{A + (\mathbf{n}+\mathbf{m}+1)I}{2}; -\left(\frac{u_1^2 + v_1^2 + \ldots + u_r^2 + v_r^2}{4} \right) \right], \tag{5.5}$$

$$\left| -\left(\frac{u_1^2 + v_1^2 + \ldots + u_r^2 + v_r^2}{4} \right) \right| < 1,$$

$$\sum_{n_1,m_1,\ldots,n_r,m_r=0}^{\infty} \prod_{j=1}^{r} \{(B_j)_{n_j} (D_j)_{m_j}\} [(A)_{\mathbf{n}+\mathbf{m}}]^{-1}$$

$$C^A_{n_1,m_1,\ldots,n_r,m_r}(\mathbf{x},\mathbf{y})\,u_1^{n_1}v_1^{m_1}\ldots u_r^{n_r}v_r^{m_r}$$

$$= \sum_{n_1,m_1,\ldots,n_r,m_r=0}^{\infty}\prod_{j=1}^{r}\left\{\frac{(2x_ju_j)^{n_j}(2y_jv_j)^{m_j}}{n_j!m_j!}\right\}$$

$$\times F^{1:2;\ldots;2}_{2:0;\ldots;0}\left[\begin{array}{l} A+(\mathbf{n}+\mathbf{m})I \quad : \frac{B_1+n_1I}{2},\frac{B_1+(n_1+1)I}{2};\cdots;\frac{B_r+n_rI}{2},\frac{B_r+(n_r+1)I}{2};\\ \frac{A+(\mathbf{n}+\mathbf{m})I}{2},\frac{A+(\mathbf{n}+\mathbf{m}+1)I}{2}:---,---;\cdots;---,---;\end{array}\right.$$

(5.6)

$$\left.\begin{array}{l}\frac{D_1+m_1I}{2},\frac{D_1+(m_1+1)I}{2};\cdots;\frac{D_r+m_rI}{2},\frac{D_r+(m_r+1)I}{2};\\ ----,----;\cdots;----,----;\end{array} -u_1^2,\cdots,-u_r^2,-v_1^2,\cdots,-v_r^2\right],$$

$$max\{|-u_1^2|,\cdots,|-u_r^2|,|-v_1^2|,\cdots,|-v_r^2|\}<1,$$

where all matrices in ${}_3F_2$ and $F^{1:2;\ldots;2}_{2:0;\ldots;0}$ commuting in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.1).

6 - Concluding remarks

Remark (i). The material developed in Sections 3-5 provides several important properties of the multi-variable Gegenbauer matrix polynomials. In this regard, the integral representation in (2.17) together with the definitions the so-called incomplete 2D Hermite polynomials (see [**29**]):

$$h_{n,m}(x,y;\tau) = n!m!\sum_{s=0}^{min(n,m)}\frac{x^{n-s}y^{m-s}\tau^s}{(n-s)!(m-s)!s!}, \tag{6.1}$$

and the Hermite family:

$$H_{n,m}(x,y;z,w|\tau) = n!m!\sum_{s=0}^{min(n,m)}\binom{n}{s}\binom{m}{s}r!\tau^rH_{n-s}(x,y)H_{m-s}(z,w), \tag{6.2}$$

suggest us to introduce the following other types of multi-variable Gegenbauer matrix polynomials:

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y};\tau) = \frac{\Gamma^{-1}(A)}{n_1!m_1!\cdots n_r!m_r!}$$

$$\times \int_0^\infty e^{-t} t^{A+(\mathbf{n}+\mathbf{m}-1)I} \prod_{j=1}^{r} \left\{ h_{n_j,m_j} \left(x_j, y_j; \frac{\tau}{t} \right) \right\} dt, \tag{6.3}$$

and

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y};\mathbf{z},\mathbf{w}|\tau) = \frac{\Gamma^{-1}(A)}{n_1!m_1!\cdots n_r!m_r!}$$

$$\times \int_0^\infty e^{-t} t^{A+(\mathbf{n}+\mathbf{m}-1)I} \prod_{j=1}^{r} \left\{ H_{n_j,m_j} \left(x_j, \frac{y_j}{t}; z_j, \frac{w_j}{t} | \frac{\tau}{t} \right) \right\} dt. \tag{6.4}$$

The evaluating the integrals in (6.3) and (6.4) leads us to the following series representations

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y};\tau)$$

$$= \sum_{s_1=0}^{min(n_1,m_1)} \cdots \sum_{s_r=0}^{min(n_r,m_r)} \prod_{j=1}^{r} \left\{ \frac{x_j^{n_j-s_j} y^{m_j-s_j} \tau^{s_j}}{(n_j-s_j)!(m_j-s_j)!s_j!} \right\} (A)_{\mathbf{n}+\mathbf{m}-\mathbf{s}}, \tag{6.5}$$

and

$$C^A_{n_1,m_1,\cdots,n_r,m_r}(\mathbf{x},\mathbf{y};\mathbf{z},\mathbf{w}|\tau)$$

$$= \prod_{j=1}^{r} \left\{ n_j m_j \sum_{s_j=0}^{min(n_j,m_j)} \sum_{p_j=0}^{\left[\frac{n_j-s_j}{2}\right]} \sum_{q_j=0}^{\left[\frac{m_j-s_j}{2}\right]} \frac{x_j^{n_j-2s_j-p_j} z_j^{m_j-2s_j-q_j} y_j^{p_j} w_j^{q_j} \tau^{s_j}}{(n_j-s_j-2p_j)!(m_j-s_j-2q_j)!s_j!p_j!q_j!} \right\}$$

$$\times (A)_{\mathbf{n}+\mathbf{m}-\mathbf{s}-\mathbf{p}-\mathbf{q}}, \tag{6.6}$$

respectively. For the matrix polynomials defined in (6.5) and (6.6), we can establish a number of properties including generating relations, recurrence relations, differential relations and hypergeometric matrix representations. The details involved in these derivations are fairly straightforward and are being left as an exercise for the interested reader.

Remark (ii). Unfortunately the technique applied in [**22**, p.278(6)] to obtain the hyper geometric representation for the classical Gegenbauer polynomials can not be applied in case of our the multi-variable Gegenbauer polynomials (2.1), where here the multinomial formula contains $4r, r \in \mathbb{N}$. Therefore, we choose here, to establish an analog result as in [**18**], for the case when $r = 2$. That is when (see (2.1)):

$$\left(1-\sum_{j=1}^{2}\left(2u_jx_j+2v_jy_j-u_j^2-v_j^2\right)\right)^{-A}$$

$$=\sum_{n_1,m_1,n_2,m_2=0}^{\infty} C^{A}_{n_1,m_1,n_2,m_2}\left(x_1,x_2,y_1,y_2\right)u_1^{n_1}v_1^{m_1}u_2^{n_2}v_2^{m_2}. \tag{6.7}$$

Now, we can write the left-hand side of (6.7) in the form:

$$\left[1-\frac{2u_1(x_1-1)}{\beta^2}-\frac{2v_1(y_1-1)}{\beta^2}-\frac{2u_2(x_2-1)}{\beta^2}-\frac{2v_2(y_2-1)}{\beta^2}-\frac{2u_1v_1}{\beta^2}\right.$$

$$\left.-\frac{2u_1v_2}{\beta^2}-\frac{2u_1v_1}{\beta^2}-\frac{2u_2v_2}{\beta^2}-\frac{2u_1u_2}{\beta^2}-\frac{2v_1v_2}{\beta^2}\right]^{-2A}(1-u_1-v_1-u_2-v_2)^{-2A}$$

$$=\sum_{n_1,m_1,n_2,m_2=0}^{\infty} C^{A}_{n_1,m_1,n_2,m_2}\left(x_1,x_2,y_1,y_2\right)u_1^{n_1}v_1^{m_1}u_2^{n_2}v_2^{m_2}, \tag{6.8}$$

where $\beta = (1-u_1-v_1-u_2-v_2)$. On expanding the multinomial expansion in (6.8), simplifying and comparing the coefficients of $u_1v_1u_2v_2$, we lead finally to the formula:

$$C^{A}_{n_1,m_1,n_2,m_2}\left(x_1,x_2,y_1,y_2\right)=\frac{(2A)_{n_1+m_1+n_2+m_2}}{n_1!m_1!n_2!m_2!}$$

$$\times\sum_{k_1,\ldots,k_{10}=0}^{\infty}\frac{(2A+(n_1+m_1+n_2+m_2)I)_{k_1+k_2+k_3+k_4}\left[(A+\frac{1}{2}I)_{k_1+\ldots+k_{10}}\right]^{-1}}{k_1!\ldots k_{10}!}$$

$$\times(-n_1)_{k_1+k_5+k_6+k_9}(-m_1)_{k_2+k_5+k_7+k_{10}}(-n_2)_{k_3+k_7+k_8+k_9}(-m_2)_{k_4+k_6+k_8+k_{10}}$$

$$\left(\frac{1-x_1}{2}\right)^{k_1}\left(\frac{1-y_1}{2}\right)^{k_2}\left(\frac{1-x_2}{2}\right)^{k_3}\left(\frac{1-y_2}{2}\right)^{k_4}\left(\frac{1}{2}\right)^{k_5}\cdots\left(\frac{1}{2}\right)^{k_{10}}. \tag{6.9}$$

It may of interest to point out that, if we let $n_2 = m_2 = k_3 = k_4 = k_6 = \ldots = k_{10} = 0$, in (6.9) and employing the matrix version of Srivastava's triple series

H_C (see Equation (1.8)), then we shall obtain a hypergeometric representation for the Gegenbauer matrix polynomials of two variables in the following form:

$$C^A_{n,m}(x,y) = \frac{(2A)_{n+m}}{n!m!}$$

$$\times H_C\left[-nI, -mI, 2A+(n+m)I; A+\frac{1}{2}I; \frac{1}{2}, \frac{1-x}{2}, \frac{1-y}{2}\right], \tag{6.10}$$

where all matrices in H_C commuting in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.1).

Remark (iii). It seems that the author in [**18**], (also as noticed in [**26**]), is not familiar with the notations and definitions of hypergeometric functions of two and three variables, these are Appell's double series and Srivastava's triple series, which illustrated fairly, fully in [**27**] and [**28**]. In this regard it is important to note that the assertions (4.5) and (6.10) are the correct versions of the Equations (49) and (51) in [**18**].

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Maged G. Bin-Saad
Aden University
Department of Mathematics
Kohrmakssar, P.O. Box 6014
Aden, Yemen
e-mail: mgbinsaad@yahoo.com